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SOME STUDIES ON 2-PRIMAL RINGS, (S,1)-RINGS AND THE CONDITION (KJ)

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Abstract. In this paper we study the connection between 2-primal rings, (S,1)-rings and related conditions. And we investigate some condition which is the special case of pseudo symmetric. We also study the condition (KJ) which is given by J. Y. Kim and H. L. Jin. We introduce some condition and we prove that our condition is equivalent to the condition (KJ) when it is an (S,1)-ring.

1. Introduction

The term 2-primal was introduced by G. F. Birkenmeier, H. E. Heatherly, and E. K. Lee [1] in the context of left near rings. Y. Hirano [2] considered the 2-primal condition in the context of strongly π-regular rings. Also G. Shin [7] proved that a ring R is 2-primal if and only if every minimal prime ideal of R is completely prime. N. K. Kim and Y. Lee [6] study quasi-duo rings which are π-regular. They also study (S,1)-rings, (S,2)-rings and related rings. J. Y. Kim and H. L. Jin [5] introduce some condition which we call a condition (KJ). And they study the connection between the condition (KJ) and 2-primal rings.

Throughout this paper, R denotes an associative ring with identity. We use $P(R)$ and $N(R)$ to represent the prime radical and the set of nilpotent elements of R, respectively. The n by n matrix ring over R is denoted by $M_n(R)$. And $r(\text{-})$ is used for the right annihilator in R. In this paper we investigate a connection between 2-primal rings, (S,1)-rings and related conditions. And we shall study the connection between weakly right duo rings and some condition.

2. 2-primal rings, (S,1)-rings and duo rings

In this section, we review 2-primal rings, (S,1)-rings and duo rings.

Definition 1. A ring R is called right (left) duo if every right (left) ideal of R is a two-sided ideal. A ring is called duo if it is both right and left duo. A ring R is called an (S,1)-ring if for $a, b \in R, ab = 0$ implies $aRb = 0$. A ring R is

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called 2-primal if $P(R) = N(R)$. And a reduced ring is a ring without nonzero nilpotent elements.

Proposition 1. *For a ring R we have the following assertions*:

(1) *If R is right* (*or left*) *duo, then R is an* (*S,1*)*-ring.*

(2) *If R is reduced, then R is an* (*S,1*)*-ring.*

(3) *If R is an* (*S,1*)*-ring, then R is* 2*-primal.*

Proof. The proof is well-known (see [6]). □

Then we shall consider a condition which is called pseudo symmetric. And we shall study the connection between the special cases of pseudo symmetric condition and several rings.

Definition 2. A ring R is called pseudo symmetric if it satisfies the following two conditions:

(PS 1) R/I is 2-primal whenever $I = 0$ or $I = r(aR)$ for some $a \in R$.

(PS 2) For any $a, b, c \in R$, if $aR(bc)^n = 0$ for a positive integer n, then it holds $a(RbR)^m c^m = 0$ for some positive integer m.

A ring R is called symmetric if $abc = 0$ implies $acb = 0$ for any $a, b, c \in R$. Symmetric rings are pseudo symmetric.

We shall consider the special case of the condition (PS 2).

(PS 2-1) For each $a \in N(R)$, it holds $(RaR)^m = 0$ for some positive integer m.

(PS 2-2) For any $a, b \in R$, if $aR(ab)^n = 0$ for a positive integer n, then it holds $a(RaR)^m b^m = 0$ for some positive integer m.

Definition 3. A ring R is called a CN-ring if every nilpotent element is central.

Proposition 2. *For a ring R we have the following assertions*:

(1) *If R is an* (*S,1*)*-ring, then R satisfies the condition* (PS 2-1)*.*

(2) *If R is a CN-ring, then R satisfies the condition* (PS 2-1)*.*

(3) *If R satisfies the condition* (PS 2-1) (*e.g. a pseudo symmetric ring*)*, then R is* 2*-primal.*

Proof. (1) Let $a^n = 0$. Since R is an (S,1)-ring, we get $aR_1aR_2a \cdots aR_{n-1}a = 0$ with $R_i = R$ for all i. So we have $(RaR)^n = 0$.

(2) Let $a^n = 0$. Since a is central, we have $(RaR)^n = 0$.

(3) $(RaR)^m = 0 \subseteq P(R)$ implies $RaR \subseteq P(R)$. Hence R is 2-primal. □

J. Y. Kim and H. L. Jin [5] studied certain 2-primal rings and introduced the following condition (KJ).

(KJ) If $aRb \subseteq P(R)$ for $a, b \in R$, it holds $a^n Rb^n = 0$ for some positive integer n.

Proposition 3. *If R satisfies the condition* (PS 2-2)*, then R satisfies the condition* (KJ)*.*

Proof. Let $aRb \subseteq P(R)$. There exsists a positive integer n such that $(ab)^n = 0$. Thus $aR(ab)^n = 0$, so by the condition (PS2-2), $a(RaR)^m b^m = 0$ for some positive integer m. Then $a^{m+1}Rb^m = 0$, hence $a^{m+1}Rb^{m+1} = 0$. Therefore the condition (PS 2-2) satisfies the condition (KJ). □

Example 1. Let R be a nonzero noncommutative domain and let

$$D_3(R) = \left\{ \begin{pmatrix} a & b & c \\ 0 & a & d \\ 0 & 0 & a \end{pmatrix} \in M_3(R) \middle| a, b, c \in R \right\}.$$

(1) $D_3(R)$ is an (S,1)-ring. So, $D_3(R)$ satisfies the condition (PS 2-1) and 2-primal by Proposition 2.

(2) $D_3(R)$ satisfies the condition (PS 2-2). So, $D_3(R)$ satisfies the condition (KJ).

(3) $D_3(R)$ is not a CN-ring (So, $D_3(R)$ is not a reduced ring).

Let $D_n(R) = \{(a_{ij}) \in M_n(R) | a_{11} = a_{22} = \cdots = a_{nn}, a_{ij} = 0 (i > j)\}$. We can see that $D_4(R)$ satisfies the condition (PS 2-1). But $D_4(R)$ is not an (S,1)-ring.

3. Weakly duo and quasi-duo rings

Definition 4. A ring R is called weakly right (left) duo if for each $a \in R$ there exists positive integer $n = n(a)$, depending on a, such that $a^n R$ (Ra^n) is a two-sided ideal. A ring is called weakly duo if it is both weakly right and weakly left duo.

A ring R is called right (left) quasi-duo if every maximal right (left) ideal of R is a two-sided ideal. A ring is called quasi-duo if it is both right and left quasi-duo.

Question 1. Do weakly duo (or quasi-duo) rings satisfy the condition (KJ)?

We shall consider the following condition in which the answer is affirmative.

($*$) If $(ab)^n = 0$, then it holds $a^m b^m = 0$ for some positive integer m.

Reduced rings, symmetric rings satisfy the condition ($*$).

Proposition 4. *If R is a CN-ring, then R satisfies the condition* ($*$).

Proof. Let $(ab)^n = 0$. When $n = 1$, it is obvious. Then suppose $(ab)^2 = 0$. We have $(ab)^2 = (ab)(ab) = a(ab)b = a^2b^2 = 0$. Hence inductively, for $(ab)^n = 0$, we get $a^n b^n = 0$. □

Let R be a reduced ring and $U_2(R) = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \in M_2(R) | a, b, c \in R\}$. Then $U_2(R)$ satisfies the condition (PS 2-1) and does not satisfy the condition ($*$). Moreover $U_2(R)$ does not satisfy the condition (KJ).

Conversely, by the following example (see [3]) there is a ring which satisfies the condition ($*$) and does not satisfy the condition (PS 2-1).

Example 2. Let S be a nonzero noncommutative domain, n be a positive integer and $R_n = D_{2^n}(S)$. Each R_n is a 2-primal ring. Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$, then R_n can be considered as a subring of R_{n+1} via σ. Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, 3, \ldots\}$. Set $R = \varinjlim R_n$ be the direct limit of D. Then R is not 2-primal. So R does not satisfy the condition (PS 2-1) by Proposition 2. But R satisfies the condition $(*)$.

Theorem 1. *If R is a weakly right duo ring which satisfies the condition $(*)$, then R satisfies the condition* (KJ).

Proof. Let $aRb \subseteq P(R)$. There exists a positive integer n such that $(ab)^n = 0$. By the condition $(*)$, $a^m b^m = 0$ for some positive integer m. Since R is a weakly right duo, $Rb^t \subseteq b^t R$ for some positive integer t. By taking positive integer s such that $st \geq m$, we get $a^{st} b^{st} = 0$. Since R is weakly right duo, we get $Rb^t \subseteq b^t R$ and $Rb^{2t} \subseteq b^{2t} R$. Inductively we get $Rb^{st} \subseteq b^{st} R$. Hence we have $a^{st} R b^{st} \subseteq a^{st} b^{st} R = 0$. □

Theorem 2. *Suppose that R is an (S,1)-ring. Then the following conditions are equivalent*:
(1) *R satisfies the condition* (KJ).
(2) *R satisfies the condition $(*)$.*

Proof. (1) ⇒ (2) Let $(ab)^n = 0$. Since R is an (S,1)-ring, we have $(aRb)^n = 0$. Thus $(arb)^n = 0$ for each $r \in R$. Then we get $arbR_1arbR_2arb \cdots arbR_{n-1}arb = 0$ with $R_i = R$ for all i, since R is an (S,1)-ring. And we get $arbR(arb)^{n-2}Rarb = 0$. Thus $(RarbR)^n = 0 \subseteq P(R)$. This implies $RarbR \subseteq P(R)$ and so $arb \in P(R)$. Then we have $aRb \subseteq P(R)$. By the condition (KJ), $a^m R b^m = 0$ for some positive integer m. Hence $a^m b^m = 0$.

(2) ⇒ (1) Firstly we show that R satisfies the condition (PS2-2). Let $aR(ab)^n = 0$. By $(ab)^{n+1} = 0$ and the condition $(*)$, we have $a^m b^m = 0$ for some positive integer m. Then since R is an (S,1)-ring, we get $a^m R b^m = 0$ and $a^{m-1} RaR b^m = 0$. Inductively, $aR_1aR_2a \cdots aR_m b^m = 0$ with $R_i = R$ for all i. So we have $(RaR)^m b^m = 0$. Thus R satisfies the condition (PS2-2). Hence by Proposition 3, R satisfies the condition (KJ). □

By Theorem 2 and the result of J. Y. Kim and H. L. Jin [5], we get following corollary.

Corollary 1. *Suppose that R is an (S,1)-ring which satisfies the condition $(*)$. Then the following conditions are equivalent*:
(1) *R is right weakly π-regular.*
(2) *$R/P(R)$ is right weakly π-regular.*
(3) *Every prime ideal of R is maximal.*

Since CN-rings satisfy the condition $(*)$, we get following corollary.

Corollary 2. *Suppose that R is an (S,1) and CN-ring. Then the following conditions are equivalent*:

(1) *R is right weakly π-regular.*

(2) *$R/P(R)$ is right weakly π-regular.*

(3) *Every prime ideal of R is maximal.*

Question 2. In Theorem 2, is the condition (S,1)-ring superfluous?

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