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GENERALIZATIONS OF TWO SUMMATION FORMULAS FOR THE GENERALIZED HYPERGEOMETRIC FUNCTION OF HIGHER ORDER DUE TO EXTON

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Abstract. In 1997, Exton, by mainly employing a widely-used process of resolving hypergeometric series into odd and even parts, obtained some new and interesting summation formulas with arguments 1 and -1. We aim at showing how easily many summation formulas can be obtained by simply combining some known summation formulas. Indeed, we present 22 results in the form of two generalized summation formulas for the generalized hypergeometric series ${}_4F_3$, including two Exton's summation formulas for ${}_4F_3$ as special cases.

1. Introduction and preliminaries

It is well known that the generalized hypergeometric function

$$(1.1)\qquad {}_pF_q\left[\begin{matrix}a_1,\ldots,\,a_p\\ b_1,\ldots,\,b_q\end{matrix};\,x\right] = {}_pF_q\left[\begin{matrix}(a_p)\\ (b_q)\end{matrix};\,x\right] = \sum_{n=0}^{\infty}\frac{(a_1)_n\cdots(a_p)_n}{(b_1)_n\cdots(b_q)_n}\,\frac{x^n}{n!},$$

where $(a)_n = \Gamma(a+n)/\Gamma(a)$ is the Pochhammer symbol, occurs in many theoretical and practical applications such as mathematics, theoretical physics, engineering, and statistics. For detailed discussion of this function, including the convergence of its series representation, see, for example, Exton [1], Slater [5], Rainville [4], or Srivastava and Choi [6]. Among other things in the theory and application of ${}_pF_q$, the summation formulas for ${}_pF_q$ such as (1.4) and (1.5) have played vital roles (see, e.g., Shen [7], Choi and Srivastava [1], Chu and de Donno [2]).

By considering the following two combinations

$${}_{q+1}F_q\left[\begin{matrix}(a_{q+1})\\ (b_q)\end{matrix};\,1\right] \pm\ {}_{q+1}F_q\left[\begin{matrix}(a_{q+1})\\ (b_q)\end{matrix};\,1\right],$$

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Exton [2] obtained some interesting summation formulas. Among those things, we recall here the following two formulas written in slightly modified form:

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}b, & \frac{1}{2}b+\frac{1}{2}\\ \frac{1}{2}+\frac{1}{2}a-\frac{1}{2}b, & 1+\frac{1}{2}a-\frac{1}{2}b, & \frac{1}{2} & \end{matrix};1\right]\\
&= \frac{2^{-a-1}\,\Gamma(1+a-b)}{\Gamma\left(1+\frac{1}{2}a-b\right)}\left[\frac{\Gamma\left(\frac{1}{2}-b\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b\right)}+\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)}\right]
\end{aligned}
\tag{1.2}
$$

and

$$
\begin{aligned}
&{}_4F_3\left[\begin{matrix}\frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}a+1, & \frac{1}{2}b+\frac{1}{2}, & \frac{1}{2}b+1\\ 1+\frac{1}{2}a-\frac{1}{2}b, & \frac{3}{2}+\frac{1}{2}a-\frac{1}{2}b, & \frac{3}{2} & \end{matrix};1\right]\\
&= \frac{2^{-a-1}\,\Gamma(2+a-b)}{ab\,\Gamma\left(1+\frac{1}{2}a-b\right)}\left[\frac{\Gamma\left(\frac{1}{2}-b\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b\right)}-\frac{\Gamma\left(\frac{1}{2}\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a\right)}\right].
\end{aligned}
\tag{1.3}
$$

Exton [2] established these results with the help of the following two summation theorems:

Gauss's theorem [4]

$$
{}_2F_1\left[\begin{matrix}a, & b\\ & c\end{matrix};1\right]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\;\Gamma(c-b)}\quad(\Re(c-a-b)>0).
\tag{1.4}
$$

Kummer's theorem [4]

$$
{}_2F_1\left[\begin{matrix}a, & b\\ 1+a-b & \end{matrix};-1\right]=\frac{\Gamma(1+a-b)\,\Gamma\left(1+\frac{1}{2}a\right)}{\Gamma(1+a)\;\Gamma\left(1+\frac{1}{2}a-b\right)}\quad(\Re(b)<1).
\tag{1.5}
$$

In 1996, Lavoie *et al.* [3] obtained a generalization of (1.5) in the form:

$$
\begin{aligned}
&{}_2F_1\left[\begin{matrix}a, & b\\ 1+a-b+i & \end{matrix};-1\right]\\
&= \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma(1-b)\,\Gamma(1+a-b+i)}{2^a\,\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)}\\
&\quad\times\left\{\frac{A_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)}\right.\\
&\qquad\left.+\frac{B_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\}\\
&\qquad(i=0,\,\pm1,\,\pm2,\,\pm3,\,\pm4,\,\pm5),
\end{aligned}
\tag{1.6}
$$

where $[x]$ denotes (as usual) the greatest integer less than or equal to x and the table of A_i and B_i is given at the end of this paper.

The main purpose of this paper is to show how easily many summation formulas for ${}_4F_3$ can be deduced by simply combining known summation formulas. In fact, we derive 22 (11 each) summation formulas for ${}_4F_3$ closely related to the Exton's results (1.2) and (1.3) with the help of the results (1.4) and (1.6). For this we shall use the following two results:

(1.7)
$$ {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ;\ 1\right] + {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ;\ -1\right] = 2\,{}_4F_3\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}b, & \frac{1}{2}b+\frac{1}{2} \\ \frac{1}{2}c, & \frac{1}{2}c+\frac{1}{2}, & \frac{1}{2} & \end{matrix} ;\ 1\right] $$

and

(1.8)
$$ \begin{aligned} &{}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ;\ 1\right] - {}_2F_1\left[\begin{matrix} a, & b \\ & c \end{matrix} ;\ -1\right] \\ &= \frac{2ab}{c}\,{}_4F_3\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}a+1, & \frac{1}{2}b+\frac{1}{2}, & \frac{1}{2}b+1 \\ \frac{1}{2}c+\frac{1}{2}, & \frac{1}{2}c+1, & \frac{3}{2} & \end{matrix} ;\ 1\right] \end{aligned} $$

which are the special cases of Equation (1.4) in [2] and the *corrected* form of Equation (1.5) in [2], respectively. Note that the $_{q+1}F_q$ appearing on the right-hand side of Equation (1.5) in the work of Exton [2] should be corrected as $_{2q+2}F_{2q+1}$.

2. Main summation formulas

The 22 (each 11) summation formulas for the generalized hypergeometric series $_4F_3$ presented in the form of the following two formulas will be established.

(2.1)
$$ \begin{aligned} &{}_4F_3\left[\begin{matrix} \frac{1}{2}a, & \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}b, & \frac{1}{2}b+\frac{1}{2} \\ \frac{1}{2}+\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}i, & 1+\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}i, & \frac{1}{2} & \end{matrix} ;\ 1\right] \\ &= \frac{\Gamma(1+a-b)\,\Gamma\left(\frac{1}{2}-b\right)}{2^{a+1}\,\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b\right)\,\Gamma\left(1+\frac{1}{2}a-b\right)} + \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma(1-b)\,\Gamma(1+a-b+i)}{2^{a+1}\,\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\quad \left\{\frac{A_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)}\right. \\ &\quad \left. + \frac{B_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\right\} (i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5) \end{aligned} $$

and

(2.2)
$$ \begin{aligned} &{}_4F_3\left[\begin{matrix} \frac{1}{2}a+\frac{1}{2}, & \frac{1}{2}a+1, & \frac{1}{2}b+\frac{1}{2}, & \frac{1}{2}b+1 \\ 1+\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}i, & \frac{3}{2}+\frac{1}{2}a-\frac{1}{2}b+\frac{1}{2}i, & \frac{3}{2} & \end{matrix} ;\ 1\right] \\ &= \frac{1+a-b+i}{a\,b\,2^{a+1}}\left[\frac{\Gamma(1+a-b)\,\Gamma\left(\frac{1}{2}-b\right)}{\Gamma\left(\frac{1}{2}+\frac{1}{2}a-b\right)\,\Gamma\left(1+\frac{1}{2}a-b\right)}\right. \\ &\quad - \frac{\Gamma\left(\frac{1}{2}\right)\,\Gamma(1-b)\,\Gamma(1+a-b+i)}{\Gamma\left(1-b+\frac{1}{2}i+\frac{1}{2}|i|\right)} \\ &\quad \left\{\frac{A_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+1\right)\,\Gamma\left(\frac{1}{2}a+\frac{1}{2}+\frac{1}{2}i-\left[\frac{1+i}{2}\right]\right)}\right. \end{aligned} $$

$$+\frac{B_i}{\Gamma\left(\frac{1}{2}a-b+\frac{1}{2}i+\frac{1}{2}\right)\Gamma\left(\frac{1}{2}a+\frac{1}{2}i-\left[\frac{i}{2}\right]\right)}\Bigg\}\Bigg]$$
$$(i=0,\ \pm1,\ \pm2,\ \pm3,\ \pm4,\ \pm5),$$

where A_i and B_i are given in the table.

Proof. The derivations of (2.1) and (2.2) are straightforward. If we take $c = 1+a-b+i$ in (1.7) and (1.8) for $i = 0, \pm1, \pm2, \pm3, \pm4, \pm5$ and use the results (1.4) and (1.6), then we get (2.1) and (2.2), respectively. □

Note that it is easily seen that the special cases of (2.1) and (2.2) when $i = 0$ reduce to the Exton's results (1.2) and (1.3), respectively.

TABLE

i	A_i	B_i
5	$-4(6+a-b)^2+2b(6+a-b)+b^2$ $+22(6+a-b)-13b-20$	$4(6+a-b)^2+2b(6+a-b)-b^2$ $-34(6+a-b)-b+62$
4	$2(a-b+3)(1+a-b)-(b-1)(b-4)$	$-4(a-b+2)$
3	$3b-2a-5$	$2a-b+1$
2	$1+a-b$	-2
1	-1	1
0	1	0
-1	1	1
-2	$a-b-1$	2
-3	$2a-3b-4$	$2a-b-2$
-4	$2(a-b-3)(a-b-1)-b(b+3)$	$4(a-b-2)$
-5	$4(a-b-4)^2-2b(a-b-4)-b^2$ $+8(a-b-4)-7b$	$4(a-b-4)^2+2b(a-b-4)-b^2$ $16(a-b-4)-b+12$

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