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CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS THROUGH RECORD STATISTICS

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Abstract. A family of continuous probability distribution has been characterized through the difference of two conditional expectations, conditioned on a non-adjacent record statistic. Also, a result based on the unconditional expectation and a conditional expectation is used to characterize a family of distributions. Further, some of its deductions are also discussed.

1. Introduction

Let $(X_n, n \geq 1)$ be a sequence of independent, identically distributed continuous random variables with the distribution function (*df*) $F(x)$ and the probability density function (*pdf*) $f(x)$. Let $X_{u(s)}$ be the s-th upper record value. Then the conditional *pdf* of $X_{u(s)}$ given $X_{u(r)} = x,\ 1 \leq r < s$ is (Ahsanullah, 1995)

$$f(X_{u(s)}|X_{u(r)} = x) = \frac{1}{\Gamma(s-r)}[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1}\frac{f(y)}{\overline{F}(x)}, \tag{1.1}$$

where $\overline{F}(x) = 1 - F(x)$.

Lee ([3]) has shown that $E[X_{u(n+1)} - X_{u(n)}|X_{u(m)} = y] = c$ and $E[X_{u(n+2)} - X_{u(n)}|X_{u(m)} = y] = 2c,\ c > 0, n \geq m+1$ if and only if the distribution is exponential. Further, Lee et al. ([4]) have extended it and showed that $E[X_{u(n+3)} - X_{u(n)}|X_{u(m)} = y] = 3c$ and $E[X_{u(n+4)} - X_{u(n)}|X_{u(m)} = y] = 4c,\ c > 0, n \geq m+1$ if and only if the distribution is exponential. We, in the present paper have extended their results in a rather very simple way and established that $E[h(X_{u(s)}) - h(X_{u(r)})|X_{u(m)} = x] = (s-r)c$ if and only if $\overline{F}(x) = e^{-\frac{h(x)}{c}},\ c > 0$, where $h(x)$ is a monotonic and differentiable function of x and $r \geq m$. Further it has also been shown that

$$E[h(X_{u(s)}) - h(X_{u(r)})] + h(x) = E[h(X_{u(s)})|X_{u(r)} = x]$$

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if and only if the *df* is

$$\overline{F}(x) = e^{-\frac{h(x)}{c}},\ c > 0. \tag{1.2}$$

2. Characterization theorems

Theorem 2.1. *Let X be an absolutely continuous random variable with the df $F(x)$ and the pdf $f(x)$ on the support (α, β), where α and β may be finite or infinite. Then for $m \leq r < s$*

$$E[h(X_{u(s)}) - h(X_{u(r)}) | X_{u(m)} = x] = (s - r)c \tag{2.1}$$

if and only if

$$\overline{F}(x) = e^{-\frac{h(x)}{c}},\ c > 0, \tag{2.2}$$

where $h(x)$ is a monotonic and differentiable function of x such that $h(x) \to 0$ as $x \to \alpha$ and $h(x)\overline{F}(x) \to 0$ as $x \to \beta$.

Proof. We have,

$$\begin{aligned} & E[h(X_{u(s)}) - h(X_{u(r)}) | X_{u(m)} = x] \\ &= \frac{1}{\Gamma(s-m)} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-m-1} \frac{f(y)}{\overline{F}(x)} dy \\ &\quad - \frac{1}{\Gamma(r-m)} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{r-m-1} \frac{f(y)}{\overline{F}(x)} dy. \end{aligned} \tag{2.3}$$

Now, it is easy to see that (2.2) implies (2.1) (Athar et al., [2]).
For sufficiency part, let $c^* = (s - r)c$, then

$$\begin{aligned} & \frac{1}{\Gamma(s-m)} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-m-1} f(y) dy \\ & - \frac{1}{\Gamma(r-m)} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{r-m-1} f(y) dy = c^* \overline{F}(x). \end{aligned} \tag{2.4}$$

Differentiating $(r - m)$ times both the sides of (2.4) with respect to x, we get

$$\frac{1}{\Gamma(s-r)} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \frac{f(y)}{\overline{F}(x)} dy = h(x) + c^*. \tag{2.5}$$

Integrating LHS of (2.5) by parts and simplifying, we have

$$\begin{aligned} & \frac{1}{\Gamma(s-r-1)[\overline{F}(x)]} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-2} f(y) dy \\ & + \frac{1}{\Gamma(s-r)[\overline{F}(x)]} \int_x^\beta h'(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \overline{F}(y) dy = h(x) + c^*. \end{aligned} \tag{2.6}$$

This in view of (2.5), reduces to

$$\frac{1}{\Gamma(s-r)} \int_x^\beta h'(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \overline{F}(y) dy = c\, \overline{F}(x). \tag{2.7}$$

Differentiating (2.7) $(s-r)$ times with respect to x, we obtain

$$h'(x)\overline{F}(x) = cf(x)$$

and hence the result. □

Remark 3.1. At $s = r+1, s = r+2$ and $h(x) = x$, we get the result as obtained by Lee ([3]).

Remark 3.2. At $s = r+3, s = r+4$ and $h(x) = x$, this reduces to the result as obtained by Lee et al. ([4]).

Remark 3.3. At $r = m, E[h(X_{u(s)})|X_{u(r)} = x] = h(x) + (s-r)c$ as obtained by Athar et al. ([2]).

Theorem 2.2. *Under the conditions as given in Theorem 2.1 and for* $1 \le r < s$

$$E[h(X_{u(s)}) - h(X_{u(r)})] + h(x) = E[h(X_{u(s)})|X_{u(r)} = x] \tag{2.8}$$

if and only if

$$\overline{F}(x) = e^{-\frac{h(x)}{c}},\ c > 0. \tag{2.9}$$

Proof. It is easy to see that (2.9) implies (2.8) and hence the necessary part. For sufficiency part we have,

$$\begin{aligned} &E[h(X_{u(s)}) - h(X_{u(r)})] + h(x) \\ &= \frac{1}{\Gamma(s-r)} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \frac{f(y)}{\overline{F}(x)} dy. \end{aligned} \tag{2.10}$$

Integrating R.H.S. of (2.10) by parts we have

$$\begin{aligned} &E[h(X_{u(s)}) - h(X_{u(r)})] + h(x) \\ &= \frac{1}{\Gamma(s-r-1)[\overline{F}(x)]} \int_x^\beta h(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-2} f(y) dy \\ &+ \frac{1}{\Gamma(s-r)[\overline{F}(x)]} \int_x^\beta h'(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \overline{F}(y) dy. \end{aligned} \tag{2.11}$$

In view of (2.10) and (2.11), we have

$$\begin{aligned} &E[h(X_{u(s)}) - h(X_{u(s-1)})]\overline{F}(x) \\ &= \frac{1}{\Gamma(s-r)} \int_x^\beta h'(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \overline{F}(y) dy. \end{aligned} \tag{2.12}$$

Since $E[h(X_{u(s)}) - h(X_{u(s-1)})] = c$ is independent of x, (2.12) can be written as

$$\frac{1}{\Gamma(s-r)} \int_x^\beta h'(y)[-\ln \overline{F}(y) + \ln \overline{F}(x)]^{s-r-1} \overline{F}(y) dy = c\overline{F}(x). \tag{2.13}$$

Differentiate (2.13) $(s-r)$ times with respect to x, to get

$$\overline{F}(x) = \frac{cf(x)}{h'(x)}$$

and hence the theorem. □

Table 1: Examples based on the $df\, F(x) = 1 - e^{-\frac{h(x)}{c}}$

Distribution	$\boldsymbol{F(x)}$	$\boldsymbol{c}$	$\boldsymbol{h(x)}$
Exponential	$1 - e^{-\theta x}$ $0 < x < \infty$	$\frac{1}{\theta}$	x
Weibull	$1 - e^{-\theta x^p}$ $0 < x < \infty$	$\frac{1}{\theta}$	x^p
Pareto	$1 - \left(\frac{x}{\alpha}\right)^{-\theta}$ $\alpha < x < \infty$	$\frac{1}{\theta}$	$\log\left(\frac{x}{\alpha}\right)$
Lomax	$1 - \left[1 + \left(\frac{x}{\alpha}\right)\right]^{-p}$ $0 < x < \infty$	$\frac{1}{p}$	$\log\left[1 + \left(\frac{x}{\alpha}\right)\right]$
Gompertz	$1 - \exp[-\frac{\lambda}{\mu}(e^{\mu x} - 1)]$ $0 < x < \infty$	$\frac{\mu}{\lambda}$	$e^{\mu x} - 1$
Beta of the I kind	$1 - (1 - x)^{\theta}$ $0 < x < 1$	$-\frac{1}{\theta}$	$\log\,(1 - x)$
Beta of the II kind	$1 - (1 + x)^{-1}$ $0 < x < \infty$	1	$\log\,(1 + x)$
Extreme value I	$1 - \exp[-e^{x}]$ $-\infty < x < \infty$	1	e^{x}
Log logistic	$1 - (1 + \theta x^p)^{-1}$ $0 < x < \infty$	1	$\log\,(1 + \theta x^p)$
Burr Type IX	$1 - \left[1 + \frac{c((1+e^{x})^{k} - 1)}{2}\right]^{-1}$ $-\infty < x < \infty$	1	$\log\left[1 + \frac{c((1+e^{x})^{k} - 1)}{2}\right]$
Burr Type XII	$1 - (1 + \theta x^p)^{-m}$ $0 < x < \infty$	$\frac{1}{m}$	$\log\,(1 + \theta x^p)$

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