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THE GEOMETRIC CONVEXITY OF A FUNCTION INVOLVING GAMMA FUNCTION WITH APPLICATIONS

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Abstract. In this paper, we prove that $(\Gamma(x))^{\frac{1}{x-1}}$ is geometrically convex on $(0,\infty)$. As its applications, we obtain some new estimates for $\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{[\Gamma(y+1)]^{\frac{1}{y}}}$.

1. Introduction

For real and positive values of x the Euler gamma function Γ and its logarithmic derivative ψ, the so-called digamma function, are defined by

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt \quad \text{and} \quad \psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}. \tag{1.1}$$

For extension of these functions to complex variable and for basic properties see [21].

Over the past half century many authors have obtained inequalities for these important functions (see [1-3, 5, 6, 9, 11, 16-18, 20] and bibliographies in those papers). In keeping with tradition, we research the geometric convexity of the gamma function, as its applications, we give some new estimates for $\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{[\Gamma(y+1)]^{\frac{1}{y}}}$.

The main purpose of this paper is to prove the following Theorem 1.

Theorem 1. $(\Gamma(x))^{\frac{1}{x-1}}$ *is geometrically convex on* $(1,\infty)$.

As applications of Theorem 1, we shall establish the following new inequalities for gamma function which improve the known results.

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Theorem 2. *If $x > y > 0$, then*

$$\begin{aligned}\left(\frac{x+1}{y+1}\right)^{\frac{(y+1)(1-\log y+y\psi(y)-\log\Gamma(y))}{y^2}} &\leq \frac{[\Gamma(x+1)]^{\frac{1}{x}}}{[\Gamma(y+1)]^{\frac{1}{y}}} \\ &\leq \left(\frac{x+1}{y+1}\right)^{\frac{(x+1)(1-\log x+x\psi(x)-\log\Gamma(x))}{x^2}}.\end{aligned} \tag{1.2}$$

Theorem 3. *Let $a = \frac{1}{2}\log(2\pi) - \frac{1}{2} = 0.4189385\cdots$. If $x > y > 0$, then*

$$\begin{aligned}\left(\frac{x+1}{y+1}\right)^{\frac{(y+1)(y-\frac{1}{2}\log y-\frac{1}{6y}-a)}{y^2}} &< \frac{[\Gamma(x+1)]^{\frac{1}{x}}}{[\Gamma(y+1)]^{\frac{1}{y}}} \\ &< \left(\frac{x+1}{y+1}\right)^{\frac{(x+1)(x-\frac{1}{2}\log x-\frac{1}{6x}+\frac{1}{90x^3}-a)}{x^2}}.\end{aligned} \tag{1.3}$$

Theorem 4. *Let $b = 2\log(2\pi) - \frac{10}{3} = 0.342420\cdots$ and $c = 15\log(2\pi) - \frac{76}{3} = 2.234822\cdots$. If $x > y \geq 1$, then*

$$\begin{aligned}\left(\frac{x+1}{y+1}\right)^{1-2\pi^2 e^{-\frac{13}{3}}} &\leq \left(\frac{x+1}{y+1}\right)^{1-\frac{\log y+b}{2y}} \\ &< \frac{[\Gamma(x+1)]^{\frac{1}{x}}}{[\Gamma(y+1)]^{\frac{1}{y}}} < \left(\frac{x+1}{y+1}\right)^{1-\frac{2\log x+c}{15x}}.\end{aligned} \tag{1.4}$$

Theorem 5. *Let $b=2\log(2\pi)-\frac{10}{3}=0.342420\cdots$, $c=15\log(2\pi)-\frac{76}{3}=2.234822\cdots$, $d=3-\log(2\pi)=1.162122\cdots$, and $g(x)=\max\{\frac{\log x-d}{2x}, \frac{2\log(x+1)+c}{15(x+1)}\}$. If $x \geq 1$, then*

$$\begin{aligned}\left(\frac{x+2}{x+1}\right)^{1-2\pi^2 e^{-\frac{13}{3}}} &\leq \left(\frac{x+2}{x+1}\right)^{1-\frac{\log x+b}{2x}} \\ &< \frac{[\Gamma(x+2)]^{\frac{1}{x+1}}}{[\Gamma(x+1)]^{\frac{1}{x}}} < \left(\frac{x+2}{x+1}\right)^{1-g(x)}.\end{aligned} \tag{1.5}$$

In particular, if $n \geq 1$, $n \in \mathbb{N}$, then

$$\begin{aligned}\left(\frac{n+2}{n+1}\right)^{\frac{22-3\log 2-6\log(2\pi)}{12}} &\leq \left(\frac{n+2}{n+1}\right)^{1-\frac{\log n+b}{2n}} \\ &\leq \frac{[(n+1)!]^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} < \left(\frac{n+2}{n+1}\right)^{1-g(n)}.\end{aligned} \tag{1.6}$$

Theorem 6. *If $x \geq 1$, then*

$$\frac{[\Gamma(x+2)]^{\frac{1}{x+1}}}{[\Gamma(x+1)]^{\frac{1}{x}}} < \left(\frac{x+2}{x+1}\right)^{\frac{4x+3}{4(x+1)}}. \tag{1.7}$$

In particular, if $n \geq 1$, $n \in \mathbb{N}$, then

$$\frac{[(n+1)!]^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} < \left(\frac{n+2}{n+1}\right)^{\frac{4n+3}{4(n+1)}}. \tag{1.8}$$

2. Preliminary knowledge on geometrically convex function

Let $I \subset (0, \infty)$ be an interval, $f : I \to (0, \infty)$ is a continuous real-valued function. f is called geometrically convex (or concave, respectively) on I if one of the following is true:

$$f(\sqrt{x_1 x_2}) \leq (\text{or} \geq, \text{ respectively}) \sqrt{f(x_1) f(x_2)} \tag{2.1}$$

for all $x_1, x_2 \in I$;

$$f(\Pi_{i=1}^{n} x_i^{\lambda_i}) \leq (\text{or} \geq, \text{ respectively}) \prod_{i=1}^{n} f(x_i)^{\lambda_i} \tag{2.2}$$

for all $x_1, x_2, \ldots, x_n \in I$ and $\lambda_1, \lambda_2, \ldots, \lambda_n \geq 0$ with $\sum_{i=1}^{n} \lambda_i = 1$.

The notion of geometric convexity (or concavity, respectively) was first introduced by P. Montel [13]. Later, the geometric convexity (or concavity, respectively) theory was developed by many authors, such as J. Matkowski [10], C. E. Finol and M. Wójtowicz [8], and C. P. Niculescu [14, 15]. The following Theorem A and Theorem B were established by C. P. Niculescu [14].

Theorem A. *Let $I \subset (0, \infty)$ be an interval. If $f : I \to (0, \infty)$ is a differentiable real-valued function, then f is geometrically convex (or concave, respectively) on I if and only if $g(x) = \frac{xf'(x)}{f(x)}$ is increasing (or decreasing, respectively) on I.*

Theorem B. *Let $I \subset (0, \infty)$ be an interval. If $f : I \to (0, \infty)$ is a differentiable real-valued function, then f is geometrically convex (or concave, respectively) on I if and only if $\frac{f(x)}{f(y)} \geq (\text{or} \leq, \text{respectively}) (\frac{x}{y})^{\frac{yf'(y)}{f(y)}}$ for all $x, y \in I$.*

It is easy to see that Theorem B is equivalent to the following Theorem C.

Theorem C. *Let $I \subset (0, \infty)$ be an interval. If $f : I \to (0, \infty)$ is a differentiable real-valued function, then f is geometrically convex (or concave, respectively) on I if and only if $(\frac{x}{y})^{\frac{yf'(y)}{f(y)}} \leq (\text{or} \geq, \text{ respectively}) \frac{f(x)}{f(y)} \leq (\text{or} \geq, \text{respectively}) (\frac{x}{y})^{\frac{xf'(x)}{f(x)}}$ for all $x, y \in I$.*

3. Lemmas

In order to prove the main results of this paper, we need to establish and introduce some lemmas in this section.

Lemma 1. *If $x \geq 1$, then*

$$\frac{3}{2}x^4 - \frac{11}{6}x^3 + \frac{2}{3}x^2 - \frac{x}{30} - \frac{2}{15} > 0. \tag{3.1}$$

Proof. Let $f(x) = \frac{3}{2}x^4 - \frac{11}{6}x^3 + \frac{2}{3}x^2 - \frac{x}{30} - \frac{2}{15}$. Then

$$f'(x) = 6x^3 - \frac{11}{2}x^2 + \frac{4}{3}x - \frac{1}{30}, \tag{3.2}$$
$$f''(x) = 18x^2 - 11x + \frac{4}{3},\ x \geq 1.$$

Equation (3.2) implies

$$f'(x) \geq f'(1) = \frac{9}{5} > 0,\ x \geq 1. \tag{3.3}$$

Inequality (3.3) leads to

$$f(x) \geq f(1) = \frac{1}{6} > 0.$$

□

Lemma 2 (see [6])**.** *If $x > 0$, then*

$$\psi'(x) > \frac{1}{x} + \frac{1}{2x^2}. \tag{3.4}$$

Lemma 3 (see [7])**.** *If $x > 0$, then*

$$\log \Gamma(x) = \frac{1}{2}\log(2\pi) + (x - \frac{1}{2})\log x - x + \frac{1}{12x} - \frac{\theta_1}{360x^3}, \tag{3.5}$$

$$\psi(x) = \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{\theta_2}{120x^4}, \tag{3.6}$$

$$\psi'(x) = \frac{1}{x} + \frac{1}{2x^2} + \frac{1}{6x^3} - \frac{1}{30x^5} + \frac{\theta_3}{42x^7}, \tag{3.7}$$

$$\psi''(x) = -\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4} + \frac{\theta_4}{6x^6}, \tag{3.8}$$

where $0 < \theta_1, \theta_2, \theta_3, \theta_4 < 1$.

Lemma 4. *If $x \geq 1$, then*

$$(x+2)\psi'(x) + x(x+1)\psi''(x) > 0. \tag{3.9}$$

Proof. Case 1: $x \geq 2$. From (3.4) and (3.8) we clearly see that

$$\begin{aligned} &(x+2)\psi'(x) + x(x+1)\psi''(x) \\ &> (x+2)(\frac{1}{x} + \frac{1}{2x^2}) + x(x+1)(-\frac{1}{x^2} - \frac{1}{x^3} - \frac{1}{2x^4}) \\ &= \frac{1}{2x}(1 - \frac{1}{x} - \frac{1}{x^2}) > 0. \end{aligned} \tag{3.10}$$

Case 2: $1 \leq x < 2$. Case 1 implies

$$(x+3)\psi'(x+1) + (x+1)(x+2)\psi''(x+1) > 0. \tag{3.11}$$

From the identity $\Gamma(x+1) = x\Gamma(x)$ we clearly see that

$$\psi'(x+1) = -\frac{1}{x^2} + \psi'(x), \quad \psi''(x+1) = \frac{2}{x^3} + \psi''(x). \tag{3.12}$$

Combining (3.11) and (3.12) we have

$$(x+3)(-\frac{1}{x^2}+\psi'(x))+(x+1)(x+2)(\frac{2}{x^3}+\psi''(x))>0,$$

which is equivalent to

$$(x+2)\psi'(x)+x(x+1)\psi''(x)>\frac{x+4}{x+2}\psi'(x)-\frac{x^2+3x+4}{x^2(x+2)}. \tag{3.13}$$

From (3.7) and (3.13) we get

$$\begin{aligned}
&(x+2)\psi'(x)+x(x+1)\psi''(x)\\
&>\frac{x+4}{x+2}\left(\frac{1}{x}+\frac{1}{2x^2}+\frac{1}{6x^3}-\frac{1}{30x^5}\right)-\frac{x^2+3x+4}{x^2(x+2)}\\
&=\frac{1}{x^5(x+2)}\left(\frac{3}{2}x^4-\frac{11}{6}x^3+\frac{2}{3}x^2-\frac{1}{30}x-\frac{2}{15}\right).
\end{aligned} \tag{3.14}$$

Hence inequality (19) follows from inequalities (3.1) and (3.14). □

4. Proof of theorems

Proof of Theorem 1. For $x\in(1,\infty)$, let $f(x)=[\Gamma(x)]^{\frac{1}{x-1}}$, then

$$\frac{xf'(x)}{f(x)}=\frac{x(x-1)\psi(x)-x\log\Gamma(x)}{(x-1)^2} \tag{4.1}$$

and

$$\begin{aligned}
[\frac{xf'(x)}{f(x)}]'&=\frac{x(x-1)^2\psi'(x)+(1-x^2)\psi(x)+(x+1)\log\Gamma(x)}{(x-1)^3}\\
&=\frac{(x+1)g(x)}{(x-1)^3},
\end{aligned} \tag{4.2}$$

where $g(x)=\frac{x(x-1)^2+\psi'(x)}{x+1}+(1-x)\psi(x)+\log\Gamma(x)$. Differentiating $g(x)$ and making use of Lemma 4 we get

$$g'(x)=\frac{(x-1)^2}{(x+1)^2}[(x+2)\psi'(x)+x(x+1)\psi''(x)]>0. \tag{4.3}$$

Inequality (4.3) implies

$$g(x)\geq\lim_{x\to1+0}g(x)=0,\quad x\in(1,+\infty). \tag{4.4}$$

Therefore, Theorem 1 follows from (4.2) and (4.4) together with Theorem A. □

Proof of Theorem 2. Let $f(x)=[\Gamma(x)]^{\frac{1}{x-1}}$, $x\in(1,\infty)$. For any $x>y>0$, Theorem 1 and Theorem C imply

$$\left(\frac{x+1}{y+1}\right)^{\frac{(y+1)f'(y+1)}{f(y+1)}}\leq\frac{[\Gamma(x+1)]^{\frac{1}{x}}}{[\Gamma(y+1)]^{\frac{1}{y}}}\leq\left(\frac{x+1}{y+1}\right)^{\frac{(x+1)f'(x+1)}{f(x+1)}}. \tag{4.5}$$

Therefore, Theorem 2 follows from the identity $\Gamma(t+1) = t\Gamma(t)$ and inequality (4.5). □

Proof of Theorem 3. For $x > y > 0$, equations (3.5) and (3.6) imply

$$\begin{cases} \frac{1}{2}\log(2\pi) + (x-\frac{1}{2})\log x - x + \frac{1}{12x} - \frac{1}{360x^3} < \log\Gamma(x) \\ \qquad < \frac{1}{2}\log(2\pi) + (x-\frac{1}{2})\log x - x + \frac{1}{12x}, \\ \log x - \frac{1}{2x} - \frac{1}{12x^2} < \psi(x) < \log x - \frac{1}{2x} - \frac{1}{12x^2} + \frac{1}{120x^4} \end{cases} \tag{4.6}$$

and

$$\begin{cases} \frac{1}{2}\log(2\pi) + (y-\frac{1}{2})\log y - y + \frac{1}{12y} - \frac{1}{360y^3} < \log\Gamma(y) \\ \qquad < \frac{1}{2}\log(2\pi) + (y-\frac{1}{2})\log y - y + \frac{1}{12y}, \\ \log y - \frac{1}{2y} - \frac{1}{12y^2} < \psi(y) < \log y - \frac{1}{2y} - \frac{1}{12y^2} + \frac{1}{120y^4}. \end{cases} \tag{4.7}$$

Therefore, Theorem 3 follows from inequalities (4.6)-(4.7) and Theorem 2. □

Proof of Theorem 4. Let $a = \frac{1}{2}\log(2\pi) - \frac{1}{2}$ and $f(y) = (\frac{b}{2} - a + 1)y - \frac{1}{2}\log y - \frac{1}{6y} - a - \frac{1}{6}, y \geq 1$. Then $f(1) = 0$ and

$$\begin{aligned} f'(y) &= \frac{1}{6y^2}[(3\log(2\pi)-1)y^2 - 3y + 1] \\ &\geq \frac{1}{6y^2}[(3\log(2\pi)-1)y - 3y + 1] \\ &\geq \frac{1}{2y^2}[\log(2\pi) - 1] > 0. \end{aligned} \tag{4.8}$$

Inequality (4.8) implies

$$f(y) = \left(\frac{b}{2} - a + 1\right)y - \frac{1}{2}\log y - \frac{1}{6y} - a - \frac{1}{6} \geq f(1) = 0,\ y \geq 1,$$

$$1 - \frac{\log y + b}{2y} \leq \frac{(y+1)(y - \frac{1}{2}\log y - \frac{1}{6y} - a)}{y^2}. \tag{4.9}$$

Taking $h(y) = 1 - \frac{\log y + b}{2y}, y \geq 1$, it is easy to see

$$\min_{y\in[1,+\infty]} h(y) = h(e^{1-b}) = 1 - 2\pi^2 e^{-\frac{13}{3}}. \tag{4.10}$$

Next let

$$g(x) = (90a - 6c - 90)x + (33x + 45)\log x + \frac{15}{x} - \frac{1}{x^2} - \frac{1}{x^3} + 90a + 15,\ x \geq 1.$$

Then

$$\begin{cases} g'(x) = (90a - 6c - 90) + 33\log x + \frac{33x+45}{x} - \frac{15}{x^2} + \frac{2}{x^3} + \frac{3}{x^4}, \\ g''(x) = \frac{3}{x^5}(x-1)(11x^3 - 4x^2 + 6x + 4) \geq 0, \\ g'(1) = 85 - 45\log(2\pi) = 2.295532\cdots > 0, \\ g(1) = 0. \end{cases} \tag{4.11}$$

From (4.11) we clearly see that $g(x) \geq 0$ for $x \geq 1$, then we get

$$\frac{(x+1)(x-\frac{1}{2}\log x-\frac{1}{6x}+\frac{1}{90x^3}-a)}{x^2} \leq 1-\frac{2\log x+c}{15x}. \tag{4.12}$$

Therefore, Theorem 4 follows from (4.9)-(4.10) and (4.12) together with Theorem 3. □

Remark 1. For any $n \geq 1, n \in \mathbb{N}$, H. Minc and L. Sathre [12] first established the following inequality:

$$1 \leq \frac{[(n+1)!]^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} \leq \frac{n+1}{n}. \tag{4.13}$$

Later, H. Alzer [4] proved

$$\frac{n+2\sqrt{2}-1}{n+1} \leq \frac{[(n+1)!]^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} \leq \frac{n+2}{n+1}. \tag{4.14}$$

From the identity $\Gamma(n+1) = n!$ we know that inequalities (4.13) and (4.14) can be rewritten as

$$1 \leq \frac{[\Gamma(n+2)]^{\frac{1}{n+1}}}{[(\Gamma(n+1)]^{\frac{1}{n}}} \leq \frac{n+1}{n} \tag{4.15}$$

and

$$\frac{n+2\sqrt{2}-1}{n+1} \leq \frac{[\Gamma(n+2)]^{\frac{1}{n+1}}}{[(\Gamma(n+1)]^{\frac{1}{n}}} \leq \frac{n+2}{n+1}, \tag{4.16}$$

respectively. Recently, F. Qi and C. P. Chen [19] gave the following result:

$$\left(\frac{x+1}{y+1}\right)^{\frac{1}{2}} < \frac{[\Gamma(x+1)]^{\frac{1}{x}}}{[\Gamma(y+1)]^{\frac{1}{y}}} < \frac{x+1}{y+1},\ x > y > 0. \tag{4.17}$$

It is obvious that inequality (1.4) is an improvement of inequality (4.17). In fact, $1-2\pi^2 e^{-\frac{13}{3}} = 0.740947\cdots$.

Proof of Theorem 5. If $x \geq 1$, then Theorem 4 implies

$$\begin{aligned}\left(\frac{x+2}{x+1}\right)^{1-2\pi^2e^{-\frac{13}{3}}} &\leq \left(\frac{x+2}{x+1}\right)^{1-\frac{\log x+b}{2x}} \\ &< \frac{[\Gamma(x+2)]^{\frac{1}{x+1}}}{[\Gamma(x+1)]^{\frac{1}{x}}} < \left(\frac{x+2}{x+1}\right)^{1-\frac{2\log(x+1)+c}{15(x+1)}}.\end{aligned} \tag{4.18}$$

Theorem 3 leads to

$$\frac{[\Gamma(x+2)]^{\frac{1}{x+1}}}{[\Gamma(x+1)]^{\frac{1}{x}}} < \left(\frac{x+2}{x+1}\right)^{\frac{(x+2)(x+1-\frac{1}{2}\log(x+1)-\frac{1}{6(x+1)}+\frac{1}{90(x+1)^3}-a)}{(x+1)^2}}, \tag{4.19}$$

where $a = \frac{1}{2}\log(2\pi) - \frac{1}{2}$. By calculation, it is not difficult to verify

$$1 + \frac{(x+1) - \frac{x+2}{2}\log x - [\frac{1}{2}\log(2\pi) - \frac{1}{2}]x + [1 - \log(2\pi)]}{(x+1)^2} < 1 - \frac{\log x - 3 + \log(2\pi)}{2x}. \tag{4.20}$$

Inequality (4.20) leads to

$$\begin{aligned} &\frac{(x+2)[(x+1) - \frac{1}{2}\log(x+1) - \frac{1}{6(x+1)} + \frac{1}{90(x+1)^3} - a]}{(x+1)^2} \\ &< \frac{(x+2)[(x+1) - \frac{1}{2}\log x - a]}{(x+1)^2} \\ &= 1 + \frac{(x+1) - \frac{x+2}{2}\log x - [\frac{1}{2}\log(2\pi) - \frac{1}{2}]x + [1 - \log(2\pi)]}{(x+1)^2} \\ &< 1 - \frac{\log x - 3 + \log(2\pi)}{2x} = 1 - \frac{\log x - d}{2x}. \end{aligned} \tag{4.21}$$

Therefore, inequality (1.5) follows from inequalities (4.18)-(4.19) and (4.21).

Next, if $n \geq 1, n \in \mathbb{N}$, then inequality (1.5) implies

$$\left(\frac{n+2}{n+1}\right)^{1-\frac{\log n + b}{2n}} < \frac{[(n+1)!]^{\frac{1}{n+1}}}{(n!)^{\frac{1}{n}}} = \frac{[\Gamma(n+2)]^{\frac{1}{n+1}}}{[\Gamma(n+1)]^{\frac{1}{n}}} < \left(\frac{n+2}{n+1}\right)^{1-g(n)}. \tag{4.22}$$

For $n \geq 1, n \in \mathbb{N}$, it is not difficult to verify

$$1 - \frac{\log n + b}{2n} \geq 1 - \frac{\ln 2 + b}{4} = \frac{22 - 3\log 2 - 6\log(2\pi)}{12}. \tag{4.23}$$

Hence inequality (1.6) follows from inequalities (4.22) and (4.23). □

Remark 2. Comparing inequality (1.6) with inequality (4.14) we know that

$$\left(\frac{n+2}{n+1}\right)^{1-g(n)} < \frac{n+2}{n+1} \quad \text{for all} \quad n \geq 1$$

and

$$\left(\frac{n+2}{n+1}\right)^{1-\frac{\log n + b}{2n}} > \frac{n+2\sqrt{2}-1}{n+1} \quad \text{for all} \quad n \geq 9. \tag{4.24}$$

In fact, if taking $g(n) = (\frac{n+2}{n+1})^{1-\frac{\log n + b}{2n}}$ and $h(n) = \frac{n+2\sqrt{2}-1}{n+1}$. Then for $n = 9$, by science computation, we get

$$g(9) = 1.085306\cdots > h(9) = 1.082842\cdots.$$

For $n \geq 10$, let

$$f(x) = 2x - \log x - b - 2(\sqrt{2}-1)\frac{2x^2+3x}{x+1}, \ x \geq 10.$$

Then

$$(4.25)\quad \begin{aligned}(x+1)^2 f'(x) &= x[(6-4\sqrt{2})x + (11-8\sqrt{2})] - \frac{1}{x} - 6(\sqrt{2}-1)\\ &> x[10(6-4\sqrt{2}) + (11-8\sqrt{2})] - \frac{1}{x} - 6(\sqrt{2}-1)\\ &> 10(71-48\sqrt{2}) - \frac{1}{10} - 6(\sqrt{2}-1)\\ &= 28.592208\cdots > 0.\end{aligned}$$

Inequality (4.25) implies

$$(4.26)\quad \begin{aligned}&2n - (\log n + b) - 2(\sqrt{2}-1)\frac{2n^2+3n}{n+1}\\ &= f(n) \geq f(10) = 0.033360\cdots > 0\end{aligned}$$

for $n \geq 10$. Inequality (4.26) leads to

$$(4.27)\quad \left(1 - \frac{\log n + b}{2n}\right)\frac{2}{2n+3} > \frac{2\sqrt{2}-2}{n+1}.$$

If $t > 0$, then it is easy to prove that

$$t > \log(1+t) > \frac{2t}{t+2},$$

hence we have

$$(4.28)\quad \log\left(\frac{n+2}{n+1}\right) > \frac{2}{2n+3}$$

and

$$(4.29)\quad \frac{2(\sqrt{2}-1)}{n+1} > \log\frac{n+2\sqrt{2}-1}{n+1}.$$

Combining inequalities (4.27)-(4.29) we get

$$\left(\frac{n+2}{n+1}\right)^{1-\frac{\log n+b}{2n}} > \frac{n+2\sqrt{2}-1}{n+1}.$$

Proof of Theorem 6. For $x \geq 1$, let

$$f(x) = \frac{28}{45(x+1)} + \frac{5}{x+2} + 2\log(x+1) + (2\log(2\pi) - 7).$$

Then

$$
\begin{aligned}
(x+1)^2 f'(x) =& -\frac{28}{45} - \frac{5(x+1)^2}{(x+2)^2} + 2(x+1)\\
>& -\frac{28}{45} - \frac{5(x+1)}{x+2} + 2(x+1)\\
=& -\frac{28}{45} + 2(x+1)\left[1 - \frac{5}{2(x+2)}\right]\\
\geq& -\frac{28}{45} + \frac{2}{3} = \frac{2}{45} > 0
\end{aligned} \tag{4.30}
$$

for all $x \geq 1$.

Inequality (4.30) implies

$$
\begin{aligned}
&\frac{28}{45(x+1)} + \frac{5}{x+2} + 2\log(x+1) + (2\log(2\pi) - 7)\\
=& f(x) \geq f(1) = 0.039826\cdots > 0,
\end{aligned}
$$

which leads to

$$
\frac{4x+3}{4x+4} > \frac{(x+2)(x+1-\frac{1}{2}\log(x+1) - \frac{1}{6x+6} + \frac{1}{90(x+1)^3} - a)}{(x+1)^2}. \tag{4.31}
$$

Therefore, inequality (1.7) follows from inequalities (4.19) and (4.31), and inequality (1.8) follows from inequality (1.7) and identity $\Gamma(n+1) = n!$. □

Remark 3. Inequality (1.8) improves the upper bound of inequality (4.14).

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