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ON MEDIAL Q-ALGEBRAS

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Abstract. In this paper, we show that the mapping $\varphi(x) = 0 * x$ is an endomorphism of a Q-algebra X, which induces a congruence relation " $\sim$ " such that X/φ is a medial Q-algebra. We also study some decompositions of ideals in Q-algebras and obtain equivalent conditions for closed ideals. Moreover, we show that if I is an ideal of a Q-algebra X, then I^g is an ignorable ideal of X.

1. Introduction

Y. Imai and K. Iséki introduced two classes of abstract algebras: BCK-algebras and BCI-algebras ([4, 5]). It is known that the class of BCK-algebras is a proper subclass of the class of BCI-algebras. In [2, 3], Q. P. Hu and X. Li introduced a wide class of abstract algebras: BCH-algebras. They have shown that the class of BCI-algebras is a proper subclass of the class of BCH-algebras. J. Neggers and H. S. Kim ([8]) introduced the notion of d-algebras, i.e., (I) $x*x = 0$; (VII) $0*x = 0$; (VI) $x*y = 0$ and $y*x = 0$ imply $x = y$, which is another useful generalization of BCK-algebras, and investigated several relations between d-algebras and BCK-algebras, and then they investigated other relations between d-algebras and oriented digraphs. On the while, Y. B. Jun, E. H. Roh, and H. S. Kim ([6]) introduced a new notion, called a BH-algebra, i.e., (I) $x * x = 0$; (II) $x * 0 = x$; (VI) $x * y = 0$ and $y * x = 0$ imply $x = y$, which is a generalization of $BCH/BCI/BCK$-algebras, and showed that there is a maximal ideal in bounded BH-algebras. J. Neggers, S. S. Ahn, and H. S. Kim ([7]) introduced a new notion, called a Q-*algebra*, which is also a generalization of $BCH/BCI/BCK$-algebras, and generalized several theorems discussed in BCI-algebras. Moreover, they introduced the notion of "quadratic" Q-algebra, and obtained the result that every quadratic Q-algebra $(X; *, e)$, $e \in X$, is of the form $x * y = x - y + e$, where $x, y \in X$ and X is a field with $|X| \geq 3$, i.e., the product is linear in a special way.

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In this paper, we show that the mapping $\varphi(x) = 0*x$ is an endomorphism of a Q-algebra X, which induces a congruence relation " $\sim$ " such that X/φ is a medial Q-algebra. We also study some decompositions of ideals in Q-algebras and obtain equivalent conditions for closed ideals. Moreover, we show that if I is an ideal of a Q-algebra X, then I^g is an ignorable ideal of X.

2. Preliminaries

A *Q-algebra* ([7]) is a non-empty set X with a constant 0 and a binary operation " $*$ " satisfying axioms:

(I) $x*x=0$,
(II) $x*0=x$,
(III) $(x*y)*z=(x*z)*y$

for all $x,y,z\in X$.
For brevity we also call X a *Q-algebra*. In X we can define a binary relation " $\leq$ " by $x\leq y$ if and only if $x*y=0$.

Example 2.1 ([1]). Let $X:=\{0,1,2,3\}$ be a set with the following table:

$*$	0	1	2	3
0	0	0	0	0
1	1	0	0	0
2	2	0	0	0
3	3	3	3	0

Then $(X;*,0)$ is a Q-algebra, which is not a $BCH/BCI/BCK$-algebra, since (VI) does not hold.

In a Q-algebra X the following property holds:
(IV) $(x*(x*y))*y=0$ for any $x,y\in X$.
A BCK-algebra is a Q-algebra X satisfying the additional axioms:

(V) $((x*y)*(x*z))*(z*y)=0$,
(VI) $x*y=0$ and $y*x=0$ imply $x=y$,
(VII) $0*x=0$

for all $x,y,z\in X$.

Definition 2.2 ([7]). Let $(X;*,0)$ be a Q-algebra and $\emptyset\neq I\subseteq X$. I is called a *subalgebra* of X if
(S) $x*y\in I$ whenever $x\in I$ and $y\in I$.
I is called an *ideal* of X if it satisfies:
(Q_0) $0\in I$,
(Q_1) $x*y\in I$ and $y\in I$ imply $x\in I$.

A Q-algebra X is called a *QS-algebra* ([1]) if it satisfies the following identity:

$$(x*y)*(x*z)=z*y$$

for any $x, y, z \in X$.

Example 2.3 ([1])**.** Let $\mathbb{Z}$ be the set of all integers and let $n\mathbb{Z} := \{nz|z \in \mathbb{Z}\}$, where $n \in \mathbb{Z}$. Then $(\mathbb{Z}; -, 0)$ and $(n\mathbb{Z}; -, 0)$ are both Q-algebras and QS-algebras, where "$-$" is the usual subtraction of integers. Also, $(\mathbb{R}; -, 0)$ and $(\mathbb{C}; -, 0)$ are both Q-algebras and QS-algebras, where $\mathbb{R}$ is the set of all real numbers and $\mathbb{C}$ is the set of all complex numbers.

Example 2.4 ([1])**.** Let $X = \{0, 1, 2\}$ with the table as follows:

$*$	0	1	2
0	0	0	0
1	1	0	0
2	2	0	0

Then X is both a Q-algebra and a QS-algebra, but not a $BCH/BCI/BCK$-algebra, since (VI) does not hold.

3. Quotient *Q*-algebras

In the following, let X denote a Q-algebra unless otherwise specified. The following lemma is useful to investigate roles of endomorphism φ of X.

Lemma 3.1. *Every Q-algebra X satisfies the following property*:

$$0 * (x * y) = (0 * x) * (0 * y)$$

for any $x, y \in X$.

Proof. For any $x, y \in X$, we have

$$\begin{aligned} 0 * (x * y) &= ((0 * y) * (0 * y)) * (x * y) \\ &= ((0 * y) * (x * y)) * (0 * y) \\ &= (((x * y) * x) * (x * y)) * (0 * y) \\ &= (((x * y) * (x * y)) * x) * (0 * y) \\ &= (0 * x) * (0 * y), \end{aligned}$$

completing the proof. □

By Lemma 3.1, the mapping $\varphi : X \to X$ defined by $\varphi(x) := 0 * x$ for any $x \in X$, is an endomorphism of Q-algebras. Note that $\varphi(0) = 0$. The kernel of this endomorphism, i.e., the set $\mathrm{Ker}\varphi = \{x \in X \mid 0 * x = 0\}$ is a subalgebra of X. If X is a Q-algebra with the additional identity $0 * x = 0$ for any $x \in X$, then $\mathrm{Ker}\varphi$ is an ideal of X.

Note that the centralizer of 0 in a Q-algebra X, i.e., the set

$$Z_0 = \{x \in X \mid 0 * x = x * 0\} = \{x \in X \mid 0 * x = x\} = \{x \in X \mid \varphi(x) = x\}$$

is a subalgebra of X which is also a group. Indeed, if $Z_0 \neq \{0\}$, then for any $x, y, z \in Z_0$ we have $x * y = (0 * x) * y = (0 * y) * x = y * x$ and as a consequence

$(x*y)*z = (y*x)*z = (y*z)*x = x*(y*z)$. Thus a Q-algebra is a group if and only if it satisfies the identity $0*x = x$ for any $x \in X$, or equivalently, if and only if it is associative.

Let " $\sim$ " be a binary operation on X defined as follows:

$$x \sim y \text{ if and only if } 0*x = 0*y,$$

in other words, $x \sim y$ if and only if $\varphi(x) = \varphi(y)$. Now we prove that " $\sim$ " is an equivalence relation on X. Since $\varphi(x) = \varphi(x)$, we have $x \sim x$. This means that " $\sim$ " is reflexive. If $x \sim y$ and $y \sim z$, then $\varphi(x) = \varphi(y)$ and $\varphi(y) = \varphi(z)$ and hence $\varphi(x) = \varphi(z)$. Therefore $x \sim z$, i.e., " $\sim$ " is transitive. Thus " $\sim$ " is an equivalence relation on X. Furthermore we have the following lemma:

Lemma 3.2. *If $x \sim y$ and $u \sim v$, then $x*u \sim y*v$, i.e., " $\sim$ " is a congruence relation in a Q-algebra X.*

Proof. Since $x \sim y$ and $u \sim v$, we have $\varphi(x) = \varphi(y)$ and $\varphi(u) = \varphi(v)$ and so by Lemma 3.1, $\varphi(x*u) = 0*(x*u) = (0*x)*(0*u) = \varphi(x)*\varphi(u) = \varphi(y)*\varphi(v) = \varphi(y*v)$. Hence $\varphi(x*u) = \varphi(y*v)$, i.e., $x*u \sim y*v$. □

We denote $[x]_\varphi := \{y \in X \mid x \sim y\} = \{y \in X \mid \varphi(x) = \varphi(y)\}$ by the equivalence class of x induced by the homomorphism $\varphi : X \to Y$. We claim that $[0]_\varphi = \mathrm{Ker}\varphi$. Indeed, if $y \in [0]_\varphi = \{y \in X \mid 0 \sim y\}$, then $\varphi(0) = \varphi(y)$. Since $\varphi(0) = 0$, $\varphi(y) = 0$ and so $y \in \mathrm{Ker}\varphi$. Conversely, if $y \in \mathrm{Ker}\varphi$, then $\varphi(y) = 0$. Since $\varphi(0) = 0$, $\varphi(0) = \varphi(y)$ and so $0 \sim y$. Hence $y \in [0]_\varphi$.

Denote $X/\varphi := \{[x]_\varphi \mid x \in X\}$ and define the following operation:

$$[x]_\varphi \circledast [y]_\varphi := [x*y]_\varphi.$$

Since " $\sim$ " is a congruence relation on X, the operation " $\circledast$ " is well-defined. In what follows, we prove that $(X/\varphi; \circledast, [0]_\varphi)$ is a Q-algebra. Let $[x]_\varphi, [y]_\varphi, [z]_\varphi$ and $[0]_\varphi \in X/\varphi$. Then we have the following properties:

(1) $[x]_\varphi \circledast [x]_\varphi = [0]_\varphi$,
(2) $[x]_\varphi \circledast [0]_\varphi = [x*0]_\varphi = [x]_\varphi$,
(3) $([x]_\varphi \circledast [y]_\varphi) \circledast [z]_\varphi = [x*y]_\varphi * [z]_\varphi = [(x*y)*z]_\varphi = [(x*z)*y]_\varphi = [x*z]_\varphi \circledast [y]_\varphi = ([x]_\varphi \circledast [z]_\varphi) \circledast [y]_\varphi$.

Summarizing the above facts we have:

Theorem 3.3. *Let $\varphi : X \to Y$ be a homomorphism of Q-algebras. Then X/φ is a Q-algebra with $[0]_\varphi = \mathrm{Ker}\varphi$.*

The Q-algebra X/φ discussed in Theorem 3.3 is called a *quotient Q-algebra induced by φ.*

Theorem 3.4. *If $\varphi : X \to Y$ is a homomorphism of Q-algebras, then $X/\varphi \cong \mathrm{Im}\varphi$.*

Proof. Define $\xi : X/\varphi \to \mathrm{Im}\varphi$ by $\xi([x]_\varphi) := \varphi(x)$. Then it is well-defined and one-one, since $[x]_\varphi = [y]_\varphi \Leftrightarrow x \in [y]_\varphi \Leftrightarrow \varphi(x) = \varphi(y) \Leftrightarrow \xi([x]_\varphi) = \xi([y]_\varphi)$ for any $[x]_\varphi, [y]_\varphi \in X/\varphi$. For any $[x]_\varphi, [y]_\varphi \in X/\varphi$, we have $\xi([x]_\varphi * [y]_\varphi) = \xi([x * y]_\varphi) = \varphi(x*y) = \varphi(x) * \varphi(y) = \xi([x]_\varphi) * \xi([y]_\varphi)$, proving that $X/\varphi \cong \mathrm{Im}\varphi$. $\square$

Definition 3.5. A Q-algebra X is said to be *medial* if it satisfies the following property:

$$(x*y)*(z*u) = (x*z)*(y*u) \text{ for any } x, y, z, u \in X.$$

Example 3.6. Let $X := \mathbb{R} - \{-n\}$, $0 \neq n \in \mathbb{Z}^+$ where $\mathbb{R}$ is the set of all real numbers and $\mathbb{Z}^+$ is the set of all positive integers. If we define $x * y := \frac{n(x-y)}{n+y}$, then $(X; *, 0)$ is a medial Q-algebra.

Lemma 3.7. *A Q-algebra X is medial if and only if it satisfies one of the following conditions: for any $x, y, z \in X$,*

(i) $y * x = 0 * (x * y)$,
(ii) $x * (y * z) = z * (y * x)$,
(iii) $x * (x * y) = y$,
(iv) $0 * (0 * y) = y$.

Proof. If a Q-algebra X is medial, then $y * x = (y * x) * 0 = (y * x) * (y * y) = (y * y) * (x * y) = 0 * (x * y)$. Let us assume (i) holds in X. Then $x * (y * z) = 0 * ((y * z) * x) = 0 * ((y * x) * z) = z * (y * x)$, which proves (ii). The condition (ii) implies mediality. Indeed, we have $(x * y) * (z * u) = u * (z * (x * y)) = u * (y * (x * z)) = (x * z) * (y * u)$, i.e., $(x * y) * (z * u) = (x * z) * (y * u)$.
Assume (i) holds. Then $x * (x * y) = 0 * ((x * y) * x)) = 0 * ((x * x) * y) = 0*(0*y) = y*0 = y$. Hence $x*(x*y) = y$, proving (iii). If we put $x := 0$ in (iii), then $0*(0*y) = y$, which proves (iv). Suppose (iv) holds. Then by Lemma 3.1 $x*y = 0*(0*(x*y)) = 0*((0*x)*(0*y)) = 0*((0*(0*y))*x) = 0*(y*x)$. Hence $x * y = 0 * (y * x)$, which completes the proof. $\square$

Corollary 3.8. *A Q-algebra is medial if and only if it is a medial QS-algebra.*

Proof. It is enough to prove the axiom $(x * y) * (x * z) = z * y$ is satisfied. In fact, by Lemma 3.7, we have

$$(x * y) * (x * z) = (x * x) * (y * z) = 0 * (y * z) = z * y,$$

proving the proof. $\square$

Lemma 3.9. *A Q-algebra X is associative if and only if $0 * x = x$ for any $x \in X$.*

Proof. If X is associative, then $(x * x) * x = x * (x * x)$ which gives $0 * x = x$ for any $x \in X$.

Conversely, assume $0*x = x$ for any $x \in X$. Then $x*(y*z) = (0*x)*(y*z) = (0*(y*z))*x = (y*z)*x = (y*x)*z = ((0*y)*x)*z = ((0*x)*y)*z = (x*y)*z$. Thus X is associative. $\square$

Corollary 3.10. *Every associative Q-algebra is medial.*

Proof. By Lemma 3.9, $0 * x = x$ for any $x \in X$. For any $x, y \in X$, we have $x * y = (0 * x) * y = (0 * y) * x = 0 * (y * x)$. It follows from Lemma 3.7 that X is a medial Q-algebra. □

Proposition 3.11. *Every QS-algebra satisfies the identity*:

$$0 * (0 * (0 * x)) = 0 * x \quad \text{for any } x \in X.$$

Proof. $0 * (0 * (0 * x)) = (0 * 0) * (0 * (0 * x)) = (0 * x) * 0 = 0 * x$. □

4. Some decompositions of ideals in Q-algebras

For any Q-algebra X and $x, y \in X$, denote

$$A(x, y) := \{z \in X \mid (z * x) * y = 0\}.$$

Theorem 4.1. *If I is an ideal of a Q-algebra X, then*

$$I = \cup_{x,y \in I} A(x, y).$$

Proof. Let I be an ideal of a Q-algebra X. If $z \in I$, then since $(z * 0) * z = (z * z) * 0 = 0 * 0 = 0$, we have $z \in A(0, z)$. Hence

$$I \subseteq \cup_{z \in I} A(0, z) \subseteq \cup_{x,y \in I} A(x, y).$$

Let $z \in \cup_{x,y \in I} A(x, y)$. Then there exist $a, b \in I$ such that $z \in A(a, b)$, so that $(z*a)*b = 0$. Since I is an ideal, it follows that $z \in I$. Thus $\cup_{x,y \in I} A(x, y) \subseteq I$, and consequently $I = \cup_{x,y \in I} A(x, y)$. □

Corollary 4.2. *If I is an ideal of a Q-algebra X, then*

$$I = \cup_{x \in I} A(0, x) = \cup_{x \in I} A(x, 0).$$

Proof. By Theorem 4.1, we have $\cup_{x \in I} A(0, x) \subseteq \cup_{x,y \in X} A(x, y) = I$. If $x \in I$, then $x \in A(0, x)$ because $(x * 0) * x = 0$. Hence $I \subseteq \cup_{x \in I} A(0, x)$. Since $(x * y) * z = (x * z) * y$, we have $\cup_{x \in I} A(0, x) = \cup_{x \in I} A(x, 0)$. This completes the proof. □

Theorem 4.3. *Let I be a subset of a Q-algebra X such that $0 \in I$ and $I = \cup_{x,y \in I} A(x, y)$. Then I is an ideal of X.*

Proof. Let $x*y, y \in I = \cup_{x,y \in I} A(x, y)$. Since $(x*(x*y))*y = (x*y)*(x*y) = 0$, we have $x \in A(x * y, y) \subseteq I$. Hence I is an ideal of X. □

Combining Theorems 4.1 and 4.3, we have the following corollary.

Corollary 4.4. *Let X be a Q-algebra and let I be a subset of X containing* 0. *Then I is an ideal of X if and only if $I = \cup_{x,y \in I} A(x, y)$.*

Definition 4.5. Let $(X; *, 0)$ be a Q-algebra and let $\emptyset \neq I \subset X$. An ideal I is said to be *closed* of X if $0 * x \in I$ for all $x \in I$.

Clearly, a closed ideal of a Q-algebra X is a subalgebra of X. Now we give a characterization of closed ideals.

Theorem 4.6. *Let I be a subset of a Q-algebra X. Then I is a closed ideal of X if and only if it satisfies*

(i) $0 \in I$,
(ii) $x * z \in I, y * z \in I$ *and* $z \in I$ *imply* $x * y \in I$.

Proof. Let I be a closed ideal of X. Clearly $0 \in I$. Assume that $x*z, y*z, z \in I$. Since I is an ideal, we have $x, y \in I$, which implies that $x * y \in I$ because I is a closed ideal and hence a subalgebra of X.

Conversely assume that I satisfies (i) and (ii). Let $x * y, y \in I$. Since $0 * 0, y * 0, 0 \in I$, by (ii) we have $0 * y \in I$. From (ii) again it follows that $x = x * 0 \in I$, so that I is an ideal of X. Now suppose that $x \in I$. Since $0 * 0, x * 0, 0 \in I$, we obtain $0 * x \in I$ by (ii). This completes the proof. □

Theorem 4.7. *Let I be an ideal of a Q-algebra X. The set*

$$I^0 := \{x \in I \mid 0 * x \in I\}$$

is the greatest closed ideal of X which is contained in I.

Proof. First we show that I^0 is an ideal of X. Clearly, $0 \in I^0$. For any $x, y \in X$, if $x * y, y \in I^0$, then $0 * y \in I$. By Lemma 3.1, we have

$$(0 * x) * (0 * y) = 0 * (x * y) \in I.$$

Since I is an ideal of X, it follows that $0 * x \in I$. Hence $x \in I^0$, which proves that I^0 is an ideal of X. If $x \in I^0$, since $I^0 \subseteq I$, we have $x \in I$ and $0 * x \in I$. Since $(0 * (0 * x)) * x = 0$, it follows from I is an ideal of X that $0 * (0 * x) \in I$, which implies $0 * x \in I^0$. This proves that I^0 is closed. Now, assume that A is a closed ideal of X which is contained in I. Let $x \in A$. Then $0 * x \in A$. Since A is contained in I, we have $x, 0 * x \in I$, and so $x \in I^0$. Thus $A \subseteq I^0$. Therefore I^0 is the greatest closed ideal of X which is contained in I. □

Definition 4.8. An ideal I of a Q-algebra X is said to be *ignorable* if $I^0 = \{0\}$.

Example 4.9. Let X be the set of all real numbers and let $C(X)$ be the set of all real-valued continuous functions on X. The operation "$*$" is defined as follows:

$$(f * g)(x) := f(x) - g(x) \text{ for all } x \in X.$$

The nullary operation 0 is the constant function 0. Then it is easy to show that $(C(X); *, 0)$ is a Q-algebra. If we define $P(X) := \{f \in C(X) \mid f(x) \geq 0, \forall x \in X\}$, then $P(X)$ is an ideal of $C(X)$, but it is not a subalgebra of $C(X)$, since if we let $f(x) := 3$ and $g(x) := 5$, where f and g are in $P(X)$, then $(f * g)(x) = f(x) - g(x) = 3 - 5 = -2 < 0$ and so $f * g \notin P(X)$. Moreover, $P(X)^0 = \{0\}$.

Theorem 4.10. *Let I be an ideal of a medial Q-algebra X. Then $I^g := (I - I^0) \cup \{0\}$ is an ignorable ideal of X.*

Proof. Let $x, y \in X$ be such that $x * y \in I^g$ and $y \in I^g$. If $y = 0$, then $x = x * 0 = x * y \in I^g$. Assume that $y \neq 0$. Clearly, $x * y, y \in I$, which implies that $x \in I$. Assume that $x \in I^0 - \{0\}$. Then $x \neq 0$ and $0 * x \in I$. Since $y \neq 0$, it follows from $y \in I^g$ that $y \in I - I^0$, so that $0 * y \notin I$. Since X is a medial Q-algebra, we have $(0 * y) * (0 * y) = (0 * (0 * x)) * y = x * y$ by Lemma 3.7. Since $x * y \in I$, we obtain $(0 * y) * (0 * x) \in I$. Since $0 * x \in I$, we have $0 * y \in I$. This is a contradiction. Hence $x \notin I^0 - \{0\}$, i.e., $x \in I^g$. This proves that I^g is an ideal of X. Now we show that $(I^g)^0 = \{0\}$. If $x \in (I^g)^0$, then $x \in I^g$ and $0 * x \in I^g$. From $x \in I^g$ it follows that $x = 0$ or $x \in I - I^0$. If $x \in I - I^0$, then $0 * x \notin I$, which is a contradiction. Thus $x = 0$. This completes the proof. □

The following corollary is obvious.

Corollary 4.11. *Let I be an ideal of a Q-algebra X. Then*

$$I^0 \cup I^g = I \text{ and } I^0 \cap I^g = \{0\}.$$

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