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THREE-STEP ITERATIVE ALGORITHMS FOR FIXED POINT PROBLEMS AND VARIATIONAL INCLUSION PROBLEMS

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Abstract. In this paper, a three-step iterative method is considered for finding a common element in the set of fixed points of a non-expansive mapping and in the set of solutions of a variational inclusion problem in a real Hilbert space.

1. Introduction and preliminaries

Let H be a real Hilbert space H and A a mapping on H. Recall that A is said to be monotone if

$$\langle Ax - Ay, x - y \rangle \geq 0, \quad \forall x, y \in H.$$

A is said to be α-strongly monotone if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \geq \alpha \|x - y\|^2, \quad \forall x, y \in H.$$

A is said to be α-strongly anti-monotone [15] if there exists a constant $\alpha > 0$ such that

$$\langle Ax - Ay, x - y \rangle \leq (-\alpha) \|x - y\|^2, \quad \forall x, y \in H.$$

A is said to be L-Lipschitz continuous if there exits a constant such that $L > 0$ such that

$$\|Ax - Ay\| \leq L \|x - y\|, \quad \forall x, y \in H.$$

A is said to be nonexpansive if

$$\|Ax - Ay\| \leq \|x - y\|, \quad \forall x, y \in H.$$

Let C be a nonempty, closed and convex subset of H. Recall that the classical variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u \rangle \geq 0, \quad \forall v \in C. \tag{1.1}$$

One can see that the variational inequality problem (1.1) is equivalent to a fixed point problem. $u \in C$ is a solution of the variational inequality (1.1)

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if and only if $u \in C$ is a fixed point of the mapping $P_C(I - \lambda A)$, where I is the identity mapping and $\lambda > 0$ is a constant. Recently, iterative methods have been applied to approximate common elements in the fixed point set of nonexpansive mappings and in the solution set of variational inequality (1.1), see [3, 4, 5, 6, 12, 13, 15, 19, 21] and the reference therein.

Noor and Huang [13] considered a three-step iterative method for finding a common element in the set of fixed points of a nonexpansive mapping and in the set of solutions of the variational inequality problem (1.1) in a real Hilbert space. To be more precise, they introduced the following algorithm:

$$(NH-1) \qquad \begin{cases} x_0 \in C, \\ z_n = (1-c_n)x_n + c_n S P_C(x_n - \rho T x_n), \\ y_n = (1-b_n)x_n + b_n S P_C(y_n - \rho T y_n), \\ x_{n+1} = (1-a_n)x_n + a_n S P_C(y_n - \rho T y_n), \quad \forall n \geq 0, \end{cases}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ for all $n \geq 0$, S is a nonexpansive mapping and T is a monotone-type operator. They showed that the sequence $\{x_n\}$ generated in the above iterative sequence converges strongly to a common element in the set of fixed points of a non-expansive mapping S and in the set of solutions of the variational inequality problem (1.1); see [13] for details.

In [14], Noor and Huang considered the following variational inclusion problem. Find an $u \in H$ such that

$$(1.2) \qquad 0 \in Au + Tu,$$

where T and A are monotone operators. They also considered the following three-step iterative algorithm:

$$(NH-2) \qquad \begin{cases} x_0 \in H, \\ z_n = (1-c_n)x_n + c_n S J_A(x_n - \rho T x_n), \\ y_n = (1-b_n)x_n + b_n S J_A(y_n - \rho T y_n), \\ x_{n+1} = (1-a_n)x_n + a_n S J_A(y_n - \rho T y_n), \quad \forall n \geq 0, \end{cases}$$

where $\{a_n\}$, $\{b_n\}$ and $\{c_n\}$ are sequences in $[0,1]$ for all $n \geq 0$, S is a nonexpansive mapping, $J_A = (I + \rho A)^{-1}$. They showed that the sequence $\{x_n\}$ generated in the above iterative sequence converges strongly to a common element in the set of fixed points of a nonexpansive mapping S and in the set of solutions of the variational inclusion problem (1.2); see [14] for details.

Recently, three-step iterative algorithms were studied by many authors; see [3, 4, 5, 6, 7, 9, 11, 12, 13, 14, 16, 17, 19, 20, 21]. In 1989, Glowinski and Le Tallec [7] used three-step iterative schemes to find the approximate solutions of the elastoviscoplasticity problem, liquid crystal theory, and eigenvalue computation, and they showed that three-step approximations perform better numerically. Haubruge et al. [9] studied the convergence analysis of three-step schemes of Glowinski and Le Tallec [7] and applied these schemes to obtain new splitting-type algorithms for solving variational inequalities, separable convex programming, and minimization of a sum of convex functions. They also proved that three-step iterations lead to highly parallelized algorithms under certain conditions.

Very recently, Noor et al. [15] considered the problem of difference of two monotone operators. More precisely, they studied the following problem. Given two operators $T, A : H \to H$, find $u \in H$ such that

$$0 \in Au - Tu, \tag{1.3}$$

where A is a maximal monotone and T is a strongly anti-monotone. They also constructed a Mann-type algorithm as follows.

$$x_0 \in H, \quad x_{n+1} = a_n x_n + (1 - a_n) J_A(x_n + \rho T x_n), \quad n \geq 0,$$

where $\{a_n\}$ is a sequence in $[0, 1]$, $J_A = (I + \rho A)^{-1}$. Strong convergence theorems of the Mann-type iterative algorithm are established in the framework of Hilbert spaces.

If $A(\cdot) = \partial f(\cdot)$, the subdifferential of a proper, convex and lower-semicontinuous function $f : H \to (-\infty, \infty]$, then the problem (1.3) is equivalent to finding $u \in H$ such that

$$0 \in \partial f(u) - Tu, \tag{1.4}$$

which was considered by Adly and Oettli [1]. We note that the problem (1.4) is equivalent to finding $u \in H$ such that

$$\langle Tu, v - u \rangle + \partial f(u) - \partial(v) \leq 0 \quad \forall v \in H, \tag{1.5}$$

which is known as the mixed variational inequality. If f is the indicator of a nonempty closed convex subset of K in a Hilbert spaces, then the problem (1.5) is reduced to finding $u \in K$ such that

$$\langle Tu, v - u \rangle \leq 0, \quad \forall v \in K, \tag{1.5}$$

which is the classical variational inequality. Variational inclusions involving the difference of two monotone operators provide us with a unified, natural, novel and simple framework to study a wide class of problems arising in DC programming, prox-regularity, multicommodity network, image restoring processing, tomograpy, molecular biology, optimization, pure and applied sciences [1, 2, 8, 10, 15] and the references therein.

In this paper, motivated by the recently research work going on in this direction, we continue to study the problem of finding a solution of the problem (1.3) by a three-step iterative algorithm. We will denote the set of solutions of the problem (1.3) by $S(A, T)$.

In order to prove our main results, we need the following lemmas.

Lemma 1.1 ([18])**.** *Suppose that $\{\delta_n\}$ is a nonnegative sequence satisfying the following inequality*

$$\delta_{n+1} \leq (1 - \lambda_n)\delta_n, \quad \forall n \geq 0,$$

where $\{\lambda_n\}$ *is a sequence in* $[0,1]$ *such that* $\sum_{n=0}^{\infty}\lambda_n=\infty$. *Then* $\lim_{n\to\infty}\delta_n=0$.

Lemma 1.2 ([15])**.** *Let* H *be a Hilbert space. An element* $u\in H$ *is a solution of the problem* (1.3) *if and only if* $u\in H$ *is a fixed point of the mapping* $J_A(I+\rho T)$, *where* $J_A=(I+\rho A)^{-1}$, I *is the identity mapping and* T *is a strongly anti-monotone mapping.*

Lemma 1.3. *Let* H *be a Hilbert space and* $S:H\to H$ *a nonexpansive mapping with a fixed point. Assume that* $F(S)\cap S(A,T)\neq\emptyset$. *If* $u\in F(S)\cap S(A,T)$, *then* $u=SJ_A(I+\rho T)u$.

Proof. Fix $u\in F(S)\cap S(A,T)$. From Lemma 1.2, we see that $u=J_A(I+\rho T)u$. We also have $u=Su$. It follows that

$$u=J_A(I+\rho T)u=Su=SJ_A(I+\rho T)u.$$

This completes the proof. □

2. Main results

Theorem 2.1. *Let* H *be a Hilbert space,* A *a maximal monotone mapping on* H *and* T *an* α*-strongly anti-monotone and* β*-Lipschitz continuous mapping on* H. *Let* $S:H\to H$ *be a nonexpansive mapping with a fixed point and let* $\{x_n\}$ *be a sequence generated by the following manner*:

$$\begin{cases} x_0\in H,\\ z_n=(1-c_n)x_n+c_nSJ_A(x_n+\rho Tx_n),\\ y_n=(1-b_n)x_n+b_nSJ_A(z_n+\rho Tz_n),\\ x_{n+1}=(1-a_n)x_n+a_nSJ_A(y_n+\rho Ty_n), \quad \forall n\geq 0, \end{cases} \tag{2.1}$$

where $\{a_n\}$, $\{b_n\}$ *and* $\{c_n\}$ *are sequences in* $[0,1]$ *for all* $n\geq 0$, $J_A=(I+\rho A)^{-1}$ *and* ρ *is a constant satisfying the restriction* $0<\rho<\frac{2\alpha}{\beta^2}$. *Assume that* $F(S)\cap S(A,T)\neq\emptyset$ *and* $\sum_{n=0}^{\infty}a_n=\infty$. *Then the sequence* $\{x_n\}$ *converges strongly to a point in* $F(S)\cap S(A,T)$.

Proof. Let $x^*\in F(S)\cap S(A,T)$. From Lemma 1.3, we see that

$$\begin{cases} x^*=(1-c_n)x^*+c_nSJ_A(x^*+\rho Tx^*), \quad \forall n\geq 0,\\ x^*=(1-b_n)x^*+b_nSJ_A(x^*+\rho Tx^*), \quad \forall n\geq 0,\\ x^*=(1-a_n)x^*+a_nSJ_A(x^*+\rho Tx^*), \quad \forall n\geq 0. \end{cases} \tag{2.2}$$

It follows from (2.1) that

$$\begin{aligned} &\|x_{n+1}-x^*\|\\ =&\|(1-a_n)(x_n-x^*)+a_n\big(SJ_A(y_n+\rho Ty_n)-SJ_A(x^*+\rho Tx^*)\big)\|\\ \leq&(1-a_n)\|x_n-x^*\|+a_n\|J_A(y_n+\rho Ty_n)-J_A(x^*+\rho Tx^*)\|\\ \leq&(1-a_n)\|x_n-x^*\|+a_n\|y_n-x^*+\rho(Ty_n-Tx^*)\|. \end{aligned} \tag{2.3}$$

From the α-strongly anti-monotone and β-Lipschitz assumptions on T, we have

$$\begin{aligned}
&\|y_n - x^* + \rho_n(Ty_n - Tx^*)\|^2 \\
&= \|y_n - x^*\|^2 + 2\rho\langle Ty_n - Tx^*, y_n - x^*\rangle + \rho^2\|Ty_n - Tx^*\|^2 \\
&\leq \|y_n - x^*\|^2 - 2\rho\alpha\|y_n - x^*\|^2 + \rho^2\beta^2\|y_n - x^*\|^2 \\
&= (1 - 2\rho\alpha + \rho^2\beta^2)\|y_n - x^*\|^2.
\end{aligned}$$

That is,

$$\|y_n - x^* + \rho(Ty_n - Tx^*)\| \leq \theta_n\|y_n - x^*\|, \tag{2.4}$$

where

$$\theta = \sqrt{1 - 2\rho\alpha + \rho^2\beta^2}.$$

From the assumption $0 < \rho < \frac{2\alpha}{\beta^2}$, we see that $\theta < 1$.

Next, we estimate $\|y_n - x^*\|$. It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\|y_n - x^*\| \\
&= \|(1 - b_n)(x_n - x^*) + b_n\big(SJ_A(z_n + \rho Tz_n) - SJ_A(x^* + \rho Tx^*)\big)\| \\
&\leq (1 - b_n)\|x_n - x^*\| + b_n\|J_A(z_n + \rho Tz_n) - J_A(x^* + \rho Tx^*)\| \\
&\leq (1 - b_n)\|x_n - x^*\| + b_n\|z_n - x^* + \rho(Tz_n - Tx^*)\|.
\end{aligned} \tag{2.5}$$

From the α-strongly anti-monotone and β-Lipschitz assumptions on T, we have

$$\begin{aligned}
&\|z_n - x^* + \rho(Tz_n - Tx^*)\|^2 \\
&= \|z_n - x^*\|^2 + 2\rho\langle Tz_n - Tx^*, y_n - x^*\rangle + \rho^2\|Tz_n - Tx^*\|^2 \\
&\leq \|z_n - x^*\|^2 - 2\rho\alpha\|z_n - x^*\|^2 + \rho^2\beta^2\|z_n - x^*\|^2 \\
&= (1 - 2\rho\alpha + \rho^2\beta^2)\|z_n - x^*\|^2.
\end{aligned}$$

That is,

$$\|z_n - x^* + \rho(Tz_n - Tx^*)\| \leq \theta\|z_n - x^*\|. \tag{2.6}$$

Finally, we estimate $\|z_n - x^*\|$. It follows from (2.1) and (2.2) that

$$\begin{aligned}
&\|z_n - x^*\| \\
&= \|(1 - c_n)(x_n - x^*) + c_n\big(SJ_A(x_n + \rho Ts_n) - SJ_A(x^* + \rho Tx^*)\big)\| \\
&\leq (1 - c_n)\|x_n - x^*\| + c_n\|J_A(x_n + \rho Tx_n) - J_A(x^* + \rho Tx^*)\| \\
&\leq (1 - c_n)\|x_n - x^*\| + c_n\|x_n - x^* + \rho(Tx_n - Tx^*)\|.
\end{aligned} \tag{2.7}$$

In a similar way, we can obtain that

$$\|x_n - x^* + \rho(Tx_n - Tx^*)\| \leq \theta\|x_n - x^*\|. \tag{2.8}$$

Substituting (2.8) into (2.7), we arrive at

$$\|z_n - x^*\| \leq [1 - c_n(1 - \theta)]\|x_n - x^*\|,$$

which combines with (2.5) and (2.6) that

$$\begin{aligned}\|y_n - x^*\| &\le \Big(1 - b_n\big(1 - \theta(1 - c_n(1-\theta))\big)\Big)\|x_n - x^*\| \\ &\le \|x_n - x^*\|.\end{aligned}$$

It follows (2.3) and (2.4) that

$$\begin{aligned}\|x_{n+1} - x^*\| &\le (1 - a_n)\|x_n - x^*\| + a_n\theta\|y_n - x^*\| \\ &\le [1 - a_n(1-\theta)]\|x_n - x^*\|.\end{aligned} \tag{2.9}$$

Applying Lemma 1.1 to (2.9), we can conclude the desired conclusion immediately. □

If $\{c_n\} \equiv 0$, then Theorem 2.1 is reduced to the following.

Corollary 2.2. *Let H be a Hilbert space, A a maximal monotone mapping on H and T an α-strongly anti-monotone and β-Lipschitz continuous mapping on H. Let $S : H \to H$ be a nonexpansive mapping with a fixed point and let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ y_n = (1 - b_n)x_n + b_n S J_A(x_n + \rho T x_n), \\ x_{n+1} = (1 - a_n)x_n + a_n S J_A(y_n + \rho T y_n), \quad \forall n \ge 0, \end{cases} \tag{2.10}$$

where $\{a_n\}$ and $\{b_n\}$ are sequences in $[0,1]$ for all $n \ge 0$, $J_A = (I + \rho A)^{-1}$ and ρ is a constant satisfying the restriction $0 < \rho < \frac{2\alpha}{\beta^2}$. Assume that $F(S) \cap S(A,T) \ne \emptyset$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point in $F(S) \cap S(A,T)$.

Remark 2.3. The algorithm (2.10) is call an Ishikawa-type iterative algorithm.

Further, if $\{b_n\} \equiv 0$, then Corollary 2.2 is reduced to the following.

Corollary 2.4. *Let H be a Hilbert space, A a maximal monotone mapping on H and T an α-strongly anti-monotone and β-Lipschitz continuous mapping on H. Let $S : H \to H$ be a nonexpansive mapping with a fixed point and let $\{x_n\}$ be a sequence generated by the following manner:*

$$\begin{cases} x_0 \in H, \\ x_{n+1} = (1 - a_n)x_n + a_n S J_A(x_n + \rho T x_n), \quad \forall n \ge 0, \end{cases} \tag{2.11}$$

where $\{a_n\}$ is a sequence in $[0,1]$ for all $n \ge 0$, $J_A = (I + \rho A)^{-1}$ and ρ is a constant satisfying the restriction $0 < \rho < \frac{2\alpha}{\beta^2}$. Assume that $F(S) \cap S(A,T) \ne \emptyset$ and $\sum_{n=0}^{\infty} a_n = \infty$. Then the sequence $\{x_n\}$ converges strongly to a point in $F(S) \cap S(A,T)$.

Remark 2.5. The algorithm (2.11) is call a Mann-type iterative algorithm. We see that Corollary 2.4 is reduced to Theorem 3.2 of Noor et al. [15] if $S = I$, where I denotes the identity operator.

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