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A NOTE ON LIE IDEALS OF PRIME RINGS

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Abstract. Let R be a 2-torsion free prime ring, U a nonzero Lie ideal of R such that $u^2 \in U$ for all $u \in U$. In the present paper, it is proved that if d is a nonzero derivation and $[[d(u), u], u] = 0$ for all $u \in U$, then $U \subseteq Z(R)$. Moreover, suppose that d_1, d_2, d_3 are nonzero derivations of R such that $d_3(y)d_1(x) = d_2(x)d_3(y)$ for all $x, y \in U$, then $U \subseteq Z(R)$. Finally, some examples are given to demonstrate that the restrictions imposed on the hypothesis of the above results are not superfluous.

Introduction

Throughout this paper, R will always denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$. Given two subsets A and B of R then $[A, B]$ will denote the additive subgroup of R generated by all elements of the form $[a, b]$ where $a \in A$, $b \in B$. For a nonempty subset S of R, we put $C_R(S) = \{x \in R \mid [x, s] = 0 \text{ for all } s \in S\}$. Recall that R is prime if $aRb = 0$ implies $a = 0$ or $b = 0$. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A mapping F from R to R is said to be commuting on a subset S of R if $[F(x), x] = 0$ for all $x \in S$, and is said to be centralizing on S if $[F(x), x] \in Z(R)$ holds for all $x \in S$. An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. The relationship between usual derivations and Lie ideals of prime rings has been extensively studied over the past 30 years. In particular, when this relationship involves the action of the derivations on Lie ideals. Many of these results extend other ones proven previously for the action of the derivations on the whole ring (see [1] for a partial references). There is a particular interest in Lie ideals U such that $u^2 \in U$ for all $u \in U$. Such a distinction already appears in several papers involving usual derivations and Lie ideals ([4], [2], [5], [3] where further references can be found). The theory of commuting and centralizing mapping on prime rings was initiated by E. C. Posner. The classical result of E. C.

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Posner ([8]) states that the existence of a nonzero centralizing derivation on a prime ring forces the ring to be commutative. Afterward J. Vukman ([9]) extend E. C. Posner's theorem by showing that if d is a nonzero derivation of a prime ring R with char $R \neq 2$ such that $[[d(x), x], x] = 0$ for all $x \in R$, then R is commutative. In [7], F. W. Niu proved that if d_1, d_2, d_3 are nonzero derivations of prime ring R with char $R \neq 2$, and if $d_3(y)d_1(x) = d_2(x)d_3(y)$ for all $x, y \in R$, then R is a commutative ring.

In this note we intend to show that the above conclusions hold for Lie ideals of prime rings. In everything that follows R will be a prime ring of char $R \neq 2$ and U will always denote a Lie ideal of R.

1. Preliminary results

We begin with the following lemmas which will be used extensively to prove our theorems.

Lemma 1.1 ([4, Lemma 2])**.** *If $U \not\subseteq Z(R)$ is a Lie ideal of R, then $C_R(U) = Z(R)$.*

Lemma 1.2 ([4, Lemma 3])**.** *If U is a Lie ideal of R, then $C_R([U, U]) = C_R(U)$.*

Lemma 1.3. *Set $V = \{u \in U |\ d(u) \in U\}$. If $U \not\subseteq Z(R)$, then $V \not\subseteq Z(R)$.*

Proof. Assume that $V \subseteq Z(R)$. Since $[U, U] \subseteq U$ and $d([U, U]) \subseteq U$, we have $[U, U] \subseteq V \subseteq Z(R)$. Hence $C_R([U, U]) = R$. From Lemma 1.1, we have $C_R(U) = Z(R)$. Application of Lemma 1.2 yields that $C_R([U, U]) = C_R(U)$. That is $R = Z(R)$, which implies a contradiction. □

Lemma 1.4. *If U is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$, then $2uv \in U$ for all u, $v \in U$.*

Proof. For all $w, u, v \in U$,

$$uv + vu = (u + v)^2 - u^2 - v^2 \in U.$$

On the other hand,

$$uv - vu \in U.$$

Adding two expressions, we have $2uv \in U$ for all $u, v \in U$. □

Lemma 1.5 ([4, Lemma 7])**.** *Let R be a prime ring. If $d \neq 0$ is a derivation of R, and $U \not\subseteq Z(R)$ is a Lie ideal, if $td(U) = 0$ or $d(U)t = 0$ we must have $t = 0$.*

Lemma 1.6 ([2, Lemma 7])**.** *Let $d \neq 0$ be a derivation of R such that $[u, d(u)] \in Z(R)$ for all $u \in U$. Then $U \subseteq Z(R)$.*

Lemma 1.7 ([4, Lemma 4])**.** *If $U \not\subseteq Z(R)$ is a Lie ideal of R and $aUb = 0$, then $a = 0$ or $b = 0$.*

Lemma 1.8 ([6, Theorem 4])**.** *Let d_1 and d_2 be nonzero derivations of R such that $d_1d_2(U) \subseteq Z(R)$. Then $U \subseteq Z(R)$.*

Lemma 1.9 ([4, Theorem 2]). *If $U \not\subseteq Z(R)$ is a Lie ideal of R and $d \neq 0$ is a derivation, then $C_R(d(U)) = Z(R)$.*

2. Main results

Theorem 2.1. *Let R be a 2-torsion free prime ring and d a nonzero derivation of R. Suppose that U is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If $[[d(u), u], u] = 0$ for all $u \in U$, then $U \subseteq Z(R)$.*

Proof. Suppose on the contrary that $U \not\subseteq Z(R)$. Thus $V \not\subseteq Z(R)$ by Lemma 1.3. By hypothesis we have $[[d(u), u], u] = 0$ for all $u \in U$. Define a mapping $B : R\times R \longrightarrow R$ by the relation $B(x, y) = [d(x), y] = [d(y), x]$ for all $x, y \in R$. It is obvious that $B(x, y) = B(y, x)$ and B is additive in two variables. Moreover, a simple calculation shows that the relation $B(xy, z) = B(x, z)y + xB(y, z) + d(x)[y, z] + [x, z]d(y)$ holds for all $x, y, z \in R$. Now write $f(x)$ for $B(x, x)$ briefly. Then we have $f(x) = 2[d(x), x]$ for all $x \in R$. It is easy to show that $f(x + y) = f(x) + f(y) + 2B(x, y)$ for all $x, y \in R$. Therefore, the assumption of the theorem can be written as follows:

$$[f(u), u] = 0 \tag{1}$$

for all $u \in U$.
The linearization of (1) gives

$$\begin{aligned}
0 &= [f(u + v), u + v] \\
&= [f(u) + f(v) + 2B(u, v), u + v] \\
&= [f(u), u] + [f(u), v] + [f(v), u] + [f(v), v] + 2[B(u, v), u] + 2[B(u, v), v]
\end{aligned}$$

which reduces to

$$[f(u), v] + [f(v), u] + 2[B(u, v), u] + 2[B(u, v), v] = 0. \tag{2}$$

Replacing u by $-u$ in (2), we have

$$[f(u), v] - [f(v), u] + 2[B(u, v), u] - 2[B(u, v), v] = 0. \tag{3}$$

From (2) and (3), we obtain that

$$[f(u), v] + 2[B(u, v), u] = 0 \tag{4}$$

for all $u, v \in U$. The substituation $2vu$ for v in (4) gives us that

$$\begin{aligned}
0 &= [f(u), 2vu] + 2[B(u, 2vu), u] \\
&= [f(u), vu] + 2[B(u, vu), u] \\
&= [f(u), v]u + v[f(u), u] + 2[B(u, v)u + vf(u) + [v, u]d(u), u].
\end{aligned}$$

Using (1) and (4) in the above relation, we arrive at

$$3[v, u]f(u) + 2[[v, u], u]d(u) = 0. \tag{5}$$

Similarly, we also obtain that

$$3f(u)[v, u] + 2d(u)[[v, u], u] = 0. \tag{6}$$

Replacing v by $2wv$ in (5) and using the fact char $R \neq 2$, we have

$$\begin{aligned} 0 &= 3[wv,u]f(u) + 2[[wv,u],u]d(u) \\ &= 3w[v,u]f(u) + 3[w,u]vf(u) + 2[w[v,u] + [w,u]v,u]d(u) \\ &= 3w[v,u]f(u) + 3[w,u]vf(u) + 4[w,u][v,u]d(u) \\ &\quad + 2w[[v,u],u]d(u) + 2[[w,u],u]vd(u) \\ &= 3[w,u]vf(u) + 4[w,u][v,u]d(u) + 2[[w,u],u]vd(u). \end{aligned}$$

Choose $v = d(u)$ for all $u \in V = \{u \in U|\ d(u) \in U\}$, we obtain $3[w,u]d(u)f(u) + 4[w,u][d(u),u]d(u) + 2[[w,u],u]d(u)^2 = 0$. That is, $3[w,u]d(u)f(u) + 2[w,u]f(u)d(u) + 2[[w,u],u]d(u)^2 = 0$. It follows that $2[[w,u],u]d(u)^2 = -3[w,u]f(u)d(u)$ from (5). And hence we conclude that $[w,u](3d(u)f(u) - f(u)d(u)) = 0$ for all $u \in V$ and $w \in U$. In other words, we have $I_u(w)(3d(u)f(u) - f(u)d(u)) = 0$ for all $w \in U$. There is nothing to prove if $u \in Z(R)$ since in this case $f(u) = 0$. Otherwise, we have

$$3d(u)f(u) - f(u)d(u) = 0 \tag{7}$$

by Lemma 1.5. Similarly, we can also prove the relation

$$3f(u)d(u) - d(u)f(u) = 0. \tag{8}$$

Combining (7) with (8), we can see easily that

$$d(u)f(u) = f(u)d(u) = 0 \tag{9}$$

for all $u \in V$. The linearization of $d(u)f(u) = 0$ gives

$$\begin{aligned} 0 &= (d(u) + d(v))(f(u) + f(v) + 2B(u,v)) \\ &= d(u)f(u) + d(u)f(v) + 2d(u)B(u,v) + d(v)f(u) + d(v)f(v) + 2d(v)B(u,v) \\ &= d(u)f(v) + 2d(u)B(u,v) + d(v)f(u) + 2d(v)B(u,v). \end{aligned}$$

Replacing u by $-u$ in the above relation, we get $-d(u)f(v) + 2d(u)B(u,v) + d(v)f(u) - 2d(v)B(u,v) = 0$. Adding the above two relations, we arrive at

$$d(v)f(u) + 2d(u)B(u,v) = 0. \tag{10}$$

Substituting $2vu$ for v in (10) and using char $R \neq 2$, we get

$$\begin{aligned} 0 &= d(vu)f(u) + 2d(u)B(u,vu) \\ &= d(v)uf(u) + vd(u)f(u) + 2d(u)B(v,u)u + 2d(u)vf(u) + 2d(u)[v,u]d(u) \\ &= d(v)uf(u) + 2d(u)B(v,u)u + 2d(u)vf(u) + 2d(u)[v,u]d(u) \\ &= d(v)uf(u) - d(v)f(u)u + 2d(u)vf(u) + 2d(u)[v,u]d(u) \\ &= d(v)[u,f(u)] + 2d(u)vf(u) + 2d(u)[v,u]d(u) \\ &= 2d(u)vf(u) + 2d(u)[v,u]d(u). \end{aligned}$$

Now we have proved that

$$d(u)vf(u) + d(u)[v,u]d(u) = 0 \tag{11}$$

for all $u, v \in V$. Replacing v by $2uv$ in (11) and using char $R \neq 2$, we get

$$d(u)uvf(u) + d(u)u[v, u]d(u) = 0. \tag{12}$$

Left multiplication by u of (11) yields that

$$ud(u)vf(u) + ud(u)[v, u]d(u) = 0. \tag{13}$$

Combining (12) with (13) we have $[d(u), u]vf(u) + [d(u), u][v, u]d(u) = 0$. More precisely, we have obtained that

$$f(u)vf(u) + f(u)[v, u]d(u) = 0 \tag{14}$$

for all $u, v \in V$. Our next goal is to prove the following

$$3f(u)vf(u) + 4f(u)[v, u]d(u) = 0. \tag{15}$$

For this purpose, writing $2vw$ in (5) instead of v and using char $R \neq 2$, we have

$$\begin{aligned} 0 &= 3[vw, u]f(u) + 2[[vw, u], u]d(u) \\ &= 3[v, u]wf(u) + 3v[w, u]f(u) + 2[v[w, u] + [v, u]w, u]d(u) \\ &= 3[v, u]wf(u) + 3v[w, u]f(u) + 2v[[w, u], u]d(u) + 4[v, u][w, u]d(u) \\ &\quad + 2[[v, u], u]wd(u) \\ &= 3[v, u]wf(u) + 4[v, u][w, u]d(u) + 2[[v, u], u]wd(u) \end{aligned}$$

for all $u, v, w \in V$. Write $L = [V, V]$ then it is easy to show that L is a Lie ideal and $d(L) \subseteq V$. Moreover, since $V \not\subseteq Z(R)$ then $L \not\subseteq Z(R)$ by Lemma 1.3. Now choosing $v = 2d(u)$ where $u \in L$ in the above relation, we have $3f(u)wf(u) + 4f(u)[w, u]d(u) = 0$ holds for all $u, v, w \in L$. Combining (14) with (15), we conclude that $f(u)vf(u) = 0$ for all $u, v \in L$. Since L is a noncentral Lie ideal, we have $f(u) = 0$ for all $u \in L$. Thus we have proved that $[d(u), u] = 0$ holds for all $u \in L$. This leads to $L \subseteq Z(R)$ by Lemma 1.6, which implies a contradiction. The proof of the theorem is complete. □

Theorem 2.2. *Let d_1, d_2, d_3 be nonzero derivations, U a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. If $d_3(y)d_1(x) = d_2(x)d_3(y)$ for all $x, y \in U$, then $U \subseteq Z(R)$.*

Proof. Suppose on the contrary that $U \not\subseteq Z(R)$. Write $V = \{x \in U \mid d_i(u) \in U, i = 1, 2, 3\}$. It is easy to see that V is a Lie ideal of R. Therefore, $V \not\subseteq Z(R)$ by Lemma 1.3. Now by hypothesis we have

$$d_3(y)d_1(x) = d_2(x)d_3(y). \tag{16}$$

Replace x by $2xp$ in (16) according to Lemma 1.1 and use char $R \neq 2$ to get $d_3(y)d_1(xp) = d_2(xp)d_3(y)$, namely,

$$d_3(y)d_1(x)p + d_3(y)xd_1(p) = d_2(x)pd_3(y) + xd_2(p)d_3(y)$$

for all $x, y, p \in U$. According to (16), we find that

$$d_2(x)d_3(y)p + d_3(y)xd_1(p) = d_2(x)pd_3(y) + xd_3(y)d_1(p).$$

This implies that

$$d_2(x)[d_3(y), p] = [x, d_3(y)]d_1(p) \tag{17}$$

for all $x, y, p \in U$. In particular, for any $y \in V$ replace x by $d_3(y)$ in (17), to get

$$d_2d_3(y)[d_3(y), p] = 0. \tag{18}$$

For all $m, n \in U$, again replace p by $2mn$ in (18) and use (18) to get

$$\begin{aligned} 0 &= d_2d_3(y)[d_3(y), 2mn] \\ &= 2d_2d_3(y)m[d_3(y), n] + 2d_2d_3(y)[d_3(y), m]n \\ &= 2d_2d_3(y)m[d_3(y), n]. \end{aligned}$$

Now we have

$$d_2d_3(y)m[d_3(y), n] = 0 \tag{19}$$

for all $m, n \in V$ and $y \in V$. By Lemma 1.7, either $d_2d_3(y) = 0$ or $[d_3(y), n] = 0$. Now let $V_1 = \{y \in V \mid d_2d_3(y) = 0\}$ and $V_2 = \{y \in V \mid [d_3(y), n] = 0\}$. Then V_1, V_2 are both additive subgroups of V and $V_1 \bigcup V_2 = V$. By Buarer's trick, either $V_1 = V$ or $V_2 = V$. If $V_1 = V$ (i.e., $d_2d_3(V) = 0$), then $V \subseteq Z(R)$ by Lemma 1.8, a contradiction. If $V_2 = V$, then for all $n \in V$, we have $n \in C_R(d_3(V)) = Z(R)$ by Lemma 1.9, again a contradiction. □

Remark 2.1. Though the assumption that $u^2 \in U$ for all $u \in U$, seems close to assuming that U is an ideal of the ring, but there exist Lie ideals with the property $u^2 \in U$ for all $u \in U$, which are not ideals. For example, let $R = \{\left(\begin{smallmatrix} a & b \\ 0 & c \end{smallmatrix}\right) \mid a, b \in Z\}$. It can be easily check that $U = \{\left(\begin{smallmatrix} a & b \\ 0 & a \end{smallmatrix}\right) \mid b \in Z\}$ is a Lie ideal of R such that $u^2 \in U$ for all $u \in U$. However, U is not an ideal of R.

The following example shows the hypothesis of primeness is essential in Theorem 2.1.

Example 2.1. Let S be any ring. Next, let $R = \left\{\left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{smallmatrix}\right) | a, b, c \in S\right\}$. It is straightforward to see that $U = \left\{\left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right) | a, b \in S\right\}$ is a Lie ideal of R. Define $d : R \longrightarrow R$ as follows: $d\left(\begin{smallmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & 0 & b \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{smallmatrix}\right)$. Then d is a derivation of R such that $[[d(u), u], u] = 0$ for all $u \in U$, however $U \not\subseteq Z(R)$.

The following example shows Theorem 2.1 fails without the condition $u^2 \in U$ for all $u \in U$.

Example 2.2. Consider the prime ring R of all 2×2 matrices over $GF(2)$.

Let $U = \{\left(\begin{smallmatrix} x & y \\ z & x \end{smallmatrix}\right) \mid x, y, z \in R\}$. Then U is a Lie ideal of R without the property $u^2 \in U$. Define $d\left(\begin{smallmatrix} x & y \\ z & w \end{smallmatrix}\right) = \left(\begin{smallmatrix} w-z & x-w \\ x-w & y-z \end{smallmatrix}\right)$. Then, d is a nonzero derivation of R such that $[[d(u), u], u] = 0$ for all $u \in U$, however $U \not\subseteq Z(R)$.

The following example shows the hypothesis of primeness is essential in Theorem 2.2

Example 2.3. Let S be any ring. Let $R = \{\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) \mid a, b \in S\}$ and $U = \{\left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right) \mid a, b \in S\}$.

Obviously, U is a Lie ideal of R with the condition $u^2 \in U$ for all $u \in U$.

Define $d\left(\begin{smallmatrix} a & b \\ 0 & 0 \end{smallmatrix}\right) = \left(\begin{smallmatrix} 0 & b \\ 0 & 0 \end{smallmatrix}\right)$. Then d is a nonzero derivation of R. Take $d_1 = d_2 = d_3 = d$, we conclude that $d(x)d(y) = d(y)d(x)$ for all $x, y \in U$. However, $U \not\subseteq Z(R)$.

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