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ON FUZZY BITOPOLOGICAL SPACES IN ŠOSTAK'S SENSE (II)

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Abstract. In this paper, we have use a fuzzy bitopological space (X, τ_1, τ_2) to create a family τ_{ij}^s which is a supra fuzzy topology on X. Also, we introduce and study the concepts of $r-(\tau_i, \tau_j)$-generalized fuzzy regular closed, $r-(\tau_i, \tau_j)$-generalized fuzzy strongly semi-closed and $r-(\tau_i, \tau_j)$-generalized fuzzy regular strongly semi-closed sets in fuzzy bitopological space in the sense of Šostak. Also, these classes of fuzzy subsets are applied for constructing several type of fuzzy closed mapping and some type of fuzzy separation axioms called fuzzy binormal, fuzzy mildly binormal and fuzzy almost pairwise normal.

1. Introduction and preliminaries

The concept of fuzzy topology was first defined in 1968 by Chang [2]. A Chang's fuzzy topology is a crisp subfamily of some family of fuzzy sets and fuzziness in the concept of openness of a fuzzy set has not been considered, which seems to be a drawback in the process of fuzzification of the concept of the topological space. Therefore, in 1985 Šostak [18], introduce the fundamental concept of a fuzzy topological structure as an extension of both crisp topology and Chang's fuzzy topology, in the sense that not only the object were fuzzified, but also the axiomatics. In [19, 20] Šostak gave some rules and showed how such an extension can be realized. Chattopadhyay et. al [3, 4] have redefined the similar concept. In [15, 5] Ramadan gave a similar definition namely "Smooth fuzzy topology" for lattice $L = [0, 1]$, it has been developed in many direction [7, 9, 10, 12, 13, 17]. In this paper, we introduce and study the concepts of $r-(\tau_i, \tau_j)$-generalized fuzzy regular closed, $r-(\tau_i, \tau_j)$-generalized fuzzy strongly semi-closed and $r-(\tau_i, \tau_j)$-generalized fuzzy regular strongly semi-closed sets which are based on the alternative effect of fuzzy closure and fuzzy interior operators with respect to two fuzzy topologies. Also, these classes of fuzzy subsets are applied for constructing several types of fuzzy closed mappings

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and some types of fuzzy separation axioms called fuzzy binormal, fuzzy mildly binormal and fuzzy almost pairwise normal.

Throughout this paper, let X be a nonempty set $I = [0,1]$, $I_0 = (0,1]$ and I^X denote the set of all fuzzy subset of X. For $\alpha \in I$, $\underline{\alpha} = \alpha$ for every $x \in X$. A fuzzy set λ is quasi-coincident with a fuzzy set μ, denoted by $\lambda q \mu$, if there exists $x \in X$ such that $\lambda(x) + \mu(x) > 1$, if λ is not quasi-coincident with μ, we denote $\lambda\ \tilde{q}\mu$, $\lambda\ \tilde{q}\mu$ if and only if $\lambda \le \underline{1} - \mu$ [14].

Definition 1.1 ([6, 18]). A mapping $\tau : I^X \longrightarrow I$ is called a supra fuzzy topology on X if it satisfies the following conditions:

(S1) $\tau(\underline{0}) = \tau(\underline{1}) = 1$.

(S2) $\tau(\bigvee_{i\in J} \mu_i) \ge \bigwedge_{i \in J} \tau(\mu_i)$ for any $\{\mu_i : i \in J\} \subseteq I^X$.

The pair (X, τ) is called supra fuzzy topological space (briefly, sfts). A supra fuzzy topology τ is called fuzzy topology on X if

(T) $\tau(\mu_1 \wedge \mu_2) \ge \tau(\mu_1) \wedge \tau(\mu_2)$ for any $\mu_1, \mu_2 \in I^X$.

The pair (X, τ) is called fuzzy topological space (briefly fts). The triple (X, τ_1, τ_2) is called fuzzy bitopological space (briefly, fbts), where τ_1 and τ_1 are fuzzy topologies on X. Throughout this paper, the indices $i, j \in \{1, 2\}$ and $i \ne j$.

Theorem 1.1 ([4]). *Let (X, τ) be a fts. For each $\lambda \in I^X$ and $r \in I_0$ we define an operator $C_\tau : I^X \times I_0 \longrightarrow I^X$ as follows*:

$$C_\tau(\lambda, r) = \bigwedge\{\mu : \lambda \le \mu, \tau(\underline{1} - \mu) \ge r\}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_0$ the operator C_τ satisfies the following conditions:

(i) $C_\tau(\underline{0}, r) = \underline{0}$.
(ii) $\lambda \le C_\tau(\lambda, r)$.
(iii) $C_\tau(\lambda, r) \vee C_\tau(\mu, r) = C_\tau(\lambda \vee \mu, r)$.
(iv) $C_\tau(\lambda, r) \le C_\tau(\lambda, s)$ *if* $r \le s$.
(v) $C_\tau(C_\tau(\lambda, r), r) = C_\tau(\lambda, r)$.

Theorem 1.2 ([8]). *Let (X, τ) be a fts. For each $\lambda \in I^X$ and $r \in I_0$, we define an operator $I_\tau : I^X \times I_0 \longrightarrow I^X$ as follows*:

$$I_\tau(\lambda, r) = \bigvee\{\mu : \mu \le \lambda, \tau(\mu) \ge r\}.$$

For each $\lambda, \mu \in I^X$ and $r, s \in I_0$ the operator I_τ satisfies the following conditions:

(i) $I_\tau(\underline{1} - \lambda, r) = \underline{1} - C_\tau(\lambda, r)$ *and* $C_\tau(\underline{1} - \lambda, r) = \underline{1} - I_\tau(\lambda, r)$.
(ii) $I_\tau(\underline{1}, r) = \underline{1}$.
(iii) $I_\tau(\lambda, r) \le \lambda$.
(iv) $I_\tau(\lambda, r) \wedge I_\tau(\mu, r) = I_\tau(\lambda \wedge \mu, r)$.
(v) $I_\tau(\lambda, r) \ge I_\tau(\lambda, s)$ *if* $r \le s$.
(vi) $I_\tau(I_\tau(\lambda, r), r) = I_\tau(\lambda, r)$.

Definition 1.2 ([16, 11]). Let (X, τ_1, τ_2) be a fbts. For $\lambda \in I^X$ and $r \in I_0$:

(i) λ is called $r-(\tau_i,\tau_j)$-fuzzy strongly semi-open (briefly, $r-(\tau_i,\tau_j)$-fsso) if there exists $\mu \in I^X$ with $\tau_i(\mu) \geq r$ such that $\mu \leq \lambda \leq I_{\tau_i}(C_{\tau_j}(\mu, r), r)$.

(ii) λ is called $r-(\tau_i,\tau_j)$-fuzzy strongly semi-closed (briefly, $r-(\tau_i,\tau_j)$-fssc) if there exists $\mu \in I^X$ with $\tau_i(\underline{1}-\mu) \geq r$ such that $C_{\tau_i}(I_{\tau_j}(\mu, r), r) \leq \lambda \leq \mu$.

(iii) λ is called $r-(\tau_i,\tau_j)$-fuzzy regular open (briefly, $r-(\tau_i,\tau_j)$-fro) if $\lambda = I_{\tau_i}(C_{\tau_j}(\lambda, r), r)$.

(iv) λ is called $r-(\tau_i,\tau_j)$-fuzzy regular closed (briefly, $r-(\tau_i,\tau_j)$-frc) if $\lambda = C_{\tau_i}(I_{\tau_j}(\lambda, r), r)$.

The set of all $r-(\tau_i,\tau_j)$-fsso, $r-(\tau_i,\tau_j)$-fssc, $r-(\tau_i,\tau_j)$-fro and $r-(\tau_i,\tau_j)$-frc sets of a fbts (X, τ_1, τ_2) will be denoted by $r-(\tau_i,\tau_j)-FSSO(X)$, $r-(\tau_i,\tau_j)-FSSC(X)$, $r-(\tau_i,\tau_j)-FRO(X)$ and $r-(\tau_i,\tau_j)-FRC(X)$ respectively.

Theorem 1.3 ([16])**.** *Let (X, τ_1, τ_2) be a fbts. For $\lambda \in I^X$ and $r \in I_0$, the following statements are equivalent*:

(i) *λ is $r-(\tau_i,\tau_j)$-fsso.*

(ii) *$\underline{1}-\lambda$ is $r-(\tau_i,\tau_j)$-fssc.*

(iii) $\lambda \leq I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(\lambda, r), r), r)$.

(iv) $C_{\tau_i}(I_{\tau_j}(C_{\tau_i}(\underline{1}-\lambda, r), r), r) \leq \underline{1}-\lambda$.

Definition 1.3 ([16])**.** Let (X, τ_1, τ_2) be a fbts. For $\lambda \in I^X$ and $r \in I_0$:

(i) The $r-(\tau_i,\tau_j)$-fuzzy strongly semi-interior of λ, denoted by $SSI_{ij}(\lambda, r)$ is defined by

$$SSI_{ij}(\lambda, r) = \bigvee\{\mu \in I^X : \mu \leq \lambda, \quad \mu \quad \text{is} \quad r-(\tau_i,\tau_j)-\text{fsso}\}.$$

(ii) The $r-(\tau_i,\tau_j)$-fuzzy strongly semi-closure of λ, denoted by $SSC_{ij}(\lambda, r)$ is defined by

$$SSC_{ij}(\lambda, r) = \bigwedge\{\mu \in I^X : \mu \geq \lambda, \quad \mu \quad \text{is} \quad r-(\tau_i,\tau_j)-\text{fssc}\}.$$

Theorem 1.4 ([16])**.** *Let (X, τ_1, τ_2) be a fbts. For $\lambda, \mu \in I^X$ and $r \in I_0$*:

(i) $SSC_{ij}(\underline{0}, r) = \underline{0}$ *and* $SSI_{ij}(\underline{1}, r) = \underline{1}$.

(ii) $I_{\tau_i}(\lambda, r) \leq SSI_{ij}(\lambda, r) \leq \lambda \leq SSC_{ij}(\lambda, r) \leq C_{\tau_i}(\lambda, r)$.

(iii) $\lambda \leq \mu \Rightarrow SSI_{ij}(\lambda, r) \leq SSI_{ij}(\mu, r)$ *and* $SSC_{ij}(\lambda, r) \leq SSC_{ij}(\mu, r)$.

(iv) *λ is $r-(\tau_i,\tau_j)$-fsso if and only if* $\lambda = SSI_{ij}(\lambda, r)$.

(v) *λ is $r-(\tau_i,\tau_j)$-fssc if and only if* $\lambda = SSC_{ij}(\lambda, r)$.

(vi) $SSC_{ij}(SSC_{ij}(\lambda, r), r) = SSC_{ij}(\lambda, r)$.

Definition 1.4 ([8])**.** A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (X, \tau_1^*, \tau_2^*)$ from a fbts (X, τ_1, τ_2) to another fbts (X, τ_1^*, τ_2^*) is said to be fuzzy pairwise continuous (fpc, for short) if and only if $\tau_i(f^{-1}(\mu)) \geq \tau_i^*(\mu)$ for each $\mu \in I^Y$ and $i = 1, 2$.

If we put $\tau = \tau_1 = \tau_2$ and $\tau^* = \tau_1^* = \tau_2^*$, then the definition of fuzzy pairwise continuous mapping reduce to the corresponding fuzzy continuous mapping due to Šostak [18].

Definition 1.5 ([1])**.** A mapping $C : I^X \times I_0 \longrightarrow I^X$ is called a supra fuzzy closure operator on X if for $\lambda, \mu \in I^X$ and $r, s \in I_0$, it satisfies the following conditions:

(C1) $C(\underline{0}, r) = \underline{0}$.
(C2) $\lambda \leq C(\lambda, r)$.
(C3) $C(\lambda, r) \vee C(\mu, r) \leq C(\lambda \vee \mu, r)$.
(C4) $C(\lambda, r) \leq C(\lambda, s)$ if $r \leq s$.
(C5) $C(C(\lambda, r), r) = C(\lambda, r)$.
The pair (X, C) is called a supra fuzzy closure space.

2. Some types of fuzzy closeness in fuzzy bitopological spaces

Theorem 2.1. *Let (X, τ_1, τ_2) be a fbts. Define a mapping $\tau^s_{ij} : I^X \longrightarrow I$ on X by*:

$$\tau^s_{ij}(\lambda) = \bigvee\{r \in I_0 : SSI_{ij}(\lambda, r) = \lambda\}.$$

Then τ^s_{ij} is a supra fuzzy topology.

Proof. (S1) It is clearly since for all $r \in I_0$, $SSI_{ij}(\underline{0}, r) = \underline{0}$ and $SSI_{ij}(\underline{1}, r) = \underline{1}$.

(S2) Suppose that there exists $\lambda = \bigvee_{k\in J} \lambda_k \in I^X$ such that

$$\tau^s_{ij}(\lambda) < \bigwedge_{k\in J} \tau^s_{ij}(\lambda_k).$$

There exists $r_0 \in I_0$ such that

$$\tau^s_{ij}(\lambda) < r_0 \leq \bigwedge_{k\in J} \tau^s_{ij}(\lambda_k).$$

This implies that for all $k \in J$, there exists $r_k \in J$ with

$$SSI_{ij}(\lambda_k, r_k) = \lambda_k$$

such that

$$\tau^s_{ij}(\lambda_k) \geq r_k \geq r_0.$$

Then

$$SSI_{ij}(\lambda_k, r_0) \geq SSI_{ij}(\lambda_k, r_k) = \lambda_k.$$

But by Theorem 1.4(ii), we have $SSI_{ij}(\lambda_k, r_0) \leq \lambda_k$, thus

$$SSI_{ij}(\lambda_k, r_0) = \lambda_k.$$

This implies that

$$SSI_{ij}(\lambda, r_0) = SSI_{ij}(\bigvee_{k\in J} \lambda_k, r_0) \geq \bigvee_{k\in J} SSI_{ij}(\lambda_k, r_0) = \bigvee_{k\in J} \lambda_k = \lambda.$$

By using Theorem 1.4(ii), we obtain

$$\lambda = SSI_{ij}(\lambda, r_0).$$

That is $\tau^s_{ij}(\lambda) \geq r_0$. It is a contradiction. Thus

$$\tau^s_{ij}(\bigvee_{k\in J} \lambda_k) \geq \bigwedge_{k\in J} \lambda_k \text{ for any } \{\lambda_k : \lambda_k \in I^X\}.$$

Hence, τ_{ij}^s is a supra fuzzy topology. □

The following example shows that τ_{ij}^s is not fuzzy topology in general.

Example 2.1. Let $X = \{a, b\}$. Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ as follows:

$$\begin{array}{ll} \lambda_1(a) = 0.7 & \lambda_1(b) = 0.1 \\ \lambda_2(a) = 0.1 & \lambda_2(b) = 0.7 \\ \lambda_3(a) = 0.7 & \lambda_3(b) = 0.5 \\ \lambda_4(a) = 0.5 & \lambda_4(b) = 0.7 \end{array}$$

Define the fuzzy topologies $\tau_1, \tau_2 : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.3, & \text{if} \quad \lambda = \lambda_1, \lambda_2 \\ 0.5, & \text{if} \quad \lambda = \underline{0.1}, \underline{0.7} \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.5, & \text{if} \quad \lambda = \underline{0.2}, \underline{0.8} \\ 0, & \text{otherwise.} \end{cases}$$

Then for $0 < r \leq 0.3$, λ_3 and λ_4 are $r-(\tau_1, \tau_2)$-fsso sets and $\lambda_3 \wedge \lambda_4$ is not $r-(\tau_1, \tau_2)$-fsso set which implies that $\tau_{12}^s(\lambda_3) \wedge \tau_{12}^s(\lambda_4) > \tau_{12}^s(\lambda_1 \wedge \lambda_3) = 0$. Then τ_{12}^s is not fuzzy topology.

Theorem 2.2. *Let (X, τ_1, τ_2) be a fbts. For each $\lambda \in I^X$ and $r \in I_0$ we define a mapping $C_{\tau_{ij}^s} : I^X \times I_0 \longrightarrow I^X$ as follows:*

$$C_{\tau_{ij}^s}(\lambda, r) = \bigwedge \{\mu : \mu \geq \lambda, \tau_{ij}^s(\underline{1} - \mu) \geq r\}.$$

Then $(X, C_{\tau_{ij}^s})$ is a supra fuzzy closure space.

Proof. (C1), (C2) and (C4) follows directly from the definition of $C_{\tau_{ij}^s}$.

(C3) Since $\lambda \leq \lambda \vee \mu$ and $\mu \leq \lambda \vee \mu$ we have $C_{\tau_{ij}^s}(\lambda, r) \leq C_{\tau_{ij}^s}(\lambda \vee \mu, r)$ and $C_{\tau_{ij}^s}(\mu, r) \leq C_{\tau_{ij}^s}(\lambda \vee \mu, r)$. Then

$$C_{\tau_{ij}^s}(\lambda, r) \vee C_{\tau_{ij}^s}(\mu, r) \leq C_{\tau_{ij}^s}(\lambda \vee \mu, r).$$

(C5) From the definition of $C_{\tau_{ij}^s}$ we obtain

$$(1) \qquad C_{\tau_{ij}^s}(\lambda, r) \leq C_{\tau_{ij}^s}(C_{\tau_{ij}^s}(\lambda, r), r) \quad \text{for each} \quad \lambda \in I^X, r \in r_0.$$

Conversely, suppose that there exist $\lambda \in I^X, r \in I_0$ and $x \in X$ such that

$$C_{\tau_{ij}^s}(C_{\tau_{ij}^s}(\lambda, r), r)(x) > C_{\tau_{ij}^s}(\lambda, r)(x).$$

By the definition of $C_{\tau_{ij}^s}(\lambda, r)$, there exists $\mu \in I^X$ with $\mu \geq \lambda$ and

$$SSI_{ij}(\underline{1} - \mu, r) = \underline{1} - \mu$$

such that

$$C_{\tau_{ij}^s}(C_{\tau_{ij}^s}(\lambda, r), r)(x) > \mu(x) \geq C_{\tau_{ij}^s}(\lambda, r)(x).$$

On the other hand, since $\mu \geq \lambda$ and $\tau_{ij}^s(\underline{1} - \mu) \geq r$, then

$$C_{\tau_{ij}^s}(\lambda, r) \leq C_{\tau_{ij}^s}(\mu, r) = \mu$$

therefore

$$C_{\tau_{ij}^s}(C_{\tau_{ij}^s}(\lambda, r), r) \leq C_{\tau_{ij}^s}(\mu, r) = \mu.$$

It is a contradiction. Thus

$$C_{\tau_{ij}^s}(C_{\tau_{ij}^s}(\lambda, r), r) \leq C_{\tau_{ij}^s}(\lambda, r) \quad \text{for each} \quad \lambda \in I^X, r \in r_0. \tag{2}$$

From (1) and (2) we obtain (C5). □

Theorem 2.3. *Let (X, τ_1, τ_2) be a fbts. For $\lambda \in I^X$ and $r \in r_0$ we define a mapping $I_{\tau_{ij}^s} : I^X \times I_0 \longrightarrow I^X$ as follows:*

$$I_{\tau_{ij}^s}(\lambda, r) = \bigvee \{\mu : \mu \leq \lambda, \tau_{ij}^s(\mu) \geq r\}.$$

Then we have

$$I_{\tau_{ij}^s}(\underline{1} - \lambda, r) = \underline{1} - C_{\tau_{ij}^s}(\lambda, r).$$

Proof. For $\lambda \in I^X$, $r \in I_0$ we have

$$\begin{aligned} \underline{1} - C_{\tau_{ij}^s}(\lambda, r) &= \underline{1} - \bigwedge \{\mu : \mu \geq \lambda, \tau_{ij}^s(\underline{1} - \mu) \geq r\} \\ &= \bigvee \{\underline{1} - \mu : \underline{1} - \mu \leq \underline{1} - \lambda, \tau_{ij}^s(\underline{1} - \mu) \geq r\} \\ &= I_{\tau_{ij}^s}(\underline{1} - \lambda, r). \end{aligned}$$

□

Definition 2.1. Let (X, τ_1, τ_2) be a fbts and $\lambda \in I^X$. Then, for $r \in r_0$, λ is called:

(i) $r - (\tau_i, \tau_j)$-generalized fuzzy closed (briefly, $r - (\tau_i, \tau_j)$-gfc) set if $\lambda \leq \mu$ and $\tau_i(\mu) \geq r$ implies $C_{\tau_j}(\lambda, r) \leq \mu$.

(ii) $r - (\tau_i, \tau_j)$-generalized fuzzy regular closed (briefly, $r - (\tau_i, \tau_j)$-gfrc) set if $\lambda \leq \mu$ and μ is $r - (\tau_i, \tau_j)$-fro set implies $C_{\tau_j}(\lambda, r) \leq \mu$.

(iii) $r - (\tau_i, \tau_j)$-generalized fuzzy strongly semi-closed (briefly, $r - (\tau_i, \tau_j)$-gfssc) set if $\lambda \leq \mu$ and $\tau_i(\mu) \geq r$ implies $C_{\tau_{ji}^s}(\lambda, r) \leq \mu$.

(iv) $r - (\tau_i, \tau_j)$-generalized fuzzy regular strongly semi-closed (briefly, $r - (\tau_i, \tau_j)$-gfrssc) set if $\lambda \leq \mu$ and μ is r–(τ_i, τ_j)-fro set implies $C_{\tau_{ji}^s}(\lambda, r) \leq \mu$.

The set of all $r - (\tau_i, \tau_j)$-gfc, $r - (\tau_i, \tau_j)$-gfrc, $r - (\tau_i, \tau_j)$-gfssc and $r - (\tau_i, \tau_j)$-gfrssc sets of a fbts (X, τ_1, τ_2) will be denoted by $r - (\tau_i, \tau_j) - GFC(X)$, $r - (\tau_i, \tau_j) - GFRC(X)$, $r - (\tau_i, \tau_j) - GFSSC(X)$ and $r - (\tau_i, \tau_j) - GFRSSC(X)$ respectively.

Remark 2.1. The following diagram shows the relation between the above different types of fuzzy closeness in a fbts (X, τ_1, τ_2).

$$\begin{array}{ccccc} & & r - (\tau_i, \tau_j)\text{-fssc} & & \\ & & \downarrow & & \\ & & r - (\tau_i, \tau_j)\text{-gfssc} & & \\ & \nearrow & & \searrow & \\ r - (\tau_i, \tau_j)\text{-gfc} & & & & r - (\tau_i, \tau_j)\text{-gfrssc} \\ & \searrow & & \nearrow & \\ & & r - (\tau_i, \tau_j)\text{-gfrc} & & \end{array}$$

Example 2.2. Let $X = \{a, b, c\}$. We define the fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ \frac{1}{3}, & \text{if} \quad \lambda = \chi_{\{a\}} \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{\chi_{\{a,b\}}, \chi_{\{a,c\}}\} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ \frac{1}{3}, & \text{if} \quad \lambda \in \{\chi_{\{c\}}, \chi_{\{a,c\}}\} \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < r \leq \frac{1}{3}$, $\chi_{\{a\}}$ is $r - (\tau_1, \tau_2)$-gfrc but not $r - (\tau_1, \tau_2)$-gfc.

Example 2.3. Let $X = \{a, b, c, d\}$. We define the fuzzy topologies $\tau_1, \tau_2 : I^X \to I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{\chi_{\{d\}}, \chi_{\{a,d\}}\} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ \frac{1}{2}, & \text{if} \quad \lambda \in \{\chi_{\{c\}}, \chi_{\{a,b\}}\} \\ \frac{3}{4}, & \text{if} \quad \lambda \in \{\chi_{\{c,d\}}, \chi_{\{a,b,c\}}\} \\ 0, & \text{otherwise.} \end{cases}$$

For $0 < r \leq \frac{1}{2}$, $\chi_{\{a\}}$ is $r - (\tau_1, \tau_2)$-gfrssc but not $r - (\tau_1, \tau_2)$-gfrc. Also, $\chi_{\{a\}}$ is $r - (\tau_1, \tau_2)$-gfssc but not $r - (\tau_1, \tau_2)$-gfc, $\chi_{\{a,c\}}$ is $r - (\tau_2, \tau_1)$-gfssc but not $r - (\tau_2, \tau_1)$-fssc and $\chi_{\{c,d\}}$ is $r - (\tau_2, \tau_1)$-gfrssc but not $r - (\tau_2, \tau_1)$-gfssc.

Remark 2.2. The complement of $r - (\tau_i, \tau_j)$-gfc (resp. $r - (\tau_i, \tau_j)$-gfrc, $r - (\tau_i, \tau_j)$-gfssc, $r - (\tau_i, \tau_j)$-gfrssc) set is called $r - (\tau_i, \tau_j)$-gfo (resp. $r - (\tau_i, \tau_j)$-gfro, $r - (\tau_i, \tau_j)$-gfsso, $r - (\tau_i, \tau_j)$-gfrsso) set.

$$\begin{array}{ccccc} & & r - (\tau_i, \tau_j)\text{-fsso} & & \\ & & \downarrow & & \\ & & r - (\tau_i, \tau_j)\text{-gfsso} & & \\ & \nearrow & & \searrow & \\ r - (\tau_i, \tau_j)\text{-gfo} & & & & r - (\tau_i, \tau_j)\text{-gfrsso} \\ & \searrow & & \nearrow & \\ & & r - (\tau_i, \tau_j)\text{-gfro} & & \end{array}$$

The set of all $r - (\tau_i, \tau_j)$-gfo, $r - (\tau_i, \tau_j)$-gfro, $r - (\tau_i, \tau_j)$-gfsso and $r - (\tau_i, \tau_j)$-gfrsso sets of a fbts (X, τ_1, τ_2) will be denoted by $r - (\tau_i, \tau_j) - GFO(X)$, $r - (\tau_i, \tau_j) - GFRO(X)$, $r - (\tau_i, \tau_j) - GFSSO(X)$ and $r - (\tau_i, \tau_j) - GFRSSO(X)$ respectively.

Lemma 2.1. *Let (X, τ_1, τ_2) be a fbts and $\lambda \in I^X$. For $r \in r_0$ λ is:*

(i) *$r-(\tau_i,\tau_j)$-gfo set if $\mu \le \lambda$ and $\tau_i(\underline{1}-\mu) \ge r$, then $\mu \le I_{\tau_j}(\lambda, r)$.*
(ii) *$r-(\tau_i,\tau_j)$-gfro set if $\mu \le \lambda$ and μ is $r-(\tau_i,\tau_j)$-frc set, then $\mu \le I_{\tau_j}(\lambda, r)$.*
(iii) *$r-(\tau_i,\tau_j)$-gfsso set if $\mu \le \lambda$ and $\tau_i(\underline{1}-\mu) \ge r$, then $\mu \le I_{\tau^s_{ji}}(\lambda, r)$.*
(iv) *$r-(\tau_i,\tau_j)$-gfrsso set if $\mu \le \lambda$ and μ is $r-(\tau_i,\tau_j)$-frc set, then $\mu \le I_{\tau^s_{ji}}(\lambda, r)$.*

Theorem 2.4. *Let λ be $r-(\tau_i,\tau_j)$-fsso set in a fbts (X,τ_1,τ_2). Then*

$$C_{\tau_j}(\lambda,r) = C_{\tau_j}(I_{\tau_i}(\lambda,r),r) = C_{\tau^s_{ij}}(\lambda,r).$$

Proof. First: Let $\lambda \in r-(\tau_i,\tau_j)-FSSO(X)$. By Theorem 1.3, we have $\lambda \le I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(\lambda,r),r),r)$. This implies that

$$C_{\tau_j}(\lambda,r) \le C_{\tau_j}(I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(\lambda,r),r),r),r) = C_{\tau_j}(I_{\tau_i}(\lambda,r),r).$$

Conversely, since $I_{\tau_i}(\lambda,r) \le \lambda \le C_{\tau_j}(\lambda,r)$, then $C_{\tau_j}(I_{\tau_i}(\lambda,r),r) \le C_{\tau_j}(\lambda,r)$. Hence, $C_{\tau_j}(\lambda,r) = C_{\tau_j}(I_{\tau_i}(\lambda,r),r)$.
Second: Since λ is $r-(\tau_i,\tau_j)$-fsso set, then

$$\lambda \le I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(\lambda,r),r),r) \le C_{\tau_j}(I_{\tau_i}(\lambda,r),r).$$

Therefore

$$C_{\tau^s_{ji}}(\lambda,r) \le C_{\tau^s_{ji}}(C_{\tau_j}(I_{\tau_i}(\lambda,r),r),r) = C_{\tau_j}(I_{\tau_i}(\lambda,r),r) = C_{\tau_j}(\lambda,r).$$

Conversely, suppose that $C_{\tau^s_{ji}}(\lambda,r) < C_{\tau_j}(\lambda,r)$. Then, there exist $\rho \in r-(\tau_j,\tau_i)-FSSC(X)$ and $x \in X$ such that $\lambda \le \rho$ and

$$C_{\tau^s_{ji}}(\lambda,r)(x) \le \rho(x) < C_{\tau_j}(\lambda,r)(x).$$

Since $\lambda \le \rho$, then $C_{\tau_j}(\lambda,r) \le C_{\tau_j}(\rho,r)$. Since $\lambda \in r-(\tau_j,\tau_i)-FSSO(X)$, then by the first part we have, $C_{\tau_j}(I_{\tau_i}(\lambda,r),r) = C_{\tau_j}(\lambda,r) \le C_{\tau_j}(\rho,r)$. Then

$$\lambda \le I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(\lambda,r),r),r) \le I_{\tau_i}(C_{\tau_j}(\rho,r),r).$$

Thus

$$C_{\tau_j}(\lambda,r) \le C_{\tau_j}(I_{\tau_i}(C_{\tau_j}(\rho,r),r),r) \le \rho.$$

It is a contradiction. Then $C_{\tau_j}(\lambda,r) \le C_{\tau^s_{ji}}(\lambda,r)$. Thus $C_{\tau_j}(\lambda,r) = C_{\tau^s_{ji}}(\lambda,r)$. From the first and second parts we have, $C_{\tau_j}(\lambda,r) = C_{\tau_j}(I_{\tau_i}(\lambda,r),r) = C_{\tau^s_{ij}}(\lambda,r)$.
□

3. Some types of fuzzy closed mappings in fuzzy bitopological spaces

Definition 3.1. Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\tau^*_1,\tau^*_2)$ be a mapping from a fbts (X,τ_1,τ_2) to another fbts (Y,τ^*_1,τ^*_2). Then f is called:

(i) ij-fuzzy almost strongly semi-closed (briefly, ij-fass closed) mapping if $f(\lambda) \in r-(\tau^*_i,\tau^*_j)-FSSC(Y)$ for each $\lambda \in r-(\tau_i,\tau_j)-FRC(X)$.
(ii) ij-fuzzy almost generalized strongly semi-closed (briefly, ij-fagss closed) mapping if $f(\lambda) \in r-(\tau^*_i,\tau^*_j)-GFSSC(Y)$ for each $\lambda \in r-(\tau_j,\tau_i)-FRC(X)$.

(iii) ij-fuzzy almost generalized regular strongly semi-closed (briefly, ij-fagrss closed) mapping if $f(\lambda) \in r - (\tau_i^*, \tau_j^*) - GFRSSC(Y)$ for each $\lambda \in r - (\tau_j, \tau_i) - FRC(X)$.

(iv) j-fuzzy closed if and only if $\tau_j^*(\underline{1} - f(\lambda)) \geq \tau_j(\underline{1} - \lambda)$ for each $\lambda \in I^X$.

Remark 3.1. From the above definition the implications contained in the following diagram are true

$$\tau_j\text{-fuzzy closed} \rightarrow ji\text{-fass closed} \rightarrow ij\text{-fagss closed} \rightarrow ij\text{-fagrss closed.}$$

The following examples, show that the reverse may not be true in general.

Example 3.1. Let $X = \{a, b, c\}$. Define the fuzzy topologies $\tau_1, \tau_2, \tau_1^*, \tau_2^*$: $I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.3, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{b\}}\} \\ 0.5, & \text{if } \lambda = \chi_{\{a,b\}} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.5, & \text{if } \lambda \in \{\chi_{\{c\}}, \chi_{\{b,c\}}\} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_1^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.4, & \text{if } \lambda \in \{\chi_{\{b\}}, \chi_{\{a,c\}}\} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2^*(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.4, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{c\}}\} \\ 0.5, & \text{if } \lambda = \chi_{\{a,c\}} \\ 0, & \text{otherwise.} \end{cases}$$

Then for $0 < r \leq 0.3$ the identity mapping $f : (X, \tau_1, \tau_2) \longrightarrow (X, \tau_1^*, \tau_2^*)$ is 21-fagrss closed but not 21-fagss closed, since $\chi_{\{c\}} \in r - (\tau_1, \tau_2) - FRC(X)$ but $f(\chi_{\{c\}}) \notin r - (\tau_2^*, \tau_1^*) - GFSSC(X)$.

Example 3.2. Let $X = \{a, b, c\}$. Define the fuzzy topologies $\tau_1, \tau_2, \tau_1^*, \tau_2^*$: $I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.7, & \text{if } \lambda \in \{\chi_{\{a\}}, \chi_{\{a,b\}}\} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0}, \underline{1} \\ 0.5, & \text{if } \lambda \in \{\chi_{\{b\}}, \chi_{\{b,c\}}\} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_1^*(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\underline{1}\\ 0.4, & \text{if } \lambda=\chi_{\{c\}}\\ 0, & \text{otherwise}\end{cases}\qquad \tau_2^*(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\underline{1}\\ 0.6, & \text{if } \lambda=\chi_{\{a,c\}}\\ 0, & \text{otherwise.}\end{cases}$$

Then for $0<r\le 0.4$ the identity mapping $f:(X,\tau_1,\tau_2)\longrightarrow(X,\tau_1^*,\tau_2^*)$ is 21-fagss closed but not 12-fass closed, since $\chi_{\{b,c\}}\in r-(\tau_1,\tau_2)-FRC(X)$ but $f(\chi_{\{b,c\}})\notin r-(\tau_1^*,\tau_2^*)-FSSC(X)$.

Example 3.3. Let $X=\{a,b,c\}$. Define $\lambda_1,\lambda_2,\lambda_3,\lambda_4\in I^X$ as follows:

$$\begin{array}{lll}\lambda_1(a)=0.8 & \lambda_1(b)=0.7 & \lambda_1(c)=0.6\\ \lambda_2(a)=1.0 & \lambda_2(b)=0.8 & \lambda_2(c)=0.7\\ \lambda_3(a)=0.3 & \lambda_3(b)=0.4 & \lambda_3(c)=0.5\\ \lambda_4(a)=0.3 & \lambda_4(b)=0.5 & \lambda_4(c)=0.6\end{array}$$

Define the fuzzy topologies $\tau_1,\tau_2,\tau_1^*,\tau_2^*:I^X\longrightarrow I$ as follows:

$$\tau_1(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\underline{1}\\ 0.5, & \text{if } \lambda=\lambda_1\\ 0, & \text{otherwise,}\end{cases}\qquad \tau_2(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\underline{1}\\ 0.4, & \text{if } \lambda=\lambda_2\\ 0, & \text{otherwise,}\end{cases}$$

$$\tau_1^*(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\underline{1}\\ 0.7, & \text{if } \lambda=\lambda_3\\ 0, & \text{otherwise,}\end{cases}\qquad \tau_2^*(\lambda)=\begin{cases}1, & \text{if } \lambda=\underline{0},\underline{1}\\ 0.6, & \text{if } \lambda=\lambda_4\\ 0, & \text{otherwise.}\end{cases}$$

Then for $0<r\le 0.4$ the identity mapping $f:(X,\tau_1,\tau_2)\longrightarrow(X,\tau_1^*,\tau_2^*)$ is 21-fass closed but not 2-fuzzy closed, since if $\nu=\underline{1}-\lambda_2$ we find that $\tau_2(\underline{1}-\nu)=0.4\nleq\tau_2^*(\underline{1}-f(\nu))=0.0$.

Theorem 3.1. *The surjective mapping $f:(X,\tau_1,\tau_2)\longrightarrow(Y,\tau_1^*,\tau_2^*)$ from a fbts (X,τ_1,τ_2) to another fbts (Y,τ_1^*,τ_2^*) is ij-fagss closed if and only if for each $\mu\in I^Y$ and $\lambda\in r-(\tau_j,\tau_i)-FRO(X)$ such that $f^{-1}(\mu)\le\lambda$, there exists $\nu\in r-(\tau_i^*,\tau_j^*)-GFSSO(Y)$ such that $\mu\le\nu$ and $f^{-1}(\nu)\le\lambda$.*

Proof. Necessity: Suppose that f is ij-fagss closed. Let $\mu\in I^Y$ and $\lambda\in r-(\tau_j,\tau_i)-FRO(X)$ such that $f^{-1}(\mu)\le\lambda$. Since $\lambda\in r-(\tau_j,\tau_i)-FRO(X)$ then $\underline{1}-\lambda\in r-(\tau_j,\tau_i)-FRC(X)$ and since f is ij-fagss closed then $f(\underline{1}-\lambda)\in r-(\tau_i^*,\tau_j^*)-GFSSC(Y)$. Let $\nu=\underline{1}-f(\underline{1}-\lambda)$. Then $\nu\in r-(\tau_i^*,\tau_j^*)-GFSSO(Y)$. Since $f^{-1}(\mu)\le\lambda$, $\underline{1}-\lambda\le\underline{1}-f^{-1}(\mu)=f^{-1}(\underline{1}-\mu)$. This implies that

$$f(\underline{1}-\lambda)\le f(f^{-1}(\underline{1}-\mu))\le\underline{1}-\mu.$$

So,

$$\mu\le\underline{1}-f(\underline{1}-\lambda)=\nu.$$

Also, we have

$$f^{-1}(\nu)=f^{-1}(\underline{1}-f(\underline{1}-\lambda))=\underline{1}-f^{-1}(f(\underline{1}-\lambda))\le\underline{1}-(\underline{1}-\lambda)=\lambda.$$

Sufficiency: Let $\eta \in r-(\tau_j,\tau_i)-FRC(X)$ and $\mu = \underline{1} - f(\eta) \in I^Y$. Then

$$f^{-1}(\mu) = f^{-1}(\underline{1} - f(\eta)) = \underline{1} - f^{-1}(f(\eta)) \leq \underline{1} - \eta$$

and since $\underline{1}-\eta \in r-(\tau_j,\tau_i)-FRO(X)$, there exists $\nu \in r-(\tau_i^*,\tau_j^*)-GFSSO(Y)$ such that $\mu \leq \nu$ and $f^{-1}(\nu) \leq \underline{1}-\eta$. Since f is surjective we have

$$\nu = f(f^{-1}(\nu)) \leq f(\underline{1}-\eta) = \underline{1} - f(\eta).$$

This implies that $f(\eta) \leq \underline{1} - \nu$. On the other hand since $\underline{1} - f(\eta) = \mu \leq \nu$, then $f(\eta) \geq \underline{1} - \nu$. Thus $f(\eta) = \underline{1} - \nu \in r-(\tau_i^*,\tau_j^*)-GFSSC(Y)$. Hence f is ij-fagss closed mapping. □

Definition 3.2. Let $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\tau_1^*,\tau_2^*)$ be a mapping from a fbts (X,τ_1,τ_2) to another fbts (Y,τ_1^*,τ_2^*). Then f is called:

- (i) ij-fuzzy strongly semi open (briefly, ij-fss open) mapping if $f(\lambda) \in r-(\tau_i^*,\tau_j^*)-FSSO(Y)$ for each $\lambda \in I^X$ with $\tau_i(\lambda) \geq r$.
- (ii) ij-fuzzy almost open (briefly, ij-fa open) mapping if $\tau_i^*(f(\lambda)) \geq r$ for each $\lambda \in r-(\tau_i,\tau_j)-FRO(X)$.
- (iii) ij-fuzzy almost strongly semi-open (briefly, ij-fass open) mapping if $f(\lambda) \in r-(\tau_i^*,\tau_j^*)-FSSO(Y)$ for each $\lambda \in r-(\tau_i,\tau_j)-FRO(X)$.

Remark 3.2. Both ij-fa openness and ij-fss openness imply ij-fass openness but the converse is not true in general as the following example shows:

Example 3.4. Let $X=\{a,b,c,d\}$ and $Y=\{x,y,z\}$. Define the fuzzy topologies $\tau_1,\tau_2: I^X \longrightarrow I$ and $\tau_1^*,\tau_2^*: I^Y \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0},\underline{1} \\ 0.5, & \text{if } \lambda \in \{\chi_{\{c\}},\chi_{\{d\}},\chi_{\{c,d\}}\} \\ 0.7, & \text{if } \lambda \in \{\chi_{\{a,c\}},\chi_{\{a,c,d\}}\} \\ 0, & \text{otherwise} \end{cases}$$

$$\tau_2(\lambda) = \begin{cases} 1, & \text{if } \lambda = \underline{0},\underline{1} \\ 0.5, & \text{if } \lambda \in \{\chi_{\{a\}},\chi_{\{b\}},\chi_{\{a,b\}}\} \\ 0, & \text{otherwise.} \end{cases}$$

$$\tau_1^*(\mu) = \begin{cases} 1, & \text{if } \mu = \underline{0},\underline{1} \\ 0.6, & \text{if } \mu = \chi_{\{z\}} \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_2^*(\mu) = \begin{cases} 1, & \text{if } \lambda = \underline{0},\underline{1} \\ 0.4, & \text{if } \mu = \chi_{\{x,z\}} \\ 0, & \text{otherwise.} \end{cases}$$

Then, for $0 < r \leq 0.4$ the mapping $f:(X,\tau_1,\tau_2) \longrightarrow (Y,\tau_1^*,\tau_2^*)$ which defined by

$$f(a)=x, \qquad f(b)=f(d)=z, \qquad f(c)=y$$

is 12-fass open but neither 12-fss open nor 12-fa open. Since, $\tau_1(\chi_{\{c\}}) = 0.5 > r$, $f(\chi_{\{c\}}) = \chi_{\{y\}} \notin r-(\tau_1^*,\tau_2^*)-FSSO(X)$ and $\chi_{\{c,d\}} \in r-(\tau_1,\tau_2)-FRO(X)$, $\tau_1^*(f(\chi_{\{c,d\}})) = \tau_1^*(\chi_{\{y,z\}}) = 0.0 \ngtr r$.

Definition 3.3. A mapping $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ from a fbts (X, τ_1, τ_2) to another fbts (Y, τ_1^*, τ_2^*) is said to be ij-fuzzy R-map if $f^{-1}(\mu) \in r - (\tau_i, \tau_j) - FRO(X)$ for every $\mu \in r - (\tau_i^*, \tau_j^*) - FRO(Y)$.

Example 3.5. Let $X = \{a, b, c\}$. Define $\lambda_1, \lambda_2, \lambda_3, \lambda_4 \in I^X$ as follows:

$$\begin{array}{lll} \lambda_1(a) = 0.4 & \lambda_1(b) = 0.4 & \lambda_1(c) = 0.2 \\ \lambda_2(a) = 0.5 & \lambda_2(b) = 0.2 & \lambda_2(c) = 0.5 \\ \lambda_3(a) = 0.4 & \lambda_3(b) = 0.4 & \lambda_3(c) = 0.3 \\ \lambda_4(a) = 0.5 & \lambda_4(b) = 0.3 & \lambda_4(c) = 0.5 \end{array}$$

Define the fuzzy topologies $\tau_1, \tau_2, \tau_1^*, \tau_2^* : I^X \longrightarrow I$ as follows:

$$\tau_1(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.3, & \text{if} \quad \lambda = \lambda_1, \lambda_3 \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_2(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.3, & \text{if} \quad \lambda = \lambda_2 \\ 0, & \text{otherwise,} \end{cases}$$

$$\tau_1^*(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.4, & \text{if} \quad \lambda = \lambda_3 \\ 0, & \text{otherwise,} \end{cases} \qquad \tau_2^*(\lambda) = \begin{cases} 1, & \text{if} \quad \lambda = \underline{0}, \underline{1} \\ 0.2, & \text{if} \quad \lambda = \lambda_4 \\ 0, & \text{otherwise.} \end{cases}$$

Then for $0 < r \leq 0.3$ the identity mapping $f : (X, \tau_1, \tau_2) \longrightarrow (X, \tau_1^*, \tau_2^*)$ is 12-fuzzy R-map.

Theorem 3.2. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be a mapping from a fbts (X, τ_1, τ_2) to another fbts (Y, τ_1^*, τ_2^*). Then f is ij-fuzzy R-map if it is fpc and ij-fass open.*

Proof. Let $\mu \in r - (\tau_i^*, \tau_j^*) - FRO(Y)$. Then $\tau_i^*(\mu) \geq r$. Since f is fpc, then $\tau_i(f^{-1}(\mu)) \geq \tau_i^*(\mu) \geq r$ and since $f^{-1}(\mu) \leq C_{\tau_j}(f^{-1}(\mu), r)$ then

(1) $$f^{-1}(\mu) = I_{\tau_i}(f^{-1}(\mu), r) \leq I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r), r).$$

Since $\mu \leq C_{\tau_j^*}(\mu, r)$, we have $f^{-1}(\mu) \leq f^{-1}(C_{\tau_j^*}(\mu, r))$. Since $\tau_j(\underline{1} - f^{-1}(C_{\tau_j^*}(\mu, r))) \geq r$, we have

$$f^{-1}(\mu) \leq C_{\tau_j}(f^{-1}(\mu), r) \leq f^{-1}(C_{\tau_j^*}(\mu, r)).$$

So,

$$\begin{aligned} f(I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r))) \leq f(C_{\tau_j}(f^{-1}(\mu), r)) &\leq f(f^{-1}(C_{\tau_j^*}(\mu, r))) \\ &\leq C_{\tau_j^*}(\mu, r). \end{aligned}$$

Since f is ij-fass open and $I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r), r) \in r - (\tau_i, \tau_j) - FRO(X)$, then $f(I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r), r)) \in r - (\tau_i^*, \tau_j^*) - FSSO(Y)$. Thus

$$f(I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r), r)) \leq I_{\tau_{ij}^{*s}}(C_{\tau_j^s}(\mu, r), r).$$

Since $I_{\tau_{ij}^{*s}}(C_{\tau_j^*}(\mu, r), r) \in r - (\tau_i^*, \tau_j^*) - FSSO(Y)$, then

$$\begin{aligned} f(I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r), r)) &\leq I_{\tau_i^*}(C_{\tau_j^*}(I_{\tau_i^*}(C_{\tau_j^*}(\mu, r), r), r), r) \\ &= I_{\tau_i^*}(C_{\tau_j^*}(\mu, r), r) \\ &= \mu. \end{aligned}$$

Then

$$(2) \qquad f^{-1}(\mu) \geq f^{-1}(f(I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r), r))) \geq I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r), r).$$

From (1) and (2) we have $f^{-1}(\mu) = I_{\tau_i}(C_{\tau_j}(f^{-1}(\mu), r), r)$ this implies that $f^{-1}(\mu) \in r - (\tau_i, \tau_j) - FRO(X)$. Hence f is ij-fuzzy R-map. □

4. Fuzzy binormal space

In this section we introduce three concepts of fuzzy normality in fuzzy bitopological spaces namely, fuzzy binormality, fuzzy mildly binormality and fuzzy almost pairwise normality.

Definition 4.1. A fbts (X, τ_1, τ_2) is said to be fuzzy binormal if $\eta = C_{\tau_i}(\eta, r)\ \tilde{q}\rho = C_{\tau_j}(\rho, r)$ implies there exist $\lambda, \mu \in I^X$ with $\tau_j(\lambda) \geq r$, $\tau_i(\mu) \geq r$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda\ \tilde{q}\mu$.

Theorem 4.1. *Let (X, τ_1, τ_2) be a fbts. Then the following statements are equivalent*:

(i) (X, τ_1, τ_2) *is fuzzy binormal.*

(ii) *If $\eta = C_{\tau_i}(\eta, r)\ \tilde{q}\rho = C_{\tau_j}(\rho, r)$, then there exist $\lambda \in r - (\tau_i, \tau_j) - GFSSO(X)$, $\mu \in r - (\tau_j, \tau_i) - GFSSO(X)$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda\ \tilde{q}\mu$.*

(iii) *For any $\eta, \mu \in I^X$ with $\tau_i(\underline{1} - \eta) \geq r$, $\tau_j(\mu) \geq r$ and $\eta \leq \mu$ there exist $\lambda \in r - (\tau_i, \tau_j) - GFSSO(X)$ such that $\eta \leq \lambda \leq C_{\tau_{ij}^s}(\lambda, r) \leq \mu$.*

Proof. (i) ⇒ (ii) It is easy.

(ii) ⇒ (iii) Let $\eta, \mu \in I^X$ with $\tau_i(\underline{1} - \eta) \geq r$, $\tau_j(\mu) \geq r$ and $\eta \leq \mu$. Then $\tau_i(\underline{1} - \eta) \geq r$, $\tau_j(\underline{1} - (\underline{1} - \mu)) \geq r$ and $\eta\ \tilde{q}(\underline{1} - \mu)$. By (ii) there exist $\lambda \in r - (\tau_i, \tau_j) - GFSSO(X)$, $\rho \in r - (\tau_j, \tau_i) - GFSSO(X)$ such that $\eta \leq \lambda$, $\underline{1} - \mu \leq \rho$ and $\lambda\ \tilde{q}\rho$. Since $\rho \in r - (\tau_j, \tau_i) - GFSSO(X)$, $\tau_j(\mu) \geq r$ and $\underline{1} - \mu \leq \rho$, then by Lemma 2.1, we have $\underline{1} - \mu \leq I_{\tau_{ij}^s}(\rho, r)$. Since $\lambda\ \tilde{q}\rho$, then $\lambda \leq \underline{1} - \rho \leq \underline{1} - I_{\tau_{ij}^s}(\rho, r)$. So,

$$\eta \leq \lambda \leq \underline{1} - \rho \leq \underline{1} - I_{\tau_{ij}^s}(\rho, r).$$

Since $\lambda \leq \underline{1} - \rho$, then

$$\lambda \leq C_{\tau_{ij}^s}(\lambda, r) \leq C_{\tau_{ij}^s}(\underline{1} - \rho, r) = \underline{1} - I_{\tau_{ij}^s}(\rho, r). \qquad \text{(from Theorem 2.3)}$$

So,

$$\eta \leq \lambda \leq C_{\tau_{ij}^s}(\lambda, r) \leq \underline{1} - I_{\tau_{ij}^s}(\rho, r) \leq \mu.$$

Thus

$$\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda, r) \leq \mu.$$

(iii) ⇒ (i) Let $\eta = C_{\tau_i}(\eta, r)\ \tilde{q}\rho = C_{\tau_j}(\rho, r)$. Then $\tau_i(\underline{1}-\eta) \geq r$, $\tau_j(\underline{1}-\rho) \geq r$ and $\eta \leq \underline{1} - \rho$. By (iii) there exists $\lambda \in r-(\tau_i, \tau_j)-GFSSO(X)$ such that

$$\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda, r) \leq \underline{1} - \rho.$$

This implies that

$$\rho \leq \underline{1} - C_{\tau^s_{ij}}(\lambda, r) = I_{\tau^s_{ij}}(\underline{1} - \lambda, r).$$

Since $\eta \leq \lambda$, $\tau_i(\underline{1} - \eta) \geq r$ and $\lambda \in r-(\tau_i, \tau_j)-GFSSO(X)$, then by using Lemma 2.1, we have $\eta \leq I_{\tau^s_{ji}}(\lambda, r)$. Since $I_{\tau^s_{ji}}(\lambda, r) \in r-(\tau_j, \tau_i)-FSSO(X)$ and $I_{\tau^s_{ij}}(\underline{1} - \lambda, r) \in r-(\tau_i, \tau_j)-FSSO(X)$, then

$$I_{\tau^s_{ji}}(\lambda, r) \leq I_{\tau_j}(C_{\tau_i}(I_{\tau_j}(I_{\tau^s_{ji}}(\lambda, r), r), r), r)$$

and

$$I_{\tau^s_{ij}}(\underline{1} - \lambda, r) \leq I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(I_{\tau^s_{ij}}(\underline{1} - \lambda, r), r), r), r).$$

Put $\mu = I_{\tau_j}(C_{\tau_i}(I_{\tau_j}(I_{\tau^s_{ji}}(\lambda, r), r), r), r)$ and $\nu = I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(I_{\tau^s_{ij}}(\underline{1} - \lambda, r), r), r), r)$. Then $\tau_j(\mu) \geq r$, $\tau_i(\nu) \geq r$, $\eta \leq \mu$, $\rho \leq \nu$ and $\mu\ \tilde{q}\nu$. Then (X, τ_1, τ_2) is fuzzy binormal. □

Theorem 4.2. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau^*_1, \tau^*_2)$ be fpc, fuzzy ij-fagss-closed and surjection mapping from a fbts (X, τ_1, τ_2) to another fbts (Y, τ^*_1, τ^*_2). If (X, τ_1, τ_2) is a fuzzy binormal space, then (Y, τ^*_1, τ^*_2) is also fuzzy binormal.*

Proof. Let $\eta = C_{\tau^*_i}(\eta, r)\ \tilde{q}\rho = \rho_{\tau^*_j}(\rho, r)$, $r \in I_0$. Then $\tau^*_i(\underline{1}-\eta) \geq r$, $\tau^*_j(\underline{1}-\rho) \geq r$ and $\eta \leq \underline{1} - \rho$. Since f is fpc then, $\tau_i(\underline{1} - f^{-1}(\eta)) \geq \tau^*_i(\underline{1} - \eta) \geq r$ and $\tau_j(\underline{1} - f^{-1}(\rho)) \geq \tau^*_j(\underline{1} - \rho) \geq r$. Thus,

$$f^{-1}(\eta) = C_{\tau_i}(f^{-1}(\eta), r)\ \tilde{q}f^{-1}(\rho) = C_{\tau_j}(f^{-1}(\rho), r).$$

Since (X, τ_1, τ_2) is fuzzy binormal, there exist $\lambda, \mu \in I^X$ with $\tau_j(\lambda) \geq r$, $\tau_i(\mu) \geq r$ such that $f^{-1}(\eta) \leq \lambda$, $f^{-1}(\rho) \leq \mu$ and $\lambda\ \tilde{q}\mu$. Let $\theta = I_{\tau_j}(C_{\tau_i}(\lambda, r), r)$ and $\delta = I_{\tau_i}(C_{\tau_j}(\mu, r), r)$. Then $\theta \in r-(\tau_j, \tau_i)-FRO(X)$ and $\delta \in r-(\tau_i, \tau_j)-FRO(X)$. Also, $f^{-1}(\eta) \leq \theta$, $f^{-1}(\rho) \leq \delta$ and $\theta\ \tilde{q}\delta$. By Theorem 3.1, there exist $\gamma \in r-(\tau^*_i, \tau^*_j)-GFSSO(Y)$ and $\nu \in r-(\tau^*_j, \tau^*_i)-GFSSO(Y)$ such that $\eta \leq \gamma$, $\rho \leq \nu$, $f^{-1}(\gamma) \leq \theta$ and $f^{-1}(\nu) \leq \delta$. Since $\theta\ \tilde{q}\delta$, then $\gamma\ \tilde{q}\nu$ and by Theorem 4.1, (Y, τ^*_1, τ^*_2) is fuzzy binormal. □

Definition 4.2. A fbts (X, τ_1, τ_2) is said to be fuzzy mildly binormal if for all $\eta \in r-(\tau_i, \tau_j)-FRC(X)$, $\rho \in r-(\tau_j, \tau_i)-FRC(X)$ such that $\eta\ \tilde{q}\rho$ there exist $\lambda, \mu \in I^X$ with $\tau_j(\lambda) \geq r$, $\tau_i(\mu) \geq r$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda\ \tilde{q}\mu$.

Theorem 4.3. *Let (X, τ_1, τ_2) be a fbts. Then the following statements are equivalent*:

(i) (X, τ_1, τ_2) *is fuzzy mildly binormal.*

(ii) *For any* $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRC(X)$ *and* $\eta\ \tilde{q}\rho$, *there exist* $\lambda \in r-(\tau_i,\tau_j)-GFSSO(X)$ *and* $\mu \in r-(\tau_j,\tau_i)-GFSSO(X)$ *such that* $\eta \leq \lambda$, $\rho \leq \mu$ *and* $\lambda\ \tilde{q}\mu$.

(iii) *For any* $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRC(X)$ *and* $\eta\ \tilde{q}\rho$, *there exist* $\lambda \in r-(\tau_i,\tau_j)-GFRSSO(X)$ *and* $\mu \in r-(\tau_j,\tau_i)-GFRSSO(X)$ *such that* $\eta \leq \lambda$, $\rho \leq \mu$ *and* $\lambda\ \tilde{q}\mu$.

(iv) *For any* $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRO(X)$ *and* $\eta \leq \rho$, *there exists* $\lambda \in r-(\tau_i,\tau_j)-GFRSSO(X)$ *such that* $\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda,r) \leq \rho$.

(v) *For any* $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRO(X)$ *and* $\eta \leq \rho$, *there exists* $\lambda \in r-(\tau_j,\tau_i)-FSSO(X)$ *such that* $\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda,r) \leq \rho$.

(vi) *For any* $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRC(X)$ *and* $\eta\ \tilde{q}\rho$, *there exist* $\lambda \in r-(\tau_j,\tau_i)-FSSO(X)$ *and* $\mu \in r-(\tau_i,\tau_j)-FSSO(X)$ *such that* $\eta \leq \lambda$, $\rho \leq \mu$ *and* $\lambda\ \tilde{q}\mu$.

Proof. (i)⇒ (ii) and (ii)⇒ (iii) are straightforward.

(iii)⇒ (iv) Let $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRO(X)$ and $\eta \leq \rho$. Consequently, $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\underline{1}-\rho \in r-(\tau_j,\tau_i)-FRC(X)$ and $\eta\ \tilde{q}(\underline{1}-\rho)$. Then there exist $\lambda \in r-(\tau_i,\tau_j)-GFRSSO(X)$ and $\mu \in r-(\tau_j,\tau_i)-GFRSSO(X)$ such that $\eta \leq \lambda$, $\underline{1}-\rho \leq \mu$ and $\lambda\ \tilde{q}\mu$. By Lemma 2.1, we have $\underline{1}-\rho \leq I_{\tau^s_{ij}}(\mu,r)$. Therefore we have $\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda,r) \leq \underline{1}-I_{\tau^s_{ij}}(\mu,r) \leq \rho$. Hence, $\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda,r) \leq \rho$.

(iv)⇒ (v) Let $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRO(X)$ and $\eta \leq \rho$. Then there exists $\mu \in r-(\tau_i,\tau_j)-GFRSSO(X)$ such that $\eta \leq \mu \leq C_{\tau^s_{ij}}(\mu,r) \leq \rho$. Since $\eta \in r-(\tau_i,\tau_j)-FRC(X)$ and $\eta \leq \mu$, then by using Lemma 2.1, we have $\eta \leq I_{\tau^s_{ji}}(\mu,r)$. Put $\lambda = I_{\tau^s_{ji}}(\mu,r)$. Then $\lambda \in r-(\tau_j,\tau_i)-FSSO(X)$ and

$$\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda,r) \leq C_{\tau^s_{ij}}(\mu,r) \leq \rho.$$

Hence, $\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda,r) \leq \rho$.

(v)⇒ (vi) Let $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRC(X)$ and $\eta\ \tilde{q}\rho$. Consequently, $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\underline{1}-\rho \in r-(\tau_j,\tau_i)-FRO(X)$ and $\eta \leq \underline{1}-\rho$. Then there exists $\lambda \in r-(\tau_j,\tau_i)-FSSO(X)$ such that

$$\eta \leq \lambda \leq C_{\tau^s_{ij}}(\lambda,r) \leq \underline{1}-\rho.$$

Then $\rho \leq \underline{1}-C_{\tau^s_{ij}}(\lambda,r)$. Put $\mu = \underline{1}-C_{\tau^s_{ij}}(\lambda,r)$. Then $\mu \in r-(\tau_i,\tau_j)-FSSO(X)$. Hence, $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda\ \tilde{q}\mu$.

(vi)⇒ (i) Let $\eta \in r-(\tau_i,\tau_j)-FRC(X)$, $\rho \in r-(\tau_j,\tau_i)-FRC(X)$ and $\eta\ \tilde{q}\rho$. Then there exist $\lambda \in r-(\tau_j,\tau_i)-FSSO(X)$ and $\mu \in r-(\tau_i,\tau_j)-FSSO(X)$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda\ \tilde{q}\mu$. Put $\theta = I_{\tau_j}(C_{\tau_i}(I_{\tau_j}(\lambda,r),r),r)$ and $\delta = I_{\tau_i}(C_{\tau_j}(I_{\tau_i}(\mu,r),r),r)$. Then we have, $\eta \leq \theta$, $\rho \leq \delta$ and $\theta\ \tilde{q}\delta$. Hence, (X,τ_1,τ_2) is fuzzy mildly binormal. □

Theorem 4.4. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be ij-fagss closed, ij-fuzzy R-map and surjection mapping from a fbts (X, τ_1, τ_2) to another fbts (Y, τ_1^*, τ_2^*). If (X, τ_1, τ_2) is a fuzzy mildly binormal space, then (Y, τ_1^*, τ_2^*) is also fuzzy mildly binormal.*

Proof. Let $\eta \in r - (\tau_i^*, \tau_j^*) - FRC(Y)$, $\rho \in r - (\tau_i^*, \tau_j^*) - FRC(Y)$ and $\eta \; \tilde{q} \rho$. Since f is ij-fuzzy R map then, $f^{-1}(\eta) \in r - (\tau_i, \tau_j) - FRC(X)$, $f^{-1}(\rho) \in r - (\tau_i, \tau_j) - FRC(X)$ furthermore, $f^{-1}(\eta) \; \tilde{q} f^{-1}(\rho)$. Since (X, τ_i, τ_j) is fuzzy mildly binormal, there exist $\theta, \delta \in I^X$ with $\tau_j(\theta) \geq r$, $\tau_i(\delta) \geq r$ such that $f^{-1}(\eta) \leq \theta$,$f^{-1}(\rho) \leq \delta$ and $\theta \; \tilde{q} \delta$. Put $\lambda = I_{\tau_j}(C_{\tau_i}(\theta, r))$ and $\mu = I_{\tau_i}(C_{\tau_j}(\delta, r))$. Then $\lambda \in r - (\tau_j, \tau_i) - FRO(X)$, $\mu \in r - (\tau_i, \tau_j) - FRO(X)$, $f^{-1}(\eta) \leq \lambda$, $f^{-1}(\rho) \leq \mu$ and $\lambda \; \tilde{q} \mu$. By Theorem 3.1, there exist $\xi \in r - (\tau_i^*, \tau_j^*) - GFSSO(Y)$, $\gamma \in r - (\tau_j^*, \tau_i^*) - GFSSO(Y)$ such that $\eta \leq \xi$, $f^{-1}(\xi) \leq \lambda$, $\rho \leq \gamma$, $f^{-1}(\gamma) \leq \mu$. Also, we can find $\xi \; \tilde{q} \gamma$. Then, from Theorem 4.3, we have (Y, τ_1^*, τ_2^*) is also fuzzy mildly binormal. □

Definition 4.3. A fbts (X, τ_1, τ_2) is said to be fuzzy almost pairwise normal if for any $\eta, \rho \in I^X$ such that $\eta \in r - (\tau_i, \tau_j) - FRC(X)$, $\tau_j(\underline{1} - \rho) \geq r$ and $\eta \; \tilde{q} \rho$ there exist $\lambda, \mu \in I^X$ with $\tau_j(\lambda) \geq r$, $\tau_i(\mu) \geq r$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda \; \tilde{q} \mu$.

Theorem 4.5. *Let (X, τ_1, τ_2) be a fbts. Then the following statements are equivalent*:

(i) *(X, τ_1, τ_2) is fuzzy almost pairwise normal.*

(ii) *For any $\eta \in r - (\tau_i, \tau_j) - FRC(X)$, $\rho \in I^X$ with $\tau_j(\underline{1} - \rho) \geq r$ and $\eta \; \tilde{q} \rho$, there exist $\lambda \in r - (\tau_i, \tau_j) - GFSSO(X)$ and $\mu \in r - (\tau_j, \tau_i) - GFSSO(X)$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda \; \tilde{q} \mu$.*

(iii) *For any $\eta \in r - (\tau_i, \tau_j) - FRC(X)$, $\rho \in I^X$ with $\tau_j(\underline{1} - \rho) \geq r$ and $\eta \; \tilde{q} \rho$, there exist $\lambda \in r - (\tau_i, \tau_j) - GFRSSO(X)$ and $\mu \in r - (\tau_j, \tau_i) - GFRSSO(X)$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda \; \tilde{q} \mu$.*

(iv) *For any $\eta \in r - (\tau_i, \tau_j) - FRC(X)$, $\rho \in I^X$ with $\tau_j(\rho) \geq r$ and $\eta \leq \rho$, there exists $\lambda \in r - (\tau_i, \tau_j) - GFRSSO(X)$ such that $\eta \leq \lambda \leq C_{\tau_{ij}^s}(\lambda, r) \leq \rho$.*

(v) *For any $\eta \in r - (\tau_i, \tau_j) - FRC(X)$, $\rho \in I^X$ with $\tau_j(\rho) \geq r$ and $\eta \leq \rho$, there exists $\lambda \in r - (\tau_j, \tau_i) - FSSO(X)$ such that $\eta \leq \lambda \leq C_{\tau_{ij}^s}(\lambda, r) \leq \rho$.*

(vi) *For any $\eta \in r - (\tau_i, \tau_j) - FRC(X)$, $\rho \in I^X$ with $\tau_j(\underline{1} - \rho) \geq r$ and $\eta \; \tilde{q} \rho$, there exist $\lambda \in r - (\tau_j, \tau_i) - FSSO(X)$ and $\mu \in r - (\tau_i, \tau_j) - FSSO(X)$ such that $\eta \leq \lambda$, $\rho \leq \mu$ and $\lambda \; \tilde{q} \mu$.*

Proof. It is similar to the proof of Theorem 4.3. □

Theorem 4.6. *Let $f : (X, \tau_1, \tau_2) \longrightarrow (Y, \tau_1^*, \tau_2^*)$ be ij-fass-open, ji-fass-closed, fpc and surjection mapping from a fbts (X, τ_1, τ_2) to another fbts (Y, τ_1^*, τ_2^*). If (X, τ_1, τ_2) is a fuzzy almost pairwise normal, then (Y, τ_1^*, τ_2^*) is also fuzzy almost pairwise normal.*

Proof. Let $\eta \in r-(\tau_i^*, \tau_j^*)-FRC(Y)$, $\rho \in I^Y$ with $\tau_j^*(\rho) \geq r$ and $\eta \leq \rho$. Since f is fpc and ij-fass open then, by Theorem 3.2, $f^{-1}(\eta) \in r-(\tau_i, \tau_j)-FRC(X)$. Since f is fpc, then $\tau_j(f^{-1}(\rho)) \geq \tau_j^*(\rho) \geq r$. Furthermore, $f^{-1}(\eta) \leq f^{-1}(\rho)$. Since (X, τ_1, τ_2) is fuzzy almost pairwise normal and by Theorem 4.5(v), there exists $\lambda \in r-(\tau_j, \tau_i)-FSSO(X)$ such that

$$f^{-1}(\eta) \leq \lambda \leq C_{\tau_{ij}^s}(\lambda, r) \leq f^{-1}(\rho).$$

Since $\lambda \in r-(\tau_j, \tau_i)-FSSO(X)$, then $\lambda \leq I_{\tau_j}(C_{\tau_i}(I_{\tau_j}(\lambda, r))) = \mu$ (say). By using Theorem 2.4, we have

$$\begin{aligned} f^{-1}(\eta) \leq \lambda \leq \mu \leq C_{\tau_i}(\mu, r) &= C_{\tau_i}(I_{\tau_j}(\lambda, r), r)) \\ &= C_{\tau_{ji}^s}(\lambda, r) \\ &\leq C_{\tau_{ji}^s}(f^{-1}(\rho), r) = f^{-1}(\rho). \end{aligned}$$

Thus

$$f^{-1}(\eta) \leq \mu \leq C_{\tau_i}(\mu, r) \leq f^{-1}(\rho).$$

This implies that

$$f^{-1}(\eta) \leq \mu \leq I_{\tau_j}(C_{\tau_i}(\mu, r), r) \leq C_{\tau_i}(\mu, r) \leq f^{-1}(\rho).$$

Since f is surjective we have

$$\eta \leq f(\mu) \leq f(I_{\tau_j}(C_{\tau_i}(\mu, r), r)) \leq f(C_{\tau_i}(\mu, r)) \leq f(f^{-1}(\rho)) = \rho.$$

Since $I_{\tau_j}(C_{\tau_i}(\mu, r), r) \in r-(\tau_j, \tau_i)-FRO(X)$ and f is ij-fass-open, we have $f(I_{\tau_j}(C_{\tau_i}(\mu, r), r)) \in r-(\tau_j^*, \tau_i^*)-FSSO(Y)$. Since $C_{\tau_i}(\mu, r) = C_{\tau_i}(I_{\tau_j}(\mu, r), r)$ $\in r-(\tau_i, \tau_j)-FRC(X)$ and f is ji-fass-closed then, $f(C_{\tau_i}(\mu, r)) \in r-(\tau_i^*, \tau_j^*)-FSSC(Y)$. Consequently

$$C_{\tau_{ij}^{*s}}(f(C_{\tau_i}(\lambda, r)), r) \leq C_{\tau_{ij}^{*s}}(f(C_{\tau_i}(\mu, r)), r) = f(C_{\tau_i}(\mu, r)) \leq \rho.$$

Thus

$$\begin{aligned} \eta \leq f(I_{\tau_j}(C_{\tau_i}(\mu, r), r)) &\leq C_{\tau_{ij}^{*s}}(f(I_{\tau_j}(C_{\tau_i}(\mu, r), r), r) \\ &\leq C_{\tau_{ij}^{*s}}(f(C_{\tau_i}(\mu, r)), r) \\ &\leq \rho. \end{aligned}$$

Then there exists $f(I_{\tau_j}(C_{\tau_i}(\mu, r), r)) \in r-(\tau_j^*, \tau_i^*)-FSSO(Y)$ such that

$$\eta \leq f(I_{\tau_j}(C_{\tau_i}(\mu, r), r)) \leq C_{\tau_{ij}^{*s}}(f(I_{\tau_j}(C_{\tau_i}(\mu, r), r), r) \leq \rho.$$

Hence from Theorem 4.5(V), we have (Y, τ_1^*, τ_2^*) is fuzzy almost pairwise normal. □

Remark 4.1. From Definition 4.1, Definition 4.2 and Definition 4.3, we have the following implications:
fuzzy mildly binormality ⇒ fuzzy almost pairwise normality ⇒ fuzzy binormality

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