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A CRITERION FOR VERTEX EXTREMAL LENGTH PARABOLIC GRAPHS AND ITS APPLICATION

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Abstract. We give a criterion for vertex extremal length parabolicity of locally finite planar graphs, and use it to show that a disk triangulation graph is circle packing parabolic if and only if its immediate finer graphs are circle packing parabolic.

1. Introduction

In 1985, Thurston gave a fascinating conjecture [12] that some 'nice' circle packing in a simply connected domain would approximate the Riemann map, which was later turned out to be true by Rodin and Sullivan [8]. Since then, together with the rapid growth of computer technology and computer related theory such as discrete mathematics, circle packings have become one of the hottest topics in the discrete version of conformal function theory, and also in other branches of mathematics.

Among all the interesting topics about circle packings, however, the one which has interested us most is the study of combinatorial structure of circle packings. A typical problem in this area is to determine the properties of circle packings that are completely, or at least partially, explained by their combinatorics. For example, let us ask the following question: "could there be two different circle packings, one of them packs the whole plane and the other packs the unit disk, with the same combinatorics?" The answer for this question is negative—it was proved by He and Schramm in [4], and we will see it in Section 2.

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Since the combinatorics of circle packings are mostly described by graphs, we first introduce some notations and terminologies for graphs. A graph $G = (V, E)$ is a pair of the vertex set V and the edge set E, as usual. The vertex set V could be either finite or infinite, but it has to be at most countable. The edge set E is definitely the set of edges, where each edge $e \in E$ corresponds to two different vertices v, w in V. In this case we use the notation $e = [v, w]$ and say that e connects the vertices v and w, or equivalently we say that the vertices v, w are the endpoints of e. Note that we do not allow more than one edge corresponding to a pair of vertices, nor an edge of the form $[v, v]$. In other words, there is no *multiple edge* nor a *self loop*. Since we do not distinguish $[v, w]$ from $[w, v]$, each edge can be considered an unordered pair in $V \times V$.

For each $v \in V$, the *neighbor* of v is defined by $N(v) = \{w \in V : [v, w] \in E\}$, and the *degree*, or *valence*, of $v \in V$ is $\deg(v) = |N(v)|$, the number of elements in $N(v)$. The graph is called *locally finite* if $\deg(v) < \infty$ for every $v \in V$, and called *uniformly bounded* or *of finite valence* if there exists $C \in \mathbb{R}$ such that $\deg(v) \leq C$ for all $v \in V$.

A finite path γ in G is a finite sequence of vertices $\gamma = [v_0, v_1, \ldots, v_n]$ such that $[v_j, v_{j+1}] \in E$, $j = 0, \ldots n-1$. In this case we say that the path γ connects v_0 and v_n. An infinite path can be defined similarly. A path is called *simple* if it visits every vertex at most once, and an infinite path is called *transient* if it visits every vertex at most finitely many. A graph is called *connected* if every two vertices can be connected by a finite path in the graph.

The following definition is due to Cannon [2].

Definition 1. Suppose $G = (V, E)$ is a connected graph. We say that G is *Vertex Extremal Length parabolic*, or *VEL-parabolic*, if there exists a function $m : V \to \mathbb{R}^+ \cup \{0\}$ such that

$$\text{(a)} \sum_{v \in V} m(v)^2 < \infty,$$

$$\text{(b)} \sum_{v \in V(\gamma)} m(v) = \infty \quad \text{for any transient path } \gamma.$$

Here $V(\gamma)$ denotes the vertices which γ visits. A function m satisfying these properties will be called a *parabolic v-metric*. If no parabolic v-metric can be defined on V, we say that G is VEL-hyperbolic.

Of course the definition of VEL-parabolicity/hyperbolicty is related to the so-called vertex extremal length between a point $v \in V$ and ∞.

Vertex extremal length, and 'Edge Extremal Length' defined by Duffin [3], are the discrete analogs of classical extremal length [1, Chap. 4]. However, edge extremal length is closely related to the random walk on graphs, while vertex extremal length is related to circle packings [5].

To describe our result, let us introduce some more definitions. A graph G is called *planar* if it can be continuously embedded into the plane $\mathbb{C} = \mathbb{R}^2$, and a planar graph is called locally finite if it can be embedded locally finitely. Finally, a measurable set $A \subset \mathbb{C}$ is called *τ-fat* if for every $x \in A$ and $r > 0$ such that $D(x,r) := \{z : |z-x| < r\}$ does not contain A, the following inequality holds:

$$\text{area}(A \cap D(x,r)) \geq \tau \cdot \text{area}(D(x,r)). \tag{1.1}$$

Now we are ready to describe our main result.

THEOREM 2. *Suppose $G = (V,E)$ is a connected infinite planar graph which is locally finite in $\mathbb{C}$, and $\mathcal{Q} = (Q_v : v \in V)$ is a collection of τ-fat sets satisfying the following properties:*

(1) for every $v \in V$, Q_v is a compact connected set in $\mathbb{C}$;
(2) $\mathcal{Q}$ is locally finite in $\mathbb{C}$; that is, for every compact set $K \subset \mathbb{C}$, there are only finitely many $v \in V$ such that $Q_v \cap K \neq \emptyset$. Also we assume that every $x \in \mathbb{C}$ is included in Q_v for at most $M < \infty$ vertices $v \in V$, where M does not depend on x;
(3) if $[v,w] \in E$, then $Q_v \cap Q_w \neq \emptyset$.

Then G is a VEL-parabolic graph.

The concept of fat sets was introduced by Schramm [9]. It means that the set is nowhere thin. For example, let us show that every disk $D = D(z,R)$ is 1/4-fat. This means that for every $x \in D$ and $0 < r < R + |x-z|$, we need to show that $\text{area}(D \cap D(x,r)) \geq (1/4)\text{area}D(x,r)$. But we may assume without loss of generality that $D = D(0,1)$, $0 \leq x < 1$, and $0 < r < 1 + x$. Then because $D(x - r/2, r/2) \subset D \cap D(x,r)$, we have

$$\text{area}(D \cap D(x,r)) \geq \text{area}(D(x-r/2, r/2)) = \frac{1}{4} \cdot \text{area}(D(x,r)),$$

as desired.

Perhaps one has noticed that the collection $\mathcal{Q}$ in Theorem 2 resembles the definition of (circle) packings. Actually in [5] He and Schramm proved Theorem 2 for the case when $\mathcal{Q}$ is a packing, hence our result should be accepted as an extension of their result.

2. Circle packing and tangency graph

Let V be an index set. With an indexed *circle packing* $\mathcal{P} = (P_v : v \in V)$, we mean a collection of closed geometric disks in the plane $\mathbb{C} = \mathbb{R}^2$ with disjoint interiors. An *interstice* of a circle packing $\mathcal{P}$ is a connected component of $\mathbb{C} \setminus (\bigcup_{v\in V} P_v)$, and the *carrier* is the union of the packed disks and the finite(bounded) interstices. In Figure 1, the region enclosed by the bold curve is the carrier, and each component of the gray part is a finite interstice. Note that when the cardinality of V is finite, the carrier is nothing but the complement of the unbounded interstice.

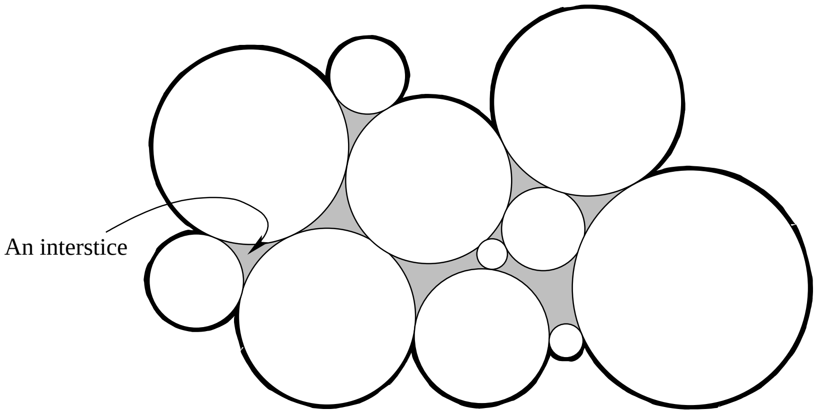

FIGURE 1. Carrier of a finite circle packing

Every circle packing $\mathcal{P}$ is associated to a graph $G(\mathcal{P}) = (V(\mathcal{P}), E(\mathcal{P}))$, called the *tangency graph* or the *nerve* of the circle packing $\mathcal{P}$, which is defined as follows: the vertex set $V(\mathcal{P})$ is nothing but the index set V, and the elements of the edge set $E(\mathcal{P})$ are exactly the pairs $[v, w]$, $v, w \in V(\mathcal{P})$, such that $P_v \cap P_w \neq \emptyset$. Clearly the tangency graph describes the combinatorial pattern of the packing, hence it inherits a lot of important properties from the associated circle packing. For example, one can easily see that $G(\mathcal{P})$ is connected if and only if the set $\bigcup_{v\in V} P_v$ is connected in $\mathbb{C}$.

Now suppose G is a disk triangulation graph. This means that G is the 1-skeleton of a topological triangulation of an open disk, thus G must be an infinite planar graph such that every face of G is a topological triangle (with three vertices on its boundary). Then there exists a

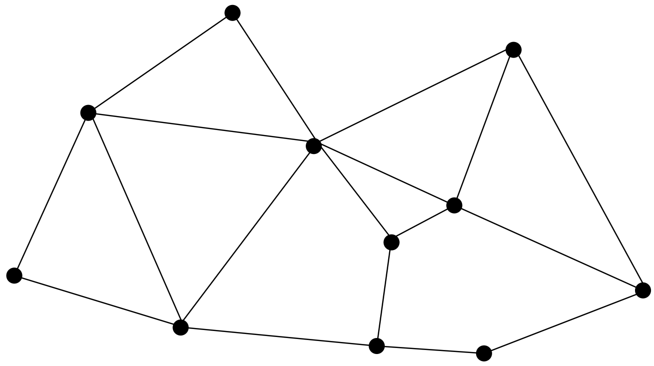

FIGURE 2. Tangency graph of the packing in Figure 1

circle packing $\mathcal{P}$ whose tangency graph is combinatorially equivalent to G and whose carrier is a simply connected domain in $\mathbb{C}$, and the simply connected domain (i.e., carrier) can be chosen as either the whole plane $\mathbb{C}$ or the unit disk $\mathbb{D}$, but not both [4]. In other words, the combinatorial property of a circle packing (i.e., the tangency graph) completely determines whether it can pack the whole complex plane or not, as we mentioned in the introduction. We say that a disk triangulation graph G is *circle packing parabolic* (cp-parabolic) if the carrier of the associated circle packing is the whole plane, and we say that G is *circle packing hyperbolic* (cp-hyperbolic) otherwise.

Now we know that there are two classes of disk triangulation graphs, cp-parabolic and cp-hyperbolic, but how can we determine its circle packing type? This is where VEL-type, either parabolic or hyperbolic, is involved.

THEOREM 3. *A disk triangulation graph is cp-parabolic if and only if it is VEL-parabolic. Equivalently, a disk triangulation graph is cp-hyperbolic if and only if it is VEL-hyperbolic.*

The above theorem is due to He and Schramm [5]. In fact, Theorem 3 says that VEL-type is an extension of cp-type, since cp-type can be defined only for disk triangulation graphs while VEL-type can be defined for every infinite connected graph.

Now we can read Theorem 2 more accurately. It says that if we have a disk triangulation graph and a collection of compact sets satisfying some properties similar to parabolic circle packings, then the triangulation graph in consideration is really associated to a parabolic circle packing.

This flexibility could save some efforts to show that a triangulation graph is cp-parabolic, as we will do in Section 4.

For those who want to study the theory of circle packings, we recommend [10] for a brief summary, and [11] for more detailed introduction.

3. Fat sets and a proof of Theorem 2

We begin this section with the following lemma.

LEMMA 4. *Suppose A and B are τ-fat sets for some $\tau > 0$. If $A \cap B \neq \emptyset$, then $A \cup B$ is $\tau/4$-fat.*

Proof. Suppose $x \in A \cup B$. Without loss of generality, we assume that $x \in A$. If $D(x, r/2)$ does not contain A, then

$$\begin{aligned}\operatorname{area}((A \cup B) \cap D(x, r)) &\geq \operatorname{area}(A \cap D(x, r)) \geq \operatorname{area}(A \cap D(x, r/2)) \\ &\geq \tau \cdot \operatorname{area}(D(x, r/2)) = (\tau/4)\operatorname{area}(D(x, r))\end{aligned}$$

as desired.

If $D(x, r/2)$ contains A but $D(x, r)$ does not contain $A \cup B$, we choose $y \in A \cap B \subset B$ such that $|x - y| < r/2$. Then since $D(y, r/2) \subset D(x, r)$, $A \subset D(x, r/2)$, and $A \cup B \not\subseteq D(x, r)$, one can easily see that $D(y, r/2)$ does not contain B. Therefore,

$$\begin{aligned}\operatorname{area}((A \cup B) \cap D(x, r)) &\geq \operatorname{area}(B \cap D(x, r)) \geq \operatorname{area}(B \cap D(y, r/2)) \\ &\geq \tau \cdot \operatorname{area}(D(y, r/2)) = (\tau/4)\operatorname{area}(D(x, r)),\end{aligned}$$

which completes the proof. □

Suppose A is a bounded τ-fat set in $\mathbb{C}$. Then for any $z \in \mathbb{C}$ and $r > 0$, the square of the diameter of $D(z, r) \cap A$ is bounded by a constant times the area of $D(z, 3r) \cap A$. To see this, suppose $x, y \in D(z, r) \cap A$. Then because $D(x, |x - y|) \subset D(z, 3r)$, the τ-fatness of A implies that

$$\begin{aligned}\operatorname{area}(D(z, 3r) \cap A) &\geq \operatorname{area}(D(x, |x - y|) \cap A) \\ &\geq \tau \cdot \operatorname{area}(D(x, |x - y|)) = \tau\pi \cdot |x - y|^2.\end{aligned}$$

Since this inequality holds for every $x, y \in D(z, r) \cap A$, we have

$$\tau\pi \cdot \operatorname{diam}(D(z, r) \cap A)^2 \leq \operatorname{area}(D(z, 3r) \cap A). \tag{3.1}$$

Now we are ready to prove Theorem 2.

Proof of Theorem 2. Suppose that $G = (V, E)$ is a connected infinite planar graph that is locally finite in $\mathbb{C}$, and $\mathcal{Q} = (Q_v : v \in V)$ is a collection of τ-fat sets satisfying the properties (1) $\sim$ (3) in Theorem 2. Let $V_0 = \{v \in V : Q_v \ni 0\}$, the subcollection of V such that for each $v \in V_0$, the corresponding Q_v contains the origin. Note that V_0 is a finite set since $\mathcal{Q}$ is locally finite. Thus there exists a positive number r_1 such that $D(0, r_1) \supset Q_v$ for all $v \in V_0$.

Next, let $V_1 = \{v \in V : Q_v \cap D(0, 2r_1) \neq \emptyset \text{ and } v \notin V_0\}$, and note that V_1 is also finite since $\mathcal{Q}$ is locally finite. Thus there exists $r_2 > 2r_1$ such that $D(0, r_2) \supset Q_v$ for all $v \in V_1$. Inductively, for every $n \geq 2$ we define

$$V_n = \{v \in V : Q_v \cap D(0, 2r_n) \neq \emptyset \text{ and } v \notin V_0 \cup V_1 \cup \cdots \cup V_{n-1}\},$$

and choose $r_{n+1} > 2r_n$ such that $D(0, r_{n+1}) \supset Q_v$ for all $v \in V_n$.

Now we define

$$m(v) := \begin{cases} \dfrac{\operatorname{diam}(Q_v \cap D(0, 2r_n))}{n r_n}, & \text{if } v \in V_n \text{ for some } n \geq 1; \\ 0, & \text{if } v \in V_0. \end{cases}$$

We claim that the function m is a parabolic v-metric. But the equation (3.1) implies that for $n \geq 1$,

$$\begin{aligned} \sum_{v \in V_n} m(v)^2 &= \sum_{v \in V_n} \frac{\operatorname{diam}(Q_v \cap D(0, 2r_n))^2}{n^2 r_n^2} \\ &\leq \frac{1}{\tau \pi n^2 r_n^2} \sum_{v \in V_n} \operatorname{area}(Q_v \cap D(0, 6r_n)) \\ &\leq \frac{M \cdot \operatorname{area}(D(0, 6r_n))}{\tau \pi n^2 r_n^2} = \frac{36M}{\tau n^2}, \end{aligned}$$

since the property (2) of Theorem 2 implies that there are at most M overlaps of Q_v in $D(0, 6r_n)$. Thus

$$\sum_{v \in V} m(v)^2 = \sum_{n=0}^{\infty} \sum_{v \in V_n} m(v)^2 \leq \sum_{n=1}^{\infty} \frac{36M}{\tau n^2} < \infty,$$

showing that m is square summable.

Next, we fix $n \in \mathbb{N}$ and assume that $[w_1, w_2, \ldots, w_s]$ is a finite path in G such that $N(w_1) \cap V_{n-1} \neq \emptyset$, $N(w_s) \cap V_{n+1} \neq \emptyset$, and $w_j \in V_n$ for all $j = 1, \ldots, s$. Then we have $Q_{w_s} \cap \{z \in \mathbb{C} : |z| = 2r_n\} \neq \emptyset$ because otherwise either w_s does not belong to V_n or $N(w_s)$ would not contain a vertex in V_{n+1}. Also note that $Q_{w_j} \cap \{z \in \mathbb{C} : |z| = r_n\} \neq \emptyset$ for some $j \in \{1, 2, \dots, s\}$ because we chose r_n so that $Q_v \subset D(0, r_n)$ for all $v \in V_1 \cup \cdots \cup V_{n-1}$. Moreover, $Q_{w_1} \cup Q_{w_2} \cup \cdots \cup Q_{w_s}$ is a compact connected set. Therefore, we must have

$$\sum_{j=1}^{s} \operatorname{diam}(Q_{w_j} \cap D(0, 2r_n)) \geq 2r_n - r_n = r_n,$$

hence

$$\sum_{j=1}^{s} m(w_j) = \sum_{j=1}^{s} \frac{\operatorname{diam}(Q_{w_j} \cap D(0, 2r_n))}{nr_n} \geq \frac{1}{n}.$$

This means that if γ is a transient path in G, then there exists k such that

$$\sum_{v \in V(\gamma) \cap V_n} m(v) \geq \frac{1}{n}$$

for all $n \geq k$. Thus

$$\sum_{v \in V(\gamma)} m(v) = \sum_{n=0}^{\infty} \left(\sum_{v \in V(\gamma) \cap V_n} m(v) \right) \geq \sum_{n=k}^{\infty} \frac{1}{n} = \infty,$$

as desired. Thus m is a parabolic v-metric defined on V, hence $G = (V, E)$ is VEL-parabolic. This completes the proof of Theorem 2. □

4. An application

Suppose $G = (V, E)$ is a disk triangulation graph. Then we divide each face of G into four triangles by connecting the midpoints of the edges (Figure 3), and get a new disk triangulation graph G_1 which we call the *first immediate finer graph of* G. Formally, the vertex set of G_1

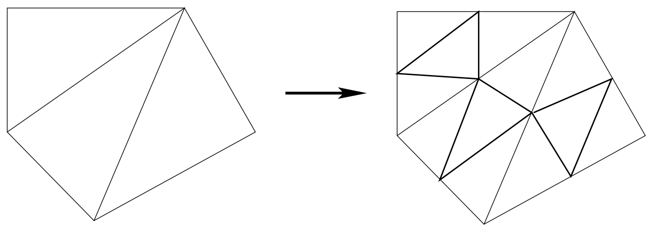

FIGURE 3. First immediate finer graph

is $V \sqcup E$, the disjoint union of V and E, and an edge $[v, w]$ appears in G_1 if and only if (i) v, w are two different edges of G which are partially bounding the same face; or (ii) $v \in V$, $w \in E$, and v is an end point of w; or (iii) $v \in E$, $w \in V$, and w is an end point of v. We repeat this process and get G_2, the first immediate finer graph of G_1, or the *second* immediate finer graph of G, and so on. If G' is an n-th immediate finer graph of G for some $n \in \mathbb{N}$, then we just say that G' is an immediate finer graph of G.

In fact, there could be two natural types of "immediate" finer graphs of a disk triangulation graph G; one is what we have just defined above, and the other is obtained by connecting each vertices of a triangle to its barycenter—we will call it the *barycentric finer graph* (Figure 4). Then

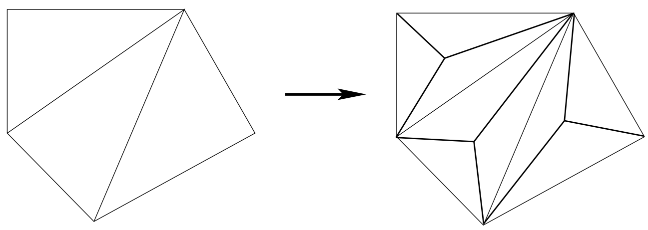

FIGURE 4. Barycentric finer graph

the question is, do the cp-types of G and its immediate finer graphs coincide? For the barycentric finer graph, the answer is positive and very easy to prove. Suppose $\mathcal{P}$ is a circle packing whose tangency graph is combinatorially the same as G. Then by inscribing a circle in each of the triangular interstice of $\mathcal{P}$ (Figure 5), we could get a new circle packing $\mathcal{P}'$ whose tangency graph is the barycentric finer graph of G. Definitely the carrier of $\mathcal{P}'$ is the same as that of $\mathcal{P}$, and this proves that the cp-types of G and its barycentric finer graph must coincide.

It is, however, not so straightforward for the first immediate finer graph we defined above, although it is very tempting to believe. Now we give a positive answer for this question as an application of our main theorem.

THEOREM 5. *Suppose $G = (V, E)$ is a disk triangulation graph and $G' = (V', E')$ is an immediate finer graph of G. Then the cp-types of G and G' are the same.*

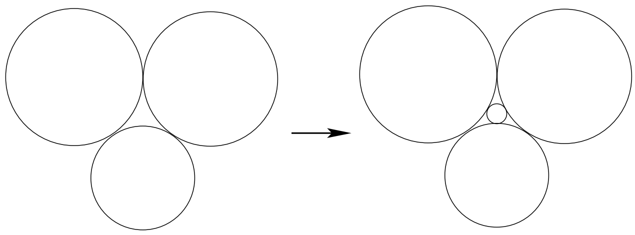

Figure 5. Circle packing related to the barycentric finer graph

Proof. Without loss of generality, we may assume that G' is the first immediate finer graph of G. First, we prove that the cp-parabolicity of G' implies the cp-parabolicity of G. Thus suppose G' is cp-parabolic, or VEL-parabolic. Then there exists a parabolic v-metric m' defined on the vertex set V'. Moreover, we can treat V as a subset of V' since there is a natural injection $V \hookrightarrow V'$. Thus for each $v \in V$ we can define $N(v) \subset V'$, the set of neighbors of v in V'. Note that $N(v)$ consists of exactly three vertices of V' for every $v \in V$, and every $w \in V'$ can belong to $N(v)$ for at most two vertices $v \in V$. Now we define

$$m(v) := 2 \cdot \max\{m'(w) : w \in \{v\} \cup N(v)\}.$$

Then we have

$$\begin{aligned}\sum_{v\in V} m(v)^2 &= \sum_{v\in V} 4 \cdot \max\{m'(w)^2 : w \in \{v\} \cup N(v)\} \\ &\leq 4 \sum_{v\in V} \left(\sum_{w\in\{v\}\cup N(v)} m'(w)^2 \right) \leq 8 \sum_{w\in V'} m'(w)^2 < \infty,\end{aligned}$$

as desired.

Now suppose $\gamma = [v_0, v_1, v_2, \ldots]$ is a transient path in G. Then it can be realized as a transient path in G'; that is, there exists a sequence of vertices $w_i \in V' \backslash V$ for $i = 0, 1, 2, \ldots$, such that $\gamma' = [v_0, w_0, v_1, w_1, v_2, \ldots]$ becomes a transient path in G'. Then because

$$m'(v_i) + m'(w_i) \leq 2 \cdot \max\{m'(v_i), m'(w_i)\} \leq m(v_i)$$

for all i, we have

$$\infty = \sum_{i=0}^{\infty} (m'(v_i) + m'(w_i)) \leq \sum_{i=0}^{\infty} m(v_i),$$

and this shows that m is a parabolic v-metric of G. We conclude that G is VEL-parabolic, hence cp-parabolic.

The converse is nontrivial, but we can prove it using Theorem 2. Suppose G is cp-parabolic. Then by Corollary 0.5 of [4] and Theorem 1.2 of [5], there exists a circle packing $\mathcal{P} = (P_v : v \in V)$ whose tangency graph is combinatorially equivalent to G and whose carrier is $\mathbb{C}$. Therefore G can be embedded in $\mathbb{C}$ so that each $v \in V$ is the center of P_v and each $[v, w] \in E$ is a straight line segment connecting the centers of P_v and P_w.

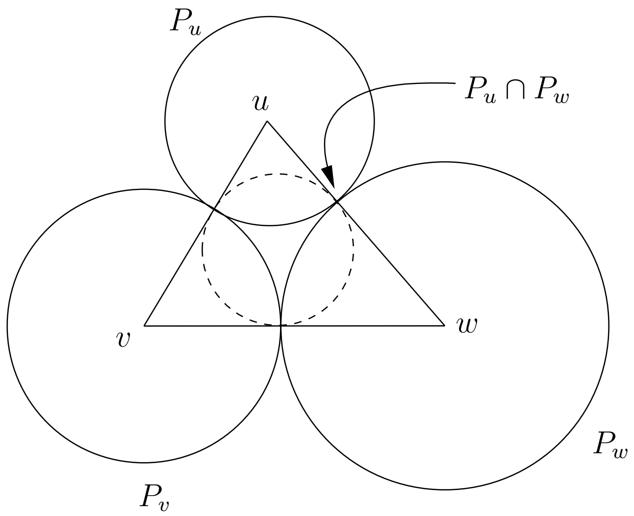

FIGURE 6. The inscribe circle and packed disks

In this embedding, each face of G is a Euclidean triangle, and one can easily check that if f is a face of G with vertices u, v, w, then the inscribed circle of f passes through the points $P_u \cap P_v$, $P_v \cap P_w$, and $P_w \cap P_u$ (Figure 6). Therefore, if $e = [v, w]$ is an edge of G and f_1 and f_2 are two faces of G sharing the edge e, the union of the closed inscribed disks of f_1 and f_2 is a connected compact set in $\mathbb{C}$, because these two disks meet at $P_v \cap P_w$. We denote by P_e this union of two disks.

Now let $\mathcal{Q} = (P_w : w \in V' = V \sqcup E)$; i.e., for $w \in V$ we assign the disk P_w of the circle packing $\mathcal{P}$, and for $w \in V' \setminus V = E$ we assign the union of the inscribed disks tangent to w. Then because each disk is 1/4-fat, every P_w is (1/16)-fat by Lemma 4. Moreover, $\mathcal{Q}$ is definitely locally finite, and for every $x \in \mathbb{C}$ there are at most 7 vertices $w \in V'$ such that $x \in P_w$. The number 7 actually occurs when $\{x\} = P_u \cap P_v$ for some $[u, v] \in E$. It is also trivial to check that if $[w, w'] \in E'$ then $P_w \cap P_{w'} \neq \emptyset$. Now by Theorem 2, we conclude G' is VEL-parabolic, or cp-parabolic. □

Theorem 5 finds its application in the author's incoming paper [6], where we study the relation between circle packings and Riemann surfaces of class $\mathcal{S}$ (cf. [7], Chap. XI). In fact, in [6] Theorem 5 plays an important role to study the relation between a disk triangulation graph and its (arbitrary) finer graphs.

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