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EXPLICIT EXAMPLES OF KÄHLER METRICS ON THE ELLIPSOIDS

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Abstract. In this paper, we construct explicitly Kähler metrics on the ellipsoids and calculate their sectional curvatures. Using MAPLE [3], we obtain some geodesic curves on an ellipsoid so that if some conditions are dropped in Question 2.1 [4], then Question 2.1 is not true.

1. Introduction

A Kähler metric g is a Riemannian metric on a complex manifold (M, I) such that $g(\ ,\ ) = g(I\ , I\ )$ and $dg(I\ ,\ ) = 0$. There are some explicit examples of Kähler metrics such as the Euclidean metric on $\mathbb{C}^n$, the Fubini-Study metric on $\mathbb{CP}^m$, the Eguchi-Hanson metric on the crepant resolution of $\mathbb{C}^2/\{\pm 1\}$, etc [2]. In this paper, we construct explicitly Kähler metrics on the ellipsoids (See Theorem 3.1) and calculate their sectional curvatures (See Theorem 3.2). With the aides of MAPLE, we find some closed geodesic curves on an ellipsoid. Using them, we show that Question 4.1 is not true.

2. Preliminaries

Let $S^2 := \{(x, y, z) \in \mathbb{R}^3 \mid x^2 + y^2 + z^2 = 1\}$. Let $p_N := (0, 0, 1)$ and $p_S := (0, 0, -1)$. Consider the Riemannian manifold (S^2, g) such that g is the induced metric on S^2 from the Euclidean space $\mathbb{R}^3$. Then we can write down the metric g explicitly in local coordinates. Define the charts $\pi_N : S^2 - \{p_N\} \mapsto \mathbb{R}^2$ and $\pi_S : S^2 - \{p_S\} \mapsto \mathbb{R}^2$, respectively, by

$$\pi_N(x, y, z) := \left(\frac{x}{1-z}, \frac{y}{1-z}\right) \quad \text{and} \quad \pi_S(x, y, z) := \left(\frac{x}{1+z}, \frac{y}{1+z}\right).$$

Then $\{(S^2 - \{p_N\}, \pi_N), (S^2 - \{p_S\}, \pi_S)\}$ is an atlas of S^2. Define the identifications $f_+ : \mathbb{R}^2 \mapsto \mathbb{C}$ and $f_- : \mathbb{R}^2 \mapsto \mathbb{C}$, respectively, by

$$f_+(x, y) = z \quad \text{and} \quad f_-(x, y) = \bar{z},$$

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where $z = x + \sqrt{-1}y$ and $\bar{z} = x - \sqrt{-1}y$. Let $\widetilde{\pi}_N := f_+ \circ \pi_N$ and $\widetilde{\pi}_S := f_- \circ \pi_S$. Since $(\widetilde{\pi}_S \circ \widetilde{\pi}_N^{-1})(z) = \frac{1}{z}$ for $z \in \mathbb{C} - \{0\}$, $\{(S^2 - \{p_N\}, \widetilde{\pi}_N), (S^2 - \{p_S\}, \widetilde{\pi}_S)\}$ is a holomorphic atlas of S^2. Thus, S^2 is a complex manifold, i.e., $S^2 \cong \mathbb{C}P^1$. Let I be the corresponding complex structure on S^2. Let $\widetilde{g}$ be the Fubini-Study metric on $(S^2, I) = \mathbb{C}P^1$ [1]. Then we easily get

$$g = \frac{4}{(1+x^2+y^2)^2}(dx^2 + dy^2) \quad \text{on } \pi_N(S^2 - \{p_N\})$$

and

$$\widetilde{g} = \frac{1}{(1+|z|^2)^2}(dx^2 + dy^2) \quad \text{on } \widetilde{\pi}_N(S^2 - \{p_N\}).$$

3. Kähler metrics

Throughout this section, we will use the notation of Section 2.
Let $S(a,b,c) := \{(x,y,z) \in \mathbb{R}^3 \mid \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2} = 1\}$ for any positive real numbers $a, b, c \in \mathbb{R}^+$. Define the maps $f_{abc} : S(a,b,c) \mapsto S^2$ by

$$f_{abc}(x,y,z) = \left(\frac{x}{a}, \frac{y}{b}, \frac{z}{c}\right)$$

for $a, b, c \in \mathbb{R}^+$. Let $\widehat{p}_N := (0,0,c)$ and $\widehat{p}_S := (0,0,-c)$. Consider the inclusion maps $i_N : S(a,b,c) - \{\widehat{p}_N\} \hookrightarrow S(a,b,c)$ and $i_S : S(a,b,c) - \{\widehat{p}_S\} \hookrightarrow S(a,b,c)$. Let $\widehat{\pi}_N := f_+ \circ \pi_N \circ f_{abc} \circ i_N$ and $\widehat{\pi}_S := f_- \circ \pi_S \circ f_{abc} \circ i_S$. Then we know that $\{(S(a,b,c) - \{\widehat{p}_N\}, \widehat{\pi}_N), (S(a,b,c) - \{\widehat{p}_S\}, \widehat{\pi}_S)\}$ is a holomorphic atlas of $S(a,b,c)$. Thus, $S(a,b,c)$ is a complex manifold. Let $\widehat{I}$ be the corresponding complex structure on $S(a,b,c)$ and $\widetilde{g}_{abc}$ the induced metric on $S(a,b,c)$ from the Euclidean space $\mathbb{R}^3$. With some computations, we get

$$\begin{aligned}\widetilde{g}_{abc} =& \frac{1}{(1+x^2+y^2)^4}(((-2ax^2 + 2ay^2 + 2a)^2 + 16b^2x^2y^2 + 16c^2x^2)dx^2 \\ & - 4xy(a(-2ax^2 + 2ay^2 + 2a) + b(2bx^2 - 2by^2 + 2b) - 4c^2) \\ & (dx \cdot dy + dy \cdot dx) + (16a^2x^2y^2 + (2bx^2 - 2by^2 + 2b)^2 + 16c^2y^2)dy^2)\end{aligned}$$

on $(\pi_N \circ f_{abc})(S(a,b,c) - \{\widehat{p}_N\})$. Since $\widehat{I}(\frac{\partial}{\partial x}) = \frac{\partial}{\partial y}$ and $\widehat{I}(\frac{\partial}{\partial y}) = -\frac{\partial}{\partial x}$, it is easy to see that $\widetilde{g}_{abc}$ is not Hermitian with respect to the complex structure $\widehat{I}$. Let $\widehat{g}_{abc} = \widetilde{g}_{abc}(\ ,\ ) + \widetilde{g}_{abc}(\widehat{I}\ , \widehat{I}\ )$ and $\omega_{abc} := \widehat{g}_{abc}(\widehat{I}\ ,\ )$. Since $\dim S(a,b,c) = 2$, we have $d\omega_{abc} = 0$. Now, we obtain:

Theorem 3.1. *Let $S(a,b,c)$ and $\widehat{g}_{abc}$ be the ellipsoid and the metric, respectively, given as above. Then the metric $\widehat{g}_{abc}$ is Kähler on the complex manifold $(S(a,b,c), \widehat{I})$. Furthermore, we get*

$$\begin{aligned}\widehat{g}_{abc} =& \frac{1}{(1+x^2+y^2)^4}(4a^2((-x^2 + y^2 + 1)^2 + 4x^2y^2) \\ & + 4b^2((x^2 - y^2 + 1)^2 + 4x^2y^2) + 16c^2(x^2 + y^2))(dx^2 + dy^2)\end{aligned}$$

on $(\pi_N \circ f_{abc})(S(a,b,c) - \{\widehat{p}_N\})$.

Remark 3.1. Define a function

$$\begin{aligned} u_{abc} :=& -\frac{2a^2+2b^2}{3(1+|z|^2)^2} - \frac{b^2-a^2}{12\bar{z}^2(1+|z|^2)^2} + \frac{4c^2}{3(1+|z|^2)^2} - \frac{(b^2-a^2)\bar{z}^2}{12(1+|z|^2)^2} \\ & - \frac{4c^2}{3(1+|z|^2)} + \frac{b^2-a^2}{6\bar{z}^2(1+|z|^2)} + \frac{2}{3}(2a^2+2b^2)\ln(1+|z|^2) \\ & + \frac{4}{3}c^2\ln(1+|z|^2) + \frac{2a^2+2b^2}{3(1+|z|^2)} - \frac{b^2-a^2}{12z^2(1+|z|^2)^2} - \frac{(b^2-a^2)z^2}{12(1+|z|^2)^2} \\ & + \frac{b^2-a^2}{6z^2(1+|z|^2)} \end{aligned}$$

on $(f_+ \circ \pi_N \circ f_{abc})(S(a,b,c) - \{\widehat{p}_N\})$ for each $a,b,c \in \mathbb{R}^+$. Then by the elementary calculation, u_{abc} is a Kähler potential for the Kähler metric $\widehat{g}_{abc}$ on $(f_+ \circ \pi_N \circ f_{abc})(S(a,b,c) - \{\widehat{p}_N\})$, i.e., $\omega_{abc} = \sqrt{-1}\partial\bar{\partial}u_{abc}$. In particular,

$$u_{aaa} = 4a^2 \ln(1+|z|^2) \quad \text{for } a \in \mathbb{R}^+.$$

Consider the pull-back metrics $(f_{abc}^{-1})^*\widehat{g}_{abc}$ on S^2 for any $a,b,c \in \mathbb{R}^+$. Conveniently, let $g_{abc} := (f_{abc}^{-1})^*\widehat{g}_{abc}$ for $a,b,c \in \mathbb{R}^+$. Then the family $\{g_{abc} \mid a,b,c \in \mathbb{R}^+\}$ is the collection of Kähler metrics g_{abc} on S^2 for $a,b,c \in \mathbb{R}^+$ so that the metrics g_{aaa} are the Fubini-Study metric on $S^2 = \mathbb{CP}^1$, up to constant, for $a \in \mathbb{R}^+$. Furthermore, we have

$$g_{\frac{1}{2\sqrt{2}}\frac{1}{2\sqrt{2}}\frac{1}{2\sqrt{2}}} = \widetilde{g} \quad \text{on } S^2.$$

With some computations, we obtain the sectional curvature K_{abc} of the Kähler metric g_{abc} as follows [5].

Theorem 3.2. *Let K_{abc} be the sectional curvature of the Kähler metric g_{abc} on S^2 for $a,b,c \in \mathbb{R}^+$. Then we have*

$$\begin{aligned} K_{abc}(x,y) =& 2(4b^4x^6y^2 + 20a^2y^4c^2x^2 - 6a^2y^4c^2x^4 + 16b^2y^4c^2x^2 \\ & - 6b^2y^4c^2x^4 + 12a^2x^4b^2y^4 + a^4 + b^4 - 4a^2y^6b^2 - 4a^4x^2y^2 \\ & - c^2a^2 - c^2b^2 + 32c^4x^2y^2 + 2a^2b^2 + 4b^4x^2y^6 + 4a^4x^2y^6 \\ & - 4b^4x^2y^2 + 4a^4x^2y^4 - 4a^2y^2b^2 - 4a^4x^4y^2 + 4b^4x^4y^2 \\ & + 8c^2a^2y^2 + 4c^2b^2y^2 + 4a^4x^6y^2 + 6a^4y^4 + 6b^4y^4 + a^4y^8 \\ & + b^4y^8 + 16c^4y^4 + 8a^2y^6c^2 + 4b^2y^6c^2 - 12a^2y^4b^2 + 18a^2y^4c^2 \\ & - 4b^4x^2y^4 - 22b^2y^4c^2 + 6a^4x^4y^4 - 12a^2x^4b^2y^2 + 8a^2x^2b^2y^2 \\ & - 4a^2x^2c^2y^2 - 4b^2x^2c^2y^2 + 2a^2y^8b^2 - a^2y^8c^2 + 6b^4x^4y^4 \\ & - b^2y^8c^2 + 8a^2y^6b^2x^2 - 4a^2y^6c^2x^2 - 4b^2y^6c^2x^2 + 16c^2a^2y^2x^4 \\ & + 20c^2b^2y^2x^4 + 8a^2x^6b^2y^2 - 4a^2x^6c^2y^2 - 4b^2x^6c^2y^2 + 4a^4y^6 \\ & + 4a^4y^2 - 4b^4y^6 - 4b^4y^2 - 12a^2x^2b^2y^4 - 4a^4x^2 + 4b^4x^6 + 6b^4x^4 \\ & + 4b^4x^2 + b^4x^8 + 16c^4x^4 + a^4x^8 - 4a^4x^6 + 6a^4x^4 - 4a^2x^6b^2 \end{aligned}$$

$$- 12a^2x^4b^2 - 4a^2x^2b^2 - 22a^2x^4c^2 + 18b^2x^4c^2 + 4c^2a^2x^2 + 8c^2b^2x^2$$
$$+ 2a^2x^8b^2 + 4a^2x^6c^2 - a^2x^8c^2 + 8b^2x^6c^2 - b^2x^8c^2)(1 + x^2 + y^2)^2$$
$$/(a^2x^4 + 2a^2x^2y^2 + a^2y^4 - 2a^2x^2 + 2a^2y^2 + a^2 + b^2x^4 + 2b^2x^2y^2$$
$$+ b^2y^4 + 2b^2x^2 - 2b^2y^2 + b^2 + 4c^2x^2 + 4c^2y^2)^3$$

on $\pi_N(S^2 - \{p_N\})$.

Remark 3.2. If $a = b = c$, then it is easy to see that

$$K_{aaa}(x, y) = \frac{1}{2a^2} \quad \text{on } \pi_N(S^2 - \{p_N\}).$$

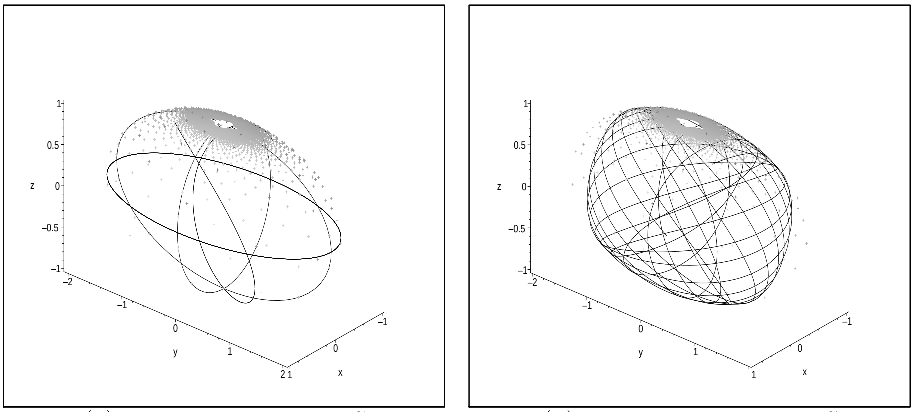
Since $g = g_{\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}\frac{1}{\sqrt{2}}}$, the sectional curvature K of the metric g is equal to 1.

4. Minimal geodesic

Throughout this section, we will use the notation of the previous sections. For convenience, denote by S, $\widehat{g}$, and f the ellipsoid $S(1, 2, 1)$, the metric $\widehat{g}_{121}$, and the map f_{121}, respectively. Then we have

$$\widehat{g} = \frac{1}{(1 + x^2 + y^2)^4}(20(1 + x^2 + y^2)^2 - 48y^2)(dx^2 + dy^2)$$

on $(\pi_N \circ f)(S - \{p_N\})$. Using MAPLE [3, See page 187], we can get Figure 1.



(a) geodesic curves on S (b) a geodesic curve on S

FIGURE 1. Geodesic curves for the metric $\widehat{g}$

From Figure 1-(a) showing some geodesic curves on S, we see that the following curves are closed geodesic in S:

$$C_1(t) := (\pi_N \circ f)^{-1}(t, 0) \quad \text{for } t \in [-\infty, \infty]$$
$$C_2(t) := (\pi_N \circ f)^{-1}(0, t) \quad \text{for } t \in [-\infty, \infty]$$
$$C_3(t) := (\pi_N \circ f)^{-1}(\cos t, \sin t) \quad \text{for } t \in [0, 2\pi].$$

Note that $C_1 = \{(\pi_N \circ f)^{-1}(\tan \frac{t}{\sqrt{20}}, 0) \mid t \in [-\sqrt{5}\pi, \sqrt{5}\pi]\}$. It is easy to see that the curve $(\pi_N \circ f)^{-1}(\tan \frac{t}{\sqrt{20}}, 0)$ is geodesic. Using MAPLE, we can also calculate the lengthes of the above closed geodesic curves. Denote by $L(C)$ the length of a curve $C(t)$. Then,

$$L(C_1) \approx 14.04962946,\ L(C_2) \approx 11.61349340,\ L(C_3) \approx 11.61349340.$$

We know that $C_1(t)$ and $C_2(t)$ are the curves in S passing through both p_N and p_S, i.e., $C_1(0) = C_2(0) = p_S$ and $C_1(\infty) = C_2(\infty) = p_N$ and

$$L(\{C_1(t) \mid 0 \le t \le \infty\}) > L(\{C_2(t) \mid 0 \le t \le \infty\}).$$

So, we have:

Proposition 4.1. *Every minimal geodesic curve in C_1 is not minimal in S*

Remind that the ellipsoid S is a simply connected complete (compact) Kähler manifold and the curve C_1 is a closed complete totally geodesic submanifold of S. Hence, Proposition 4.1 implies that the following question is not true.

Question 4.1. Let M be a simply connected complete Kähler manifold and N a closed complete totally geodesic submanifold of M. Then every minimal geodesic in N is also minimal in M.

That is, the condition that N is complex in Question 2.1 [4] can not be dropped in order to be a true statement, although we don't know whether it is true or not.

Remark 4.1. Using MAPLE [3], we see that the below curves are not geodesic in the ellipsoid S.

$$C_4(t) := (\pi_N \circ f)^{-1}(t, t) \quad \text{for } t \in [-\infty, \infty]$$
$$C_5(t) := (\pi_N \circ f)^{-1}(t, -t) \quad \text{for } t \in [-\infty, \infty].$$

Note that Figure 1-(b) is the picture of a geodesic curve $(\pi_N \circ f)^{-1}C(t)$ on S such that $C(0) = (0, 0)$ and $C'(0) = (1, 1)$. But they are all the closed curves in S passing through both p_N and p_S, i.e., $C_4(0) = C_5(0) = p_S$ and $C_4(\infty) = C_5(\infty) = p_N$. Furthermore,

$$L(C_4) = L(C_5) \approx 12.92772026.$$

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