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CHARACTERIZATIONS OF RAPIDLY DECREASING GENERALIZED FUNCTIONS

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Abstract. The well-known characterizations of the Schwartz space of rapidly decreasing functions is extended to new algebras of rapidly decreasing generalized functions.

1. Introduction

The Schwartz space $\mathcal{S}$ of rapidly decreasing functions on $\mathbb{R}^n$ and its different generalizations, in view of their importance in analysis, have been characterized differently by many authors, e.g. see [17], [12], [14], [4], [9] and [1]. Recall that

$$\mathcal{S} = \left\{ f \in \mathcal{C}^\infty : \forall \alpha \in \mathbb{Z}_+^n, \forall \beta \in \mathbb{Z}_+^n, \sup_{x \in \mathbb{R}^n} \left| x^\beta \partial^\alpha f(x) \right| < \infty \right\},$$

and let

$$\mathcal{S}^* = \left\{ f \in \mathcal{C}^\infty : \forall \alpha \in \mathbb{Z}_+^n, \sup_{x \in \mathbb{R}^n} \left| \partial^\alpha f(x) \right| < \infty \right\},$$

$$\mathcal{S}_* = \left\{ f \in \mathcal{C}^\infty : \forall \beta \in \mathbb{Z}_+^n, \sup_{x \in \mathbb{R}^n} \left| x^\beta f(x) \right| < \infty \right\},$$

then the characterization of $\mathcal{S}$ given in [4] is the result:

$$\mathcal{S} = \mathcal{S}^* \cap \mathcal{S}_*. \tag{1}$$

To built a Fourier analysis within the new generalized functions of [5], the algebra of rapidly decreasing generalized functions on $\mathbb{R}^n$, denoted $\mathcal{G}_\mathcal{S}$, was first constructed in [15] and recently studied in [7] and [6]. The algebra of regular rapidly decreasing generalized functions on $\mathbb{R}^n$, denoted $\mathcal{G}_\mathcal{S}^\infty$, is fundamental in the study of local regularity of a Colombeau generalized functions and also for developing a generalized microlocal analysis.

The purpose of this work is to lift the characterizations of the Schwartz space $\mathcal{S} = \mathcal{S}^* \cap \mathcal{S}_*$ to the algebras $\mathcal{G}_\mathcal{S}$ and $\mathcal{G}_\mathcal{S}^\infty$. Actually we do more, these characterizations are given in the general context of the algebras $\mathcal{G}_\mathcal{S}^\mathcal{R}(\Omega)$ of

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$\mathcal{R}$-rapidly decreasing generalized functions on an open set Ω of $\mathbb{R}^n$, see [6] and [2]. Section 2 recall the notion of regular set of sequences. Sections 3, 4 and 5 introduce, respectively, the algebra of $\mathcal{R}$-bounded generalized functions, the algebra of $\mathcal{R}$-roughly decreasing generalized functions and the algebra of $\widetilde{\mathcal{R}}$-rapidly decreasing generalized functions. Section 6 gives the characterization of the algebra $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$, provided Ω is a box of $\mathbb{R}^n$, and as corollaries of this result we obtain the characterizations of the classical algebras $\mathcal{G}_{\mathcal{S}}$ and $\mathcal{G}_{\mathcal{S}}^{\infty}$. Finally, Section 7 gives the characterization of $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\mathbb{R}^n)$ using the Fourier transform.

2. Regular set of sequences

Recall the definition of a regular set of sequences introduced in [6], see [2].

Definition. A non void subset $\mathcal{R}$ of $\mathbb{R}_+^{\mathbb{Z}_+}$ is said to be regular, if

For all $(N_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}$ and $(k,k') \in \mathbb{Z}_+^2$, there exists $(N'_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}$ such that

(R1) $$N_{m+k} + k' \leq N'_m \ , \forall m \in \mathbb{Z}_+ \ .$$

For all $(N_m)_{m\in\mathbb{Z}_+}$ and $(N'_m)_{m\in\mathbb{Z}_+}$ in $\mathcal{R}$, there exists $(N"_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}$ such that

(R2) $$\max(N_m, N'_m) \leq N"_m \ , \forall m \in \mathbb{Z}_+ \ .$$

For all $(N_m)_{m\in\mathbb{Z}_+}$ and $(N'_m)_{m\in\mathbb{Z}_+}$ in $\mathcal{R}$, there exists $(N"_m)_{m\in\mathbb{Z}_+} \in \mathcal{R}$ such that

(R3) $$N_{m_1} + N'_{m_2} \leq N"_{m_1+m_2} \ , \forall (m_1, m_2) \in \mathbb{Z}_+^2 \ .$$

The notion of regular set is extended to the sets of double sequences.

Definition. A non void subset $\widetilde{\mathcal{R}}$ of $\mathbb{R}_+^{\mathbb{Z}_+^2}$ is said to be regular if

For all $(N_{q,l})_{(q,l)\in\mathbb{Z}_+^2} \in \widetilde{\mathcal{R}}$ and $(k,k',k'') \in \mathbb{Z}_+^3$, there exists $\left(N'_{q,l}\right)_{(q,l)\in\mathbb{Z}_+^2} \in \widetilde{\mathcal{R}}$ such that

($\widetilde{R}$1) $$N_{q+k,l+k'} + k'' \leq N'_{q,l} \ , \forall (q,l) \in \mathbb{Z}_+^2 \ .$$

For all $(N_{q,l})_{(q,l)\in\mathbb{Z}_+^2}$ and $\left(N'_{q,l}\right)_{(q,l)\in\mathbb{Z}_+^2}$ in $\widetilde{\mathcal{R}}$, there exists $(N"_{q,l})_{(q,l)\in\mathbb{Z}_+^2} \in \widetilde{\mathcal{R}}$ such that

($\widetilde{R}$2) $$\max\left(N_{q,l}, N'_{q,l}\right) \leq N"_{q,l} \ , \forall (q,l) \in \mathbb{Z}_+^2 \ .$$

For all $(N_{q,l})_{(q,l)\in\mathbb{Z}_+^2}$ and $\left(N'_{q,l}\right)_{(q,l)\in\mathbb{Z}_+^2}$ in $\widetilde{\mathcal{R}}$, there exists $(N"_{q,l})_{(q,l)\in\mathbb{Z}_+^2} \in \widetilde{\mathcal{R}}$ such that

($\widetilde{R}$3) $$N_{q_1,l_1} + N'_{q_2,l_2} \leq N"_{q_1+q_2,l_1+l_2} \ , \forall (q_1,q_2,l_1,l_2) \in \mathbb{Z}_+^4 \ .$$

Example 2.1. i) The set $\mathbb{R}_+^{\mathbb{Z}_+}$ of all positive sequences is regular.

ii) The set $\mathcal{A}$ of affine sequences defined by

$$\mathcal{A} = \left\{ (N_m)_{m\in\mathbb{N}} \in \mathbb{R}_+^{\mathbb{Z}_+} : \exists a \geq 0, \exists b \geq 0, \forall m \in \mathbb{Z}_+, N_m \leq am + b \right\}$$

is regular.

iii) The set $\mathcal{B}$ of all bounded sequences of $\mathbb{R}_+^{\mathbb{Z}_+}$ is regular.

iv) The set $\mathbb{R}_+^{\mathbb{Z}_+^2}$ of all positive double sequences is regular.

v) The set $\widetilde{\mathcal{B}}$ of all bounded sequences of $\mathbb{R}_+^{\mathbb{Z}_+^2}$ is regular.

We give the following results, easy to prove, needed in the formulation of the principal theorems of this paper.

Lemma 2.2. *Let $\widetilde{\mathcal{R}}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+^2}$. Then*

(i) *The subset $\widetilde{\mathcal{R}}^0 := \left\{ N_{.,0} : N \in \widetilde{\mathcal{R}} \right\}$ is regular in $\mathbb{R}_+^{\mathbb{Z}_+}$.*

(ii) *The subset $\widetilde{\mathcal{R}}_0 := \left\{ N_{0,.} : N \in \widetilde{\mathcal{R}} \right\}$ is regular in $\mathbb{R}_+^{\mathbb{Z}_+}$.*

3. The algebra of $\mathcal{R}$-bounded generalized functions

We adopt the standard notations and definitions of distributions and Colombeau algebra, see [11] and [10].

Let

$$\mathcal{S}^* (\Omega) = \left\{ f \in \mathcal{C}^\infty (\Omega) : \forall\alpha \in \mathbb{Z}_+^n, \sup_{x\in\Omega} |\partial^\alpha f(x)| < \infty \right\},$$

and $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$, if we define

$$\mathcal{E}_{\mathcal{S}^*}^{\mathcal{R}} (\Omega) = \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}^* (\Omega)^I : \exists N \in \mathcal{R}, \forall\alpha \in \mathbb{Z}_+^n, \sup_{x\in\Omega} |\partial^\alpha u_\epsilon (x)| = O\left(\epsilon^{-N_{|\alpha|}}\right), \epsilon \to 0 \right\},$$

$$\mathcal{N}_{\mathcal{S}^*}^{\mathcal{R}} (\Omega) = \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}^* (\Omega)^I : \forall N \in \mathcal{R}, \forall\alpha \in \mathbb{Z}_+^n, \sup_{x\in\Omega} |\partial^\alpha u_\epsilon (x)| = O\left(\epsilon^{N_{|\alpha|}}\right), \epsilon \to 0 \right\},$$

where $I =]0,1]$, then the properties of $\mathcal{E}_{\mathcal{S}^*}^{\mathcal{R}} (\Omega)$ and $\mathcal{N}_{\mathcal{S}^*}^{\mathcal{R}} (\Omega)$ are given by the following results.

Proposition 3.1. (i) *The space $\mathcal{E}_{\mathcal{S}^*}^{\mathcal{R}} (\Omega)$ is a subalgebra of $\mathcal{S}^* (\Omega)^I$.*

(ii) *The space $\mathcal{N}_{\mathcal{S}^*}^{\mathcal{R}} (\Omega)$ is an ideal of $\mathcal{E}_{\mathcal{S}^*}^{\mathcal{R}} (\Omega)$.*

(iii) *We have $\mathcal{N}_{\mathcal{S}^*}^{\mathcal{R}} (\Omega) = \mathcal{N}_{\mathcal{S}^*} (\Omega)$, where*

$$\mathcal{N}_{\mathcal{S}^*} (\Omega) = \left\{ (u_\epsilon)_\epsilon \in \mathcal{S}^* (\Omega)^I : \forall m \in \mathbb{Z}_+, \forall\alpha \in \mathbb{Z}_+^n, \sup_{x\in\Omega} |\partial^\alpha u_\epsilon (x)| = O(\epsilon^m), \epsilon \to 0 \right\}.$$

Proof. Follows easily from the definitions and standard arguments of Colombeau algebras, see [10]. □

Definition. An open subset Ω of $\mathbb{R}^n$ is said to be a box, if

$$\Omega = \mathbf{I}_1 \times \mathbf{I}_2 \times \cdots \times \mathbf{I}_n,$$

where each $\mathbf{I}_i$ is a finite or infinite open interval of $\mathbb{R}$.

We have also the null characterization of the ideal $\mathcal{N}_{\mathcal{S}^*}(\Omega)$ provided Ω is a box.

Proposition 3.2. *Let Ω be a box. Then an element $(u_\epsilon)_\epsilon \in \mathcal{E}^{\mathcal{R}}_{\mathcal{S}^*}(\Omega)$ belongs to $\mathcal{N}_{\mathcal{S}^*}(\Omega)$ if and only if the following condition is satisfied,*

$$\forall m \in \mathbb{Z}_+, \sup_{x\in\Omega} |u_\epsilon(x)| = O(\epsilon^m), \ \epsilon \to 0. \tag{2}$$

Proof. Suppose that $(u_\epsilon)_\epsilon \in \mathcal{E}^{\mathcal{R}}_{\mathcal{S}^*}(\Omega)$ satisfies (2). It suffices to show that $(\partial_i u_\epsilon)_\epsilon$ satisfies the $\mathcal{N}_{\mathcal{S}^*}(\Omega)$ estimates for all $i = 1, \ldots, n$. Suppose that u_ϵ is real valued, in the complex case, we shall carry out the following calculus separately on its real and imaginary part. Let $m \in \mathbb{Z}_+$, we have to show that

$$\sup_{x\in\Omega} |\partial_i u_\epsilon(x)| = O(\epsilon^m), \ \epsilon \to 0.$$

Since $(u_\epsilon)_\epsilon \in \mathcal{E}^{\mathcal{R}}_{\mathcal{S}^*}(\Omega)$, then

$$\exists N \in \mathcal{R}, \sup_{x\in\Omega} \left|\partial_i^2 u_\epsilon(x)\right| = O\left(\epsilon^{-N_2}\right), \ \epsilon \to 0. \tag{3}$$

Since $(u_\epsilon)_\epsilon$ satisfies (2), we have

$$\sup_{x\in\Omega} |u_\epsilon(x)| = O\left(\epsilon^{N_2+2m}\right), \ \epsilon \to 0. \tag{4}$$

By Taylor's formula, we have

$$u_\epsilon\left(x + \epsilon^{N_2+m} e_i\right) = u_\epsilon(x) + \partial_i u_\epsilon(x)\,\epsilon^{N_2+m} + \frac{1}{2}\partial_i^2 u_\epsilon\left(x + \theta\epsilon^{N_2+m} e_i\right)\epsilon^{2(N_2+m)},$$

where $\theta \in]0,1[$ and ϵ is sufficiently small, as Ω is a box. It follows that

$$\begin{aligned} |\partial_i u_\epsilon(x)| \quad &\le \quad \underbrace{\left|u_\epsilon\left(x + \epsilon^{N_2+m} e_i\right)\right| \epsilon^{-N_2-m}}_{(*)} + \underbrace{|u_\epsilon(x)|\,\epsilon^{-N_2-m}}_{(**)} + \\ &\quad + \underbrace{\epsilon^{N_2+m}\left|\partial_i^2 u_\epsilon\left(x + \theta\epsilon^{N_2+m} e_i\right)\right|}_{(***)} . \end{aligned}$$

From (4), we have that $(*)$ and $(**)$ are of order $O(\epsilon^m)$, $\epsilon \to 0$, and from (3), we have that $(***)$ is of order $O(\epsilon^m)$, $\epsilon \to 0$. □

Definition. Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$, the algebra of $\mathcal{R}$-bounded generalized functions, denoted by $\mathcal{G}^{\mathcal{R}}_{\mathcal{S}^*}(\Omega)$, is the quotient algebra

$$\mathcal{G}^{\mathcal{R}}_{\mathcal{S}^*}(\Omega) = \frac{\mathcal{E}^{\mathcal{R}}_{\mathcal{S}^*}(\Omega)}{\mathcal{N}_{\mathcal{S}^*}(\Omega)} . \tag{5}$$

Remark 3.3. When $\mathcal{R}$ is the set of all positive sequences, the algebra $\mathcal{G}^{\mathcal{R}}_{\mathcal{S}^*}(\Omega)$ is denoted by $\mathcal{G}_{L^\infty}(\Omega)$ in [3], this algebra is constructed on the differential algebra $D_{L^\infty}(\Omega)$ of Schwartz [16]. So it is more correct to write $\mathcal{G}^{\mathcal{R}}_{L^\infty}(\Omega)$ instead of $\mathcal{G}^{\mathcal{R}}_{\mathcal{S}^*}(\Omega)$.

4. The algebra of $\mathcal{R}$-roughly decreasing generalized functions

Let

$$\mathcal{S}_* (\Omega) = \left\{ f \in \mathcal{C}^\infty (\Omega) : \forall \beta \in \mathbb{Z}_+^n, \sup_{x\in\Omega} \left| x^\beta f (x) \right| < \infty \right\},$$

and $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$, if we define

$$\mathcal{E}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega) = \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathcal{S}_* (\Omega)^I : \exists N \in \mathcal{R}, \forall \beta \in \mathbb{Z}_+^n, \\ \sup_{x\in\Omega} \left| x^\beta u_\epsilon (x) \right| = O \left(\epsilon^{-N_{|\beta|}} \right), \epsilon \to 0 \end{array} \right\},$$

$$\mathcal{N}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega) = \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathcal{S}_* (\Omega)^I : \forall N \in \mathcal{R}, \forall \beta \in \mathbb{Z}_+^n, \\ \sup_{x\in\Omega} \left| x^\beta u_\epsilon (x) \right| = O \left(\epsilon^{N_{|\beta|}} \right), \epsilon \to 0 \end{array} \right\},$$

then the following properties of $\mathcal{E}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$ and $\mathcal{N}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$ are easy to verify.

Proposition 4.1. (i) *The space* $\mathcal{E}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$ *is a subalgebra of* $\mathcal{S}_* (\Omega)^I$.

(ii) *The space* $\mathcal{N}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$ *is an ideal of* $\mathcal{E}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$.

(iii) *We have* $\mathcal{N}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega) = \mathcal{N}_{\mathcal{S}_*} (\Omega)$, *where*

$$\mathcal{N}_{\mathcal{S}_*} (\Omega) = \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathcal{S}_* (\Omega)^I : \forall m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}_+^n, \\ \sup_{x\in\Omega} \left| x^\beta u_\epsilon (x) \right| = O (\epsilon^m), \epsilon \to 0 \end{array} \right\}.$$

The following proposition characterizes $\mathcal{N}_{\mathcal{S}_*} (\Omega)$.

Proposition 4.2. *Let* $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$. *Then* $(u_\epsilon)_\epsilon \in \mathcal{N}_{\mathcal{S}_*} (\Omega)$ *if and only if the following condition is satisfied,*

$$\forall m \in \mathbb{Z}_+, \sup_{x\in\Omega} |u_\epsilon (x)| = O (\epsilon^m), \ \epsilon \to 0. \tag{6}$$

Proof. Suppose that $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$ satisfies (6). Since $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$, then $\exists N \in \mathcal{R}$, $\forall \beta \in \mathbb{Z}_+^n$,

$$\sup_{x\in\Omega} \left| x^{2\beta} u_\epsilon (x) \right| = O \left(\epsilon^{-N_{2|\beta|}} \right), \ \epsilon \to 0.$$

From (6), for all $m \in \mathbb{Z}_+$ we have

$$\sup_{x\in\Omega} |u_\epsilon (x)| = O \left(\epsilon^{2m+N_{2|\beta|}} \right), \ \epsilon \to 0.$$

Therefore $\forall x \in \Omega$,

$$\left| x^\beta u_\epsilon (x) \right|^2 = \left| x^{2\beta} u_\epsilon (x) \right| |u_\epsilon (x)| = O \left(\epsilon^{2m} \right), \ \epsilon \to 0,$$

hence

$$\left| x^\beta u_\epsilon (x) \right| = O (\epsilon^m), \ \epsilon \to 0. \qquad \square$$

Definition. Let $\mathcal{R}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+}$, the algebra of $\mathcal{R}$-roughly decreasing generalized functions, denoted by $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)$, is the quotient algebra

$$\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega) = \frac{\mathcal{E}_{\mathcal{S}_*}^{\mathcal{R}} (\Omega)}{\mathcal{N}_{\mathcal{S}_*} (\Omega)}.$$

Remark 4.3. The C^∞ regularity in the definition of elements of $\mathcal{G}_{\mathcal{S}_*}^{\mathcal{R}}(\Omega)$ is not in fact needed in the proof of the principal results of this work.

5. The algebra of $\widetilde{\mathcal{R}}$-rapidly decreasing generalized functions

Recall

$$\mathcal{S}(\Omega) = \left\{ f \in \mathcal{C}^\infty(\Omega) : \forall (\alpha,\beta) \in \mathbb{Z}_+^{2n}, \sup_{x\in\Omega} \left| x^\beta \partial^\alpha f(x) \right| < \infty \right\},$$

the space of rapidly decreasing functions on Ω, and let $\widetilde{\mathcal{R}}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+^2}$, if we define

$$\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega) = \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathcal{S}(\Omega)^I : \exists N \in \widetilde{\mathcal{R}}, \forall (\alpha,\beta) \in \mathbb{Z}_+^{2n}, \\ \sup_{x\in\Omega} \left| x^\beta \partial^\alpha u_\epsilon(x) \right| = O\left(\epsilon^{-N_{|\alpha|,|\beta|}}\right), \epsilon \to 0 \end{array} \right\},$$

$$\mathcal{N}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega) = \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathcal{S}(\Omega)^I : \forall N \in \widetilde{\mathcal{R}}, \forall (\alpha,\beta) \in \mathbb{Z}_+^{2n}, \\ \sup_{x\in\Omega} \left| x^\beta \partial^\alpha u_\epsilon(x) \right| = O\left(\epsilon^{N_{|\alpha|,|\beta|}}\right), \epsilon \to 0 \end{array} \right\},$$

then we have the following results.

Proposition 5.1. *We have the following assertions*:

(i) *The space* $\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$ *is a subalgebra of* $\mathcal{S}(\Omega)^I$.
(ii) *The space* $\mathcal{N}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$ *is an ideal of* $\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$.
(iii) *We have* $\mathcal{N}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega) = \mathcal{N}_{\mathcal{S}}(\Omega)$, *where*

$$\mathcal{N}_{\mathcal{S}}(\Omega) = \left\{ \begin{array}{c} (u_\epsilon)_\epsilon \in \mathcal{S}(\Omega)^I : \forall m \in \mathbb{Z}_+, \forall (\alpha,\beta) \in \mathbb{Z}_+^{2n}, \\ \sup_{x\in\Omega} \left| x^\beta \partial^\alpha u_\epsilon(x) \right| = O(\epsilon^m), \epsilon \to 0 \end{array} \right\}.$$

Proof. The proof is not difficult, it follows from the properties of the set $\widetilde{\mathcal{R}}$. □

Definition. Let $\widetilde{\mathcal{R}}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+^2}$, the algebra of $\widetilde{\mathcal{R}}$-rapidly decreasing generalized functions on Ω, denoted by $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$, is the quotient algebra

$$\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega) = \frac{\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)}{\mathcal{N}_{\mathcal{S}}(\Omega)}.$$

Example 5.2. (i) For $\widetilde{\mathcal{R}} = \mathbb{R}_+^{\mathbb{Z}_+^2}$, we obtain the algebra $\mathcal{G}_{\mathcal{S}}(\Omega)$ of rapidly decreasing generalized functions on Ω, see [10].

(ii) For $\widetilde{\mathcal{R}} = \widetilde{\mathcal{B}}$, we obtain the algebra $\mathcal{G}_{\mathcal{S}}^\infty(\Omega)$ of regular rapidly decreasing generalized functions on Ω, see [7].

6. Characterization of $\widetilde{\mathcal{R}}$-rapidly decreasing generalized functions

Let us mention that the theorem of [4] can be extended to an open subset Ω of $\mathbb{R}^n$, provided Ω is a box.

Proposition 6.1. *If Ω is a box of $\mathbb{R}^n$, then*

$$\mathcal{S}(\Omega) = \mathcal{S}^*(\Omega) \cap \mathcal{S}_*(\Omega). \tag{7}$$

Proof. The proof is the same as in [4], noting that in the Taylor's expansion, the hypothesis that Ω is a box assures that $(x_1 + h, x')$ stays in Ω for all $(x_1, x') \in \Omega$ and $h > 0$ sufficiently small. □

The principal result of this section is an extension of (7) to the algebra of $\widetilde{\mathcal{R}}$-rapidly decreasing generalized functions. It is the first characterization of the algebra $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$ in the spirit of (7).

Theorem 6.2. *If Ω is a box, then*

$$\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega) = \mathcal{G}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0}(\Omega) \cap \mathcal{G}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}(\Omega). \tag{8}$$

Proof. We have to show that $\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega) = \mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0}(\Omega) \cap \mathcal{E}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}(\Omega)$ and $\mathcal{N}_{\mathcal{S}}(\Omega) = \mathcal{N}_{\mathcal{S}_*}(\Omega) \cap \mathcal{N}_{\mathcal{S}^*}(\Omega)$. The inclusions $\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega) \subset \mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0}(\Omega) \cap \mathcal{E}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}(\Omega)$ and $\mathcal{N}_{\mathcal{S}}(\Omega) \subset \mathcal{N}_{\mathcal{S}_*}(\Omega) \cap \mathcal{N}_{\mathcal{S}^*}(\Omega)$ are obvious. In order to show the inclusion $\mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0}(\Omega) \cap \mathcal{E}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}(\Omega) \subset \mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$, let $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0}(\Omega) \cap \mathcal{E}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}(\Omega)$, then $(u_\epsilon)_\epsilon \in \mathcal{S}^*(\Omega)^I \cap \mathcal{S}_*(\Omega)^I = \mathcal{S}(\Omega)^I$. In order to show that $(u_\epsilon)_\epsilon$ satisfies the estimates of $\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$, set $x = (x_1, x') \in \mathbf{I}_1 \times (\mathbf{I}_2 \times \mathbf{I}_3 \times \cdots \times \mathbf{I}_n) := \Omega$ and consider in first the case $x_1 > 0$. For $h > 0$ sufficiently small, the Taylor's expansion of u_ϵ with respect to x_1 gives

$$u_\epsilon(x_1 + h, x') = u_\epsilon(x_1, x') + h\partial_1 u_\epsilon(x_1, x') + \frac{h^2}{2}\partial_1^2 u_\epsilon(\xi, x') \tag{9}$$

for $\xi \in \left]x_1, x_1 + h\right[$. The hypothesis $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0}(\Omega) \cap \mathcal{E}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}(\Omega)$ gives

$$\exists L \in \widetilde{\mathcal{R}}_0; \forall k \in \mathbb{Z}_+, \sup_{x_1 > 0}\left(1 + |x|^2\right)^k |u_\epsilon(x)| = O\left(\epsilon^{-L_k}\right), \epsilon \to 0,$$

$$\begin{aligned}\sup_{x_1 > 0}\left(1 + |x|^2\right)^k |u_\epsilon(x_1 + h, x')| &\leq \sup_{x_1 > 0}\left(1 + |(x_1 + h, x')|^2\right)^k |u_\epsilon(x_1 + h, x')| \\ &= O\left(\epsilon^{-L_k}\right), \epsilon \to 0,\end{aligned}$$

$$\exists M \in \widetilde{\mathcal{R}}^0, \sup_{x_1 > 0}\left|\partial_1^2 u_\epsilon(x)\right| = O\left(\epsilon^{-M_2}\right), \epsilon \to 0.$$

It follows from (9) that

$$|\partial_1 u_\epsilon(x_1, x')| \leq \frac{1}{h}\left[|u_\epsilon(x_1 + h, x')| + |u_\epsilon(x_1, x')|\right] + \frac{h}{2}\left|\partial_1^2 u_\epsilon(\xi, x')\right|.$$

Therefore

$$\sup_{x_1>0}\left(1+|x|^2\right)^k|\partial_1 u_\epsilon(x)|^2 = O\left(\epsilon^{-L_k-M_2}\right), \epsilon\to 0.$$

From $(\widetilde{R}3)$ of Definition, there exists $N'\in\widetilde{\mathcal{R}}$ such that

$$L_k+M_2\le N'_{2,k}\ .$$

Consequently

$$\sup_{x_1>0}\left(1+|x|^2\right)^k|\partial_1 u_\epsilon(x)|^2 = O\left(\epsilon^{-N'_{2,k}}\right), \epsilon\to 0.$$

So if $\beta\in\mathbb{Z}_+^n$, then

$$\sup_{x_1>0}\left|x^\beta\partial_1 u_\epsilon(x)\right|^2 \le C\sup_{x_1>0}\left(1+|x|^2\right)^{|\beta|}|\partial_1 u_\epsilon(x)|^2 = O\left(\epsilon^{-N'_{2,|\beta|}}\right), \epsilon\to 0.$$

If $x_1<0$, one considers v_ϵ such that $v_\epsilon(x)=u_\epsilon(-x_1,x')$. We see that $(v_\epsilon)_\epsilon\in\mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0}(\Omega)\cap\mathcal{E}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}(\Omega)$ and consequently the precedent argument gives the existence of $N"\in\widetilde{\mathcal{R}}$ such that

$$\sup_{x_1>0}\left|x^\beta\partial_1 v_\epsilon(x)\right|^2 = \sup_{x_1<0}\left|x^\beta\partial_1 u_\epsilon(x)\right|^2 = O\left(\epsilon^{-N"_{2,|\beta|}}\right), \epsilon\to 0.$$

Now from $(\widetilde{R}1)$ and $(\widetilde{R}2)$ of Definition, there exists $N\in\widetilde{\mathcal{R}}$ such that

$$\max\left(N'_{2,|\beta|}, N"_{2,|\beta|}\right)\le N_{1,|\beta|},$$

consequently

$$\sup_{x\in\Omega}\left|x^\beta\partial_1 u_\epsilon(x)\right| = O\left(\epsilon^{-N_{1,|\beta|}}\right), \epsilon\to 0.$$

In a similar way, we show

$$\exists N\in\widetilde{\mathcal{R}};\forall\beta\in\mathbb{Z}_+^n, \sup_{x\in\Omega}\left|x^\beta\partial_i u_\epsilon(x)\right| = O\left(\epsilon^{-N_{1,|\beta|}}\right), i=2,\ldots,n.$$

Therefore, by induction, we obtain

$$\exists N\in\widetilde{\mathcal{R}};\forall\alpha\in\mathbb{Z}_+^n,\forall\beta\in\mathbb{Z}_+^n, \sup_{x\in\Omega}\left|x^\beta\partial^\alpha u_\epsilon(x)\right| = O\left(\epsilon^{-N_{|\alpha|,|\beta|}}\right), \epsilon\to 0,$$

i.e., $(u_\epsilon)_\epsilon\in\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$.

Suppose now that $(u_\epsilon)_\epsilon\in\mathcal{N}_{\mathcal{S}_*}(\Omega)\cap\mathcal{N}_{\mathcal{S}^*}(\Omega)$. Then

$$\forall m\in\mathbb{Z}_+,\forall k\in\mathbb{Z}_+, \sup_{x_1>0}\left(1+|x|^2\right)^k|u_\epsilon(x)| = O\left(\epsilon^{\frac{m}{2}}\right), \epsilon\to 0,$$

$$\begin{aligned}\sup_{x_1>0}\left(1+|x|^2\right)^k|u_\epsilon(x_1+h,x')| &\le \sup_{x_1>0}\left(1+|(x_1+h,x')|^2\right)^k|u_\epsilon(x_1+h,x')|\\ &= O\left(\epsilon^{\frac{m}{2}}\right), \epsilon\to 0,\end{aligned}$$

$$\forall m\in\mathbb{Z}_+; \sup_{x_1>0}\left|\partial_1^2 u_\epsilon(x)\right| = O\left(\epsilon^{\frac{m}{2}}\right), \epsilon\to 0.$$

It follows from (9) that

$$\sup_{x_1>0}\left(1+|x|^2\right)^k |\partial_1 u_\epsilon(x)|^2 = O(\epsilon^m), \epsilon \to 0.$$

Consequently, if $\beta \in \mathbb{Z}_+^n$, then

$$\sup_{x_1>0}\left|x^\beta \partial_1 u_\epsilon(x)\right|^2 \leq C_1 \sup_{x_1>0}\left(1+|x|^2\right)^{|\beta|} |\partial_1 u_\epsilon(x)|^2 = O(\epsilon^m), \epsilon \to 0.$$

If $x_1 < 0$, one considers v_ϵ such that $v_\epsilon(x) = u_\epsilon(-x_1, x')$ as above, then we obtain

$$\sup_{x_1>0}\left|x^\beta \partial_1 v_\epsilon(x)\right|^2 = \sup_{x_1<0}\left|x^\beta \partial_1 u_\epsilon(x)\right|^2 = O(\epsilon^m), \epsilon \to 0.$$

Therefore, by induction, we have

$$\forall m \in \mathbb{Z}_+, \forall \alpha \in \mathbb{Z}_+^n, \forall \beta \in \mathbb{Z}_+^n, \sup_{x\in\Omega}\left|x^\beta \partial^\alpha u_\epsilon(x)\right| = O(\epsilon^m), \epsilon \to 0.$$

Thus $\mathcal{N}_{\mathcal{S}_*}(\Omega) \cap \mathcal{N}_{\mathcal{S}^*}(\Omega) \subset \mathcal{N}_{\mathcal{S}}(\Omega)$ and consequently $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega) = \mathcal{G}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0}(\Omega) \cap \mathcal{G}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}(\Omega)$. □

Propositions 3.2 and 4.2 give the following result characterizing the negligible elements of the algebra $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$.

Corollary 6.3. *Let Ω be a box. Then an element $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}(\Omega)$ is in $\mathcal{N}_{\mathcal{S}}(\Omega)$ if and only if the following condition is satisfied,*

$$\forall m \in \mathbb{Z}_+, \sup_{x\in\Omega}|u_\epsilon(x)| = O(\epsilon^m), \epsilon \to 0. \tag{10}$$

Theorem 6.2 gives the following corollaries characterizing the algebra of rapidly decreasing generalized functions $\mathcal{G}_{\mathcal{S}}$ and the algebra of regular rapidly decreasing generalized functions $\mathcal{G}_{\mathcal{S}}^\infty$.

Corollary 6.4. (i) *When $\widetilde{\mathcal{R}} = \mathbb{R}_+^{\mathbb{Z}_+^2}$ we obtain $\mathcal{G}_{\mathcal{S}}^{\mathbb{R}_+^{\mathbb{Z}_+^2}} = \mathcal{G}_{\mathcal{S}}$ and we have*

$$\mathcal{G}_{\mathcal{S}} = \mathcal{G}_{\mathcal{S}^*} \cap \mathcal{G}_{\mathcal{S}_*}, \tag{11}$$

where

$$\mathcal{G}_{\mathcal{S}^*} := \frac{\left\{(u_\epsilon)_\epsilon \in \mathcal{S}^{*I} : \forall \alpha \in \mathbb{Z}_+^n, \exists m \in \mathbb{Z}_+, \sup_{x\in\mathbb{R}^n}|\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-m}), \epsilon \to 0\right\}}{\left\{(u_\epsilon)_\epsilon \in \mathcal{S}^{*I} : \forall \alpha \in \mathbb{Z}_+^n, \forall m \in \mathbb{Z}_+, \sup_{x\in\mathbb{R}^n}|\partial^\alpha u_\epsilon(x)| = O(\epsilon^{m}), \epsilon \to 0\right\}}$$

and

$$\mathcal{G}_{\mathcal{S}_*} := \frac{\left\{(u_\epsilon)_\epsilon \in \mathcal{S}_*^{I} : \forall \beta \in \mathbb{Z}_+^n, \exists m \in \mathbb{Z}_+, \sup_{x\in\mathbb{R}^n}\left|x^\beta u_\epsilon(x)\right| = O(\epsilon^{-m}), \epsilon \to 0\right\}}{\left\{(u_\epsilon)_\epsilon \in \mathcal{S}_*^{I} : \forall \beta \in \mathbb{Z}_+^n, \forall m \in \mathbb{Z}_+, \sup_{x\in\mathbb{R}^n}|x^\beta u_\epsilon(x)| = O(\epsilon^{m}), \epsilon \to 0\right\}}.$$

(ii) *When* $\widetilde{\mathcal{R}} = \widetilde{\mathcal{B}}$ *we obtain* $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{B}}} = \mathcal{G}_{\mathcal{S}}^{\infty}$ *and we have*

$$\mathcal{G}_{\mathcal{S}}^{\infty} = \mathcal{G}_{\mathcal{S}^*}^{\infty} \cap \mathcal{G}_{\mathcal{S}_*}^{\infty}, \tag{12}$$

where

$$\mathcal{G}_{\mathcal{S}^*}^{\infty} := \frac{\left\{ (u_\epsilon)_\epsilon \in \mathcal{S}^{*I} : \exists m \in \mathbb{Z}_+, \forall \alpha \in \mathbb{Z}_+^n, \sup\limits_{x \in \mathbb{R}^n} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{-m}), \epsilon \to 0 \right\}}{\left\{ (u_\epsilon)_\epsilon \in \mathcal{S}^{*I} : \forall m \in \mathbb{Z}_+, \forall \alpha \in \mathbb{Z}_+^n, \sup\limits_{x \in \mathbb{R}^n} |\partial^\alpha u_\epsilon(x)| = O(\epsilon^{m}), \epsilon \to 0 \right\}}$$

and

$$\mathcal{G}_{\mathcal{S}_*}^{\infty} := \frac{\left\{ (u_\epsilon)_\epsilon \in \mathcal{S}_*^{I} : \exists m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}_+^n, \sup\limits_{x \in \mathbb{R}^n} \left|x^\beta u_\epsilon(x)\right| = O(\epsilon^{-m}), \epsilon \to 0 \right\}}{\left\{ (u_\epsilon)_\epsilon \in \mathcal{S}_*^{I} : \forall m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}_+^n, \sup\limits_{x \in \mathbb{R}^n} |x^\beta u_\epsilon(x)| = O(\epsilon^{m}), \epsilon \to 0 \right\}} .$$

7. Characterization of $\widetilde{\mathcal{R}}$-rapidly decreasing generalized functions via the Fourier transform

The Fourier transform of $u \in \mathcal{S}$, denoted by $\widehat{u}$ or $\mathcal{F}(u)$, is defined by

$$\widehat{u}(\xi) = (2\pi)^{-\frac{n}{2}} \int e^{-ix\xi} u(x)\, dx .$$

Definition. The Fourier transform of $u = [(u_\epsilon)_\epsilon] \in \mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}$, denoted by $\mathcal{F}_{\mathcal{S}}(u)$, is defined by

$$\mathcal{F}_{\mathcal{S}}(u) = \widehat{u} = [(\widehat{u_\epsilon})_\epsilon] \text{ in } \mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}} .$$

Remark 7.1. The inverse Fourier transform of $u \in \mathcal{S}$, denoted $\widetilde{u}$ or $\mathcal{F}_{\mathcal{S}}^{-1}(u)$, is defined as usually.

The following proposition gives the main results of the Fourier transform $\mathcal{F}_{\mathcal{S}}$, its proof is standard.

Proposition 7.2. *The map*

$$\mathcal{F}_{\mathcal{S}} : \mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}} \to \mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}$$

is an algebraic isomorphism.

Let

$$\widehat{\mathcal{S}^*} = \left\{ f \in \mathcal{C}^\infty : \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \widehat{f}(\xi) \right| < \infty \right\},$$

and let $\widetilde{\mathcal{R}}$ be a regular subset of $\mathbb{R}_+^{\mathbb{Z}_+^2}$, if we define

$$\mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0} = \left\{ (u_\epsilon)_\epsilon \in \widehat{\mathcal{S}^*}^I : \exists N \in \widetilde{\mathcal{R}}^0, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \widehat{u_\epsilon}(\xi) \right| = O\left(\epsilon^{-N_{|\beta|}}\right), \epsilon \to 0 \right\},$$

$$\mathcal{N}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0} = \left\{ (u_\epsilon)_\epsilon \in \widehat{\mathcal{S}^*}^I : \forall N \in \widetilde{\mathcal{R}}^0, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \widehat{u_\epsilon}(\xi) \right| = O\left(\epsilon^{N_{|\beta|}}\right), \epsilon \to 0 \right\},$$

then the following proposition is easy to prove.

Proposition 7.3. (i) *The space* $\mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$ *is a subalgebra of* $\widehat{\mathcal{S}^*}^I$.

(ii) *The space* $\mathcal{N}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$ *is an ideal of* $\mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$.

(iii) *The ideal* $\mathcal{N}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0} = \mathcal{N}_{\widehat{\mathcal{S}^*}}$ *, where*

$$\mathcal{N}_{\widehat{\mathcal{S}^*}} := \left\{ (u_\epsilon)_\epsilon \in \widehat{\mathcal{S}^*}^I : \forall m \in \mathbb{Z}, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \widehat{u_\epsilon}(\xi) \right| = O\left(\epsilon^m\right), \epsilon \to 0 \right\}.$$

The following proposition characterizes $\mathcal{N}_{\widehat{\mathcal{S}^*}}$.

Proposition 7.4. *Let* $(u_\epsilon)_\epsilon \in \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$. *Then* $(u_\epsilon)_\epsilon \in \mathcal{N}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$ *if and only if the following condition is satisfied,*

$$\forall m \in \mathbb{Z}_+, \sup_{\xi \in \mathbb{R}^n} \left| \widehat{u_\epsilon}(\xi) \right| = O\left(\epsilon^m\right), \ \epsilon \to 0. \tag{13}$$

Proof. The proof is similar to that of Proposition 4.2. □

Definition. The algebra $\mathcal{G}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$ is defined as the quotient algebra

$$\mathcal{G}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0} = \frac{\mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}}{\mathcal{N}_{\widehat{\mathcal{S}^*}}} .$$

The next theorem is the second characterization of $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}$.

Theorem 7.5. *We have*

$$\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}} = \mathcal{G}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0} \cap \mathcal{G}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0} . \tag{14}$$

Proof. Let $(u_\epsilon)_\epsilon \in \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$. It follows that

$$\begin{aligned} \int \left| x^\beta \widehat{u_\epsilon}(x) \right| dx &\leq C \sup_{x \in \mathbb{R}^n} \left(1 + |x|^2\right)^n \left| x^\beta \widehat{u_\epsilon}(x) \right|, \\ &= O\left(\epsilon^{-N'_{|\beta|+2n}}\right), \ \epsilon \to 0, \\ &= O\left(\epsilon^{-N_{|\beta|}}\right), \ \epsilon \to 0, \end{aligned}$$

for some $N \in \widetilde{\mathcal{R}}^0$. The continuity of the Fourier transformation $\mathcal{F}$ from the Lebesgue space of integrable functions $\mathbb{L}^1$ to the Lebesgue space of essentially bounded functions $\mathbb{L}^\infty$ gives

$$\left|\left| \partial^\beta u_\epsilon \right|\right|_{\mathbb{L}^\infty} = O\left(\epsilon^{-N_{|\beta|}}\right), \ \epsilon \to 0,$$

which shows that $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}$ and therefore $\mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0} \subset \mathcal{E}_{\mathcal{S}^*}^{\widetilde{\mathcal{R}}^0}$. Consequently $\mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0} \cap \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0} \subset \mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}$. In order to show the inverse inclusion let us mention in first, that from [4], we have

$$(u_\epsilon)_\epsilon \in \mathcal{S}^I \Longleftrightarrow (u_\epsilon)_\epsilon \in \mathcal{S}_*^I \cap \widehat{\mathcal{S}^*}^I,$$

which implies in particular that $\mathcal{S} \subset \widehat{\mathcal{S}^*}$. On the other hand if $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}$, then

$$\begin{aligned} \int \left|\partial^\beta u_\epsilon(x)\right| dx &\leq C \sup_{x\in\mathbb{R}^n} \left(1+|x|^2\right)^n \left|\partial^\beta u_\epsilon(x)\right|, \\ &= O\left(\epsilon^{-N'_{|\beta|,2n}}\right),\ \epsilon \to 0, \\ &= O\left(\epsilon^{-N_{|\beta|,0}}\right),\ \epsilon \to 0, \end{aligned}$$

for some $N \in \widetilde{\mathcal{R}}$, i.e.,

$$\int \left|\partial^\beta u_\epsilon(x)\right| dx = O\left(\epsilon^{-N_{|\beta|}}\right),\ \epsilon \to 0,$$

for some $N \in \widetilde{\mathcal{R}}^0$. The continuity $\mathcal{F}$ from $\mathbb{L}^1$ to $\mathbb{L}^\infty$ gives

$$\left|\left|\xi^\beta \widehat{u_\epsilon}\right|\right|_{\mathbb{L}^\infty} = O\left(\epsilon^{-N_{|\beta|}}\right),\ \epsilon \to 0,$$

which shows that $(u_\epsilon)_\epsilon \in \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$ and consequently $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0} \cap \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$. Thus $\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}} \subset \mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0} \cap \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$, so we have $\mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}} = \mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0} \cap \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$. A similar proof shows that $\mathcal{N}_{\mathcal{S}} = \mathcal{N}_{\mathcal{S}_*} \cap \mathcal{N}_{\widehat{\mathcal{S}^*}}$. Therefore $\mathcal{G}_{\mathcal{S}}^{\widetilde{\mathcal{R}}} = \mathcal{G}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0} \cap \mathcal{G}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$. □

The following corollary gives a second characterization of the space $\mathcal{N}_{\mathcal{S}}$.

Corollary 7.6. *An element $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}$ is in $\mathcal{N}_{\mathcal{S}}$ if and only if the following condition is satisfied,*

$$\forall m \in \mathbb{Z}_+, \sup_{\xi\in\mathbb{R}^n} \left|\widehat{u_\epsilon}(\xi)\right| = O\left(\epsilon^m\right), \epsilon \to 0. \tag{15}$$

Proof. If $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}}^{\widetilde{\mathcal{R}}}$, then $(u_\epsilon)_\epsilon \in \mathcal{E}_{\mathcal{S}_*}^{\widetilde{\mathcal{R}}_0} \cap \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$ by Theorem 7.5. From $(u_\epsilon)_\epsilon \in \mathcal{E}_{\widehat{\mathcal{S}^*}}^{\widetilde{\mathcal{R}}^0}$ and (15), we have by Proposition 7.4 that $(u_\epsilon)_\epsilon \in \mathcal{N}_{\widehat{\mathcal{S}^*}}$. In order to show that $(u_\epsilon)_\epsilon \in \mathcal{N}_{\mathcal{S}_*}$, we have

$$\begin{aligned} \int \left|\widehat{u_\epsilon}(x)\right| dx &\leq C \sup_{x\in\mathbb{R}^n} \left(1+|x|^2\right)^n \left|\widehat{u_\epsilon}(x)\right|, \\ &= O\left(\epsilon^m\right), \epsilon \to 0, \end{aligned}$$

for all $m \in \mathbb{Z}_+$. The continuity of $\mathcal{F}$ from $\mathbb{L}^1$ to $\mathbb{L}^\infty$ gives

$$||u_\epsilon||_{\mathbb{L}^\infty} = O\left(\epsilon^m\right), \epsilon \to 0,$$

this implies, by Proposition 4.2, that $(u_\epsilon)_\epsilon \in \mathcal{N}_{\mathcal{S}_*}$. Consequently $(u_\epsilon)_\epsilon \in \mathcal{N}_{\mathcal{S}_*} \cap \mathcal{N}_{\widehat{\mathcal{S}^*}} = \mathcal{N}_{\mathcal{S}}$ □

We have also an other characterization of the algebra $\mathcal{G}_{\mathcal{S}}$.

Corollary 7.7. *We have*

$$\mathcal{G}_{\mathcal{S}} = \mathcal{G}_{\mathcal{S}_*} \cap \mathcal{G}_{\widehat{\mathcal{S}^*}}, \tag{16}$$

where

$$\mathcal{G}_{\widehat{\mathcal{S}^*}} := \frac{\left\{ (u_\epsilon)_\epsilon \in \widehat{\mathcal{S}^*}^I : \forall \beta \in \mathbb{Z}_+^n, \exists m \in \mathbb{Z}_+, \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \widehat{u_\epsilon}(\xi) \right| = O\left(\epsilon^{-m}\right), \epsilon \to 0 \right\}}{\left\{ (u_\epsilon)_\epsilon \in \widehat{\mathcal{S}^*}^I : \forall \beta \in \mathbb{Z}_+^n, \forall m \in \mathbb{Z}_+, \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \widehat{u_\epsilon}(\xi) \right| = O\left(\epsilon^{m}\right), \epsilon \to 0 \right\}}.$$

The following result is obtained as a corollary of Theorem 7.5.

Corollary 7.8. *We have*

$$\mathcal{G}_{\mathcal{S}}^\infty = \mathcal{G}_{\mathcal{S}_*}^\infty \cap \mathcal{G}_{\widehat{\mathcal{S}^*}}^\infty, \tag{17}$$

where

$$\mathcal{G}_{\widehat{\mathcal{S}^*}}^\infty := \frac{\left\{ (u_\epsilon)_\epsilon \in \widehat{\mathcal{S}^*}^I : \exists m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \widehat{u_\epsilon}(\xi) \right| = O\left(\epsilon^{-m}\right), \epsilon \to 0 \right\}}{\left\{ (u_\epsilon)_\epsilon \in \widehat{\mathcal{S}^*}^I : \forall m \in \mathbb{Z}_+, \forall \beta \in \mathbb{Z}_+^n, \sup_{\xi \in \mathbb{R}^n} \left| \xi^\beta \widehat{u_\epsilon}(\xi) \right| = O\left(\epsilon^{m}\right), \epsilon \to 0 \right\}}.$$

References

[1] J. Alvarez and H. Obiedat, *Characterizations of the Schwartz space S and the Beurling-Bjorck space S_w*, Cubo **6** (2004), no. 4, 167–183.

[2] K. Benmeriem and C. Bouzar, *Ultraregular generalized functions of Colombeau type*, J. Math. Sci. Univ. Tokyo **15** (2008), no. 4, 427–447.

[3] H. A. Biagioni and M. Oberguggenberger, *Generalized solutions to the Korteweg-de Vries and the regularized long-wave equations*, SIAM J. Math. Anal. **23** (1992), no. 4, 923–940.

[4] J. Chung, S. Y. Chung, and D. Kim, *Une caractérisation de l'espace $\mathcal{S}$ de Schwartz*, C. R. Acad. Sci. Paris Sér. I Math. **316** (1993), no. 1, 23–25.

[5] J. F. Colombeau, *New Generalized Functions and Multiplication of Distributions*, North-Holland Publishing Co., Amsterdam, 1984.

[6] A. Delcroix, *Regular rapidly decreasing nonlinear generalized functions. Application to microlocal regularity*, J. Math. Anal. Appl. **327** (2007), no. 1, 564–584.

[7] C. Garetto, *Pseudo-differential operators in algebras of generalized functions and global hypoellipticity*, Acta Appl. Math. **80** (2004), no. 2, 123–174.

[8] I. M. Gel'fand and G. E. Shilov, *Generalized Functions, Vol. 2*, Academic Press, 1967.

[9] K. Gröchenig and G. Zimmermann, *Spaces of test functions via the STFT*, J. Funct. Spaces Appl. **2** (2004), no. 1, 25–53.

[10] M. Grosser, M. Kunzinger, M. Oberguggenberger, and R. Steinbauer, *Geometric Theory of Generalized Functions with Applications to General Relativity*, Kluwer Academic Publishers, Dordrecht, 2001.

[11] L. Hörmander, *Distributions Theory And Fourier Analysis*, Springer, 1983.

[12] A. I. Kashpirovski, *Equality of the spaces S_β^α and $S^\alpha \cap S_\beta$*, Funct. Anal. Appl. **14** (1980), p. 129.

[13] M. Oberguggenberger, *Regularity theory in Colombeau algebras*, Bull. Cl. Sci. Math. Nat. Sci. Math. No. **31** (2006), 147–162.

[14] N. Ortner and P. Wagner, *Applications of weighted $\mathcal{D}'_{L^p}$-spaces to the convolution of distributions*, Bull. Polish Acad. Sci. Math. **37** (1989), no. 7-12, 579–595.
[15] Ya. V. Radyno, N. F. Tkhan, and S. Ramdan, *The Fourier transformation in an algebra of new generalized functions*, Dokl. Akad. Nauk **327** (1992), no. 1, 20–24; translation in Russian Acad. Sci. Dokl. Math. **46** (1993), no. 3, 414–417.
[16] L. Schwartz, *Théorie des distributions*, Hermann, Paris, 1966.
[17] L. Volevich and S. Gindikin, *The Cauchy problem and related problems for convolution equations*, Uspehi Mat. Nauk **27** (1972), no. 4(166), 65–143.

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