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HELICOIDAL SURFACES AND THEIR GAUSS MAP IN MINKOWSKI 3-SPACE

Miekyung Choi, Young Ho Kim, Huili Liu, and Dae Won Yoon

Abstract. The helicoidal surface is a generalization of rotation surface in a Minkowski space. We study helicoidal surfaces in a Minkowski 3-space in terms of their Gauss map and provide some examples of new classes of helicoidal surfaces with constant mean curvature in a Minkowski 3-space.

1. Introduction

As is well known that the helicoidal surface is a kind of generalization of some ruled surfaces and rotation surfaces in a Euclidean space or a Minkowski space. Quite a few works have been recently done with the helicoidal surfaces in Minkowski space with prescribed mean or Gaussian curvature ([3, 10]).

The notion of finite type immersion of submanifolds of a Euclidean space or a pseudo-Euclidean space has been widely used in classifying and characterizing well known Riemannian or pseudo-Riemannian submanifolds ([5, 6]). In some sense, it is a generalization of solving eigenvalue problems formed with the Laplace operator in the set of submanifolds in a Euclidean or a pseudo-Euclidean space. It gives a nice relationship between some algebraic properties and geometric properties.

On the other hand, the Gauss map is a very useful tool to look into submanifolds of a Euclidean space or a pseudo-Euclidean space ([1, 2, 8, 13]). Thus, it is interesting to examine the behavior of the Gauss map of given submanifolds in a Euclidean space or a pseudo-Euclidean space based on the Laplacian. In particular, the Gauss map G of some minimal (or maximal) surfaces including the catenoid in Euclidean 3-space and the Enneper's surface of the second kind in a Minkowski 3-space satisfies a unique partial differential equation similar to an eigenvalue problem that is not an actual eigenvalue problem.

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Three of the present authors defined and used the notion of pointwise 1-type Gauss map to study certain surfaces in a Euclidean space or a Minkowski space ([7, 9, 12, 13]).

The Gauss map G on a submanifold M of a pseudo-Euclidean space E^m_s of index s is said to be of *pointwise* 1*-type* if

$$\Delta G = F(G + C) \tag{1.1}$$

for a nonzero smooth function F on M and a constant vector C, where Δ denotes the Laplace operator defined on M. Especially, it is called *proper* if the function F defined by (1.1) is non-constant. The non-proper pointwise 1-type Gauss map is just of 1-type in the usual sense ([5, 8]). A submanifold with pointwise 1-type Gauss map is said to be of the *first kind* if the vector C in (1.1) is the zero vector. Otherwise, the pointwise 1-type Gauss map is said to be of the *second kind* ([7, 9, 12]).

In this article we study the helicoidal surfaces with pointwise 1-type Gauss map in a Minkowski 3-space. We also provide some new examples of helicoidal surfaces in a Minkowski 3-space by solving some ordinary differential equations related to mean curvature.

2. Preliminaries

Let $\mathbb{E}^3_1$ be a Minkowski 3-space with the Lorentz metric

$$\langle \cdot, \cdot \rangle = -dx_0^2 + dx_1^2 + dx_2^2,$$

where (x_0, x_1, x_2) is a system of the canonical coordinates in $\mathbb{R}^3$. Let M be a connected 2-dimensional surface in $\mathbb{E}^3_1$ and $x : M \to \mathbb{E}^3_1$ a smooth nondegenerate isometric immersion. A surface M is said to be *spacelike* (resp. *timelike*) if the induced metric on M is positive definite (resp. indefinite). Assuming that M is orientable, we can always choose a unit normal vector field G globally defined on M. On the other hand, the unit normal vector field G can be regarded as a map $G : M \to \mathbb{H}^2_+$ if M is spacelike, and as a map $G : M \to \mathbb{S}^2_1$ if M is timelike. Here, $\mathbb{H}^2_+ = \{x \in \mathbb{E}^3_1 \mid \langle x, x\rangle = -1, x_2 > 0\}$ is the *hyperbolic space* and $\mathbb{S}^2_1 = \{x \in \mathbb{E}^3_1 \mid \langle x, x\rangle = 1\}$ is the *de Sitter space.* The map G is also called the *Gauss map* of the surface M. For the matrix $\tilde{g} = (\tilde{g}_{ij})$ consisting of the components of the induced metric on M, we denote by $\tilde{g}^{-1} = (\tilde{g}^{ij})$ (resp. $\mathcal{G}$) the inverse matrix (resp. the determinant) of the matrix $(\tilde{g}_{ij})$. The Laplacian Δ on M is, in turn, given by

$$\Delta = -\frac{1}{\sqrt{|\mathcal{G}|}} \sum_{i,j} \frac{\partial}{\partial x^i}\Big(\sqrt{|\mathcal{G}|}\ \tilde{g}^{ij} \frac{\partial}{\partial x^j}\Big).$$

Let e be a nonzero vector in $\mathbb{E}^3_1$ and $\mathbf{S}(e)$ be the set of screw motions fixing e in $\mathbb{E}^3_1$. In particular, if e is non-null, the screw motions fixing e belong to $\mathbf{O}(e)$, the set of orthogonal transformations with positive determinant. Then a *helicoidal motion* around the axis in the e-direction can be defined by

$$g_t(x) = A(t)x^T + (ht)e^T, \quad x = (x_0, x_1, x_2) \in \mathbb{E}^3_1,\ t \in \mathbb{R},\ A \in \mathbf{S}(e),$$

where h is a constant.

Let $\gamma : I = (a, b) \subset \mathbb{R} \to \Pi$ be a plane curve in $\mathbb{E}^3_1$ and l a straight line in Π which does not intersect the curve γ. A *helicoidal surface* M with the axis l and pitch h in $\mathbb{E}^3_1$ is a nondegenerate surface which is invariant under the action of the helicoidal motion g_t. Depending on the axis being spacelike, timelike or null, there are three types of screw motions. If the axis l is spacelike (resp. timelike), then l is transformed to the x_1-axis or x_2-axis (resp. x_0-axis) by the Lorentz transformation. Therefore, we may consider x_2-axis (resp. x_0-axis) as the axis if l is spacelike (resp. timelike). If the axis l is null, then we may assume that the axis is the line spanned by the vector $(1, 1, 0)$.

We now consider the helicoidal surfaces in $\mathbb{E}^3_1$ with spacelike, timelike or null axis respectively.

Case 1. The axis l is spacelike.

Without loss of generality we may assume that the profile curve γ lies in the x_1x_2-plane or x_0x_2-plane. Hence the curve γ can be represented by

$$\gamma(u) = (0, f(u), g(u)) \quad \text{or} \quad \gamma(u) = (f(u), 0, g(u))$$

for smooth functions f and g on an open interval $I = (a, b)$. Therefore, the surface M may be parameterized by

$$x(u, v) = (f(u)\sinh v, f(u)\cosh v, g(u) + hv),\ f(u) > 0,\ h \in \mathbb{R} \tag{2.1}$$

or

$$x(u, v) = (f(u)\cosh v, f(u)\sinh v, g(u) + hv),\ f(u) > 0,\ h \in \mathbb{R}. \tag{2.2}$$

Case 2. The axis l is timelike.

In this case, we may assume that the curve γ lies in the x_0x_1-plane. So the curve γ is given by $\gamma(u) = (g(u), f(u), 0)$ for a positive function $f = f(u)$ on an open interval $I = (a, b)$. Hence the surface M can be expressed by

$$x(u, v) = (g(u) + hv, f(u)\cos v, f(u)\sin v),\ f(u) > 0,\ h \in \mathbb{R}. \tag{2.3}$$

Case 3. The axis l is null.

In this case, we may assume that the curve γ lies in the x_0x_1-plane of the form $\gamma(u) = (f(u), g(u), 0)$, where $f = f(u)$ is a positive function and $g = g(u)$ is a function satisfying $p(u) = f(u) - g(u) \neq 0$ for all $u \in I$. Under the cubic screw motions, its parametrization has the form

$$x(u, v) = (f(u) + \frac{v^2}{2}p(u) + hv,\ g(u) + \frac{v^2}{2}p(u) + hv,\ p(u)v),\ h \in \mathbb{R}. \tag{2.4}$$

3. Examples of helicoidal surfaces with pointwise 1-type Gauss map

In this section, we provide some examples of helicoidal surfaces with pointwise 1-type Gauss map in Minkowski 3-space and we give the names of them upon their invariance under the orthogonal transformations related to those of rotation surfaces in $\mathbb{E}^3_1$.

Example 3.1 (Right helicoid of type I)**.** A right helicoid with spacelike axis in $\mathbb{E}^3_1$ is parameterized by

$$x(u,v) = (u\sinh v,\ u\cosh v,\ a+hv),\ u>0,\ h\neq 0$$

for some constant a. It could be spacelike or timelike depending on the region satisfying either $-u^2+h^2>0$ or $-u^2+h^2<0$. If it is spacelike, its Gauss map G is given by

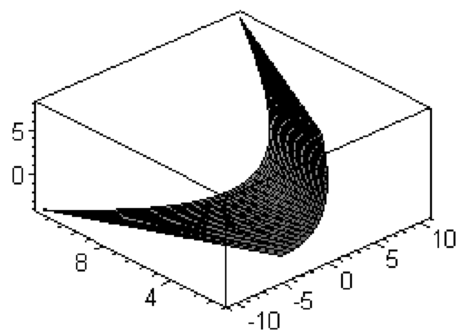
$$G = \frac{1}{\sqrt{-u^2+h^2}}(-h\cosh v,\ -h\sinh v,\ -u).$$

Hence the Laplacian ΔG of the Gauss map G satisfies

$$\Delta G = -\frac{2h^2}{(-u^2+h^2)^2}G,$$

which has pointwise 1-type Gauss map of the first kind. In case of M being timelike, we can have a similar result.

Fig 1 : Right helicoid of type I



Example 3.2 (Right helicoid of type II)**.** A right helicoid M with timelike axis in $\mathbb{E}^3_1$ is parameterized by

$$x(u,v) = (a+hv, u\cos v,\ u\sin v),\ u>0,\ h\neq 0$$

for some constant a. The surface M is spacelike or timelike upon the sign of h^2-u^2. Assuming that M is spacelike, the Gauss map G is given by

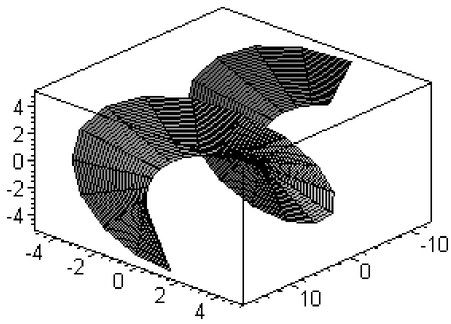
$$G = \frac{1}{\sqrt{u^2-h^2}}(-u,\ h\sin v,\ -h\cos v).$$

By a direct computation, we see that its Laplacian satisfies

$$\Delta G = -\frac{2h^2}{(u^2 - h^2)^2} G$$

and hence, the Gauss map is of pointwise 1-type of the first kind.

Fig 2 : Right helicoid of type II



Example 3.3 (Helicoidal surface of elliptic type). Let M be a surface with timelike axis in $\mathbb{E}^3_1$ parameterized by

$$x(u, v) = (\pm u + a + hv,\ u\cos v,\ u\sin v),\ u > 0,\ h \neq 0$$

for some constant a. Then, the Gauss map G is obtained by

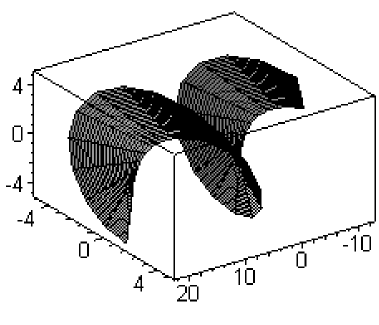
$$G = \frac{1}{|h|}(-u,\ \mp u\cos v + h\sin v,\ \mp u\sin v - h\cos v).$$

Therefore, its Laplacian satisfies

$$\Delta G = \frac{2}{h^2} G$$

and thus, the Gauss map G of M is of pointwise 1-type of the first kind. Indeed, it is non-proper. We call such a surface M a *helicoidal surface of elliptic type.*

Fig 3 : Helicoidal surface of elliptic type



The following helicoidal surfaces are generated by a null axis.

Example 3.4 (Spacelike helicoidal surface of Enneper type)**.** For a positive constant a and a constant b, a helicoidal surface parameterized by

$$x(u,v) = (au^3 - \frac{h^2}{4u} + b - u - uv^2 + hv,\ au^3 - \frac{h^2}{4u} + b + u - uv^2 + hv,\ -2uv)$$

has the Gauss map

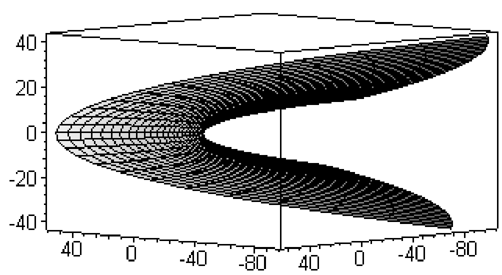
$$G = \frac{1}{2\sqrt{3a}\ u^2}\ (3au^3 + \frac{h^2}{4u} + u + uv^2 - hv,\ 3au^3 + \frac{h^2}{4u} - u + uv^2 - hv,\ 2uv - h).$$

Then, the Gauss map satisfies

$$\Delta G = -\frac{1}{6au^4} G.$$

Thus, it has pointwise 1-type Gauss map of the first kind and it is proved to be minimal. We call such a surface a *helicoidal surface of Enneper type.*

Fig 4 : Helicoidal surface of Enneper type



Example 3.5 (Helicoidal surface of hyperbolic type or de Sitter type)**.** Let M be a helicoidal surface parameterized by

$$x(u,v) = (-\frac{a}{u} - \frac{h^2}{4u} + b - u - uv^2 + hv,\ -\frac{a}{u} - \frac{h^2}{4u} + b + u - uv^2 + hv,\ -2uv)$$

for some positive constant a and some constant b. Then, the Gauss map G is given by

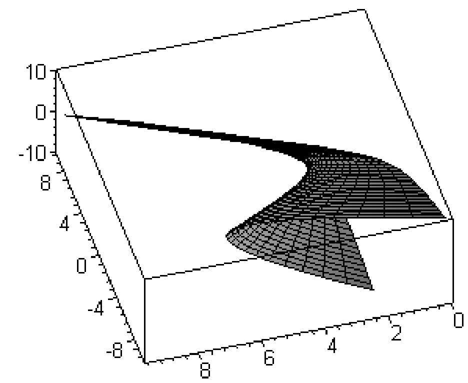
$$G = \frac{1}{2\sqrt{a}}\ (\frac{a}{u} + \frac{h^2}{4u} + u + uv^2 - vh,\ \frac{a}{u} + \frac{h^2}{4u} - u + uv^2 - vh,\ 2uv - h).$$

Hence, the Laplacian ΔG is given by

$$\Delta G = -\frac{1}{2a} G.$$

Thus, it has non-proper pointwise 1-type Gauss map of the first kind. In fact, it is easily proved that M has nonzero constant mean curvature. Such a surface M is called a *helicoidal surface of hyperbolic type.* The surface M is called a helicoidal surface of *de Sitter type* if the constant a is negative.

Fig 5 : Helicoidal surface of de Sitter type



Example 3.6 (Helicoidal surface of parabolic type I^+)**.** Consider a spacelike helicoidal surface parameterized by

$$x(u,v) = (k(u) - u - uv^2 + hv, k(u) + u - uv^2 + hv, -2uv), \ h \neq 0,$$

where $k(u) = -\frac{1}{4}\left(\frac{h^2}{u} + \frac{u}{2(u^2+1)} - \frac{\tan^{-1} u}{2}\right)$. Then, the Gauss map is given by

$$G = \frac{u^2+1}{u^2}\left(\frac{h^2}{4u} + \frac{u^3}{4(u^2+1)^2} + u + uv^2 - vh, \ \frac{h^2}{4u} + \frac{u^3}{4(u^2+1)^2} - u + uv^2 - vh, \ 2uv - h\right).$$

Moreover, its Laplacian satisfies

$$\Delta G = \frac{-2(u^4+1)}{u^4} G.$$

Thus, G is of pointwise 1-type of the first kind. The helicoidal surface parameterized as above is called a *helicoidal surface of parabolic type I^+*.

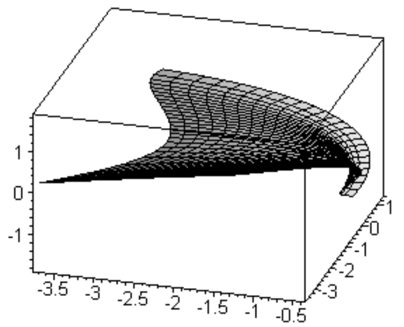
Example 3.7 (Helicoidal surface of parabolic type II^+)**.** Let a helicoidal surface be parameterized by

$$x(u,v) = (k(u) - u - uv^2 + hv, k(u) + u - uv^2 + hv, -2uv), \ h \neq 0,$$

where $k(u) = -\frac{1}{4}\left(\frac{h^2}{u} - \frac{u}{2(u^2+1)} + \frac{\tan^{-1} u}{2}\right)$. Then, it is timelike and the Gauss map G can be expressed by

$$G = \frac{u^2+1}{u^2}\left(\frac{h^2}{4u} - \frac{u^3}{4(u^2+1)^2} + u + uv^2 - vh, \ \frac{h^2}{4u} - \frac{u^3}{4(u^2+1)^2} - u + uv^2 - vh, \ 2uv - h\right).$$

Fig 6 : Helicoidal surface of parabolic type I^+

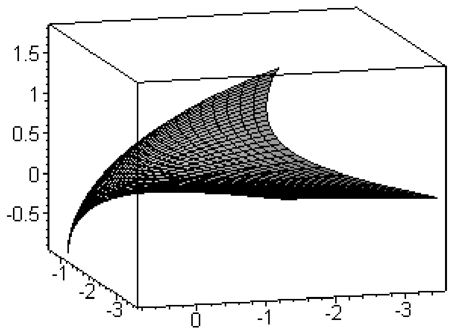


A direct computation yields

$$\Delta G = \frac{2(u^4+1)}{u^4} G.$$

So, it has pointwise 1-type Gauss map of the first kind. The helicoidal surface above is called a *helicoidal surface of parabolic type II^+*.

Fig 7 : Helicoidal surface of parabolic type II^+



4. Fundamental lemma

We prove the following lemma for later use.

Lemma 4.1. *Let M be a helicoidal surface in a Minkowski* 3*-space. If M has pointwise 1-type Gauss map, then the function F in* (1.1) *depends only on u and the vector C is parallel to the axis in $\mathbb{E}^3_1$.*

Proof. We separate three cases of proof according to the character of the axis.

Case 1. Suppose that M is a helicoidal surface in $\mathbb{E}^3_1$ with spacelike axis parameterized by (2.1) for some smooth functions f and g.

First, if f is constant, then the parametrization of M can be written as

$$x(u,v) = (a\sinh v,\ a\cosh v,\ g(u)+hv),\ h\in\mathbb{R} \tag{4.1}$$

for a nonzero constant a. From a direct computation, we see that the Laplacian ΔG of the Gauss map G satisfies $\Delta G = \frac{1}{a^2}G$. Therefore M has non-proper pointwise 1-type Gauss map of the first kind, that is, the constant vector C is zero vector. In this case, M is part of a hyperbolic cylinder.

From now on, we assume that f is not a constant function. Then, we may put $f(u)=u$. Thus, M is parameterized by

$$x(u,v) = (u\sinh v,\ u\cosh v,\ g(u)+hv),\ u>0,\ h\in\mathbb{R}. \tag{4.2}$$

We now suppose that M is timelike. By a direct computation, the Gauss map G and its Laplacian ΔG are obtained as follows:

$$G = \frac{1}{\sqrt{u^2+u^2{g'}^2-h^2}}(-h\cosh v+ug'\sinh v, -h\sinh v+ug'\cosh v, -u) \tag{4.3}$$

and

$$\Delta G = -\frac{1}{(u^2+u^2{g'}^2-h^2)^{\frac{7}{2}}}(A_1\cosh v+B_1\sinh v, A_1\sinh v+B_1\cosh v, D_1), \tag{4.4}$$

where we have put

$$\begin{aligned} A_1 &= A_1(u)\\ &= h\ \{2h^4+4h^4{g'}^2+7h^4g'g''u+(-2h^2-2h^2{g'}^2+h^4{g''}^2+h^4g'g''')u^2\\ &\quad +(-8h^2g'g''+h^2{g'}^3g'')u^3\\ &\quad +(3h^2{g'}^2{g''}^2-h^2{g'}^3g'''-2h^2{g''}^2-2h^2g'g''')u^4\\ &\quad +(g'g''+{g'}^3g'')u^5+({g''}^2-3{g'}^2{g''}^2+g'g'''+{g'}^3g''')u^6\}, \end{aligned} \tag{4.5}$$

$$\begin{aligned} B_1 &= B_1(u)\\ &= -3h^6g''+(-6h^4g'-8h^4{g'}^3-h^6g''')u+(7h^4g''-7h^4{g'}^2g'')u^2\\ &\quad +(7h^2g'+12h^2{g'}^3+5h^2{g'}^5-4h^4g'{g''}^2+3h^4g'''+h^4{g'}^2g''')u^3\\ &\quad +(-5h^2g''+6h^2{g'}^2g''+2h^2{g'}^4g'')u^4+\big(-g'(1+{g'}^2)^3+8h^2g'{g''}^2\\ &\quad -3h^2g'''-2h^2{g'}^2g'''\big)u^5+(g''+{g'}^2g'')u^6+({g'}^2g'''-4g'{g''}^2+g''')u^7 \end{aligned} \tag{4.6}$$

and
(4.7)
$$\begin{aligned} D_1 &= D_1(u) \\ &= u\ \{(2h^4+4h^4{g'}^2)+7h^4g'g''u+(-2h^2-2h^2{g'}^2+h^4{g''}^2+h^4g'g''')u^2 \\ &\quad +(-8h^2g'g''+h^2{g'}^3g'')u^3+(3h^2{g'}^2{g''}^2-h^2{g'}^3g'''-2h^2{g''}^2 \\ &\quad -2h^2g'g''')u^4+(g'g''+{g'}^3g'')u^5+({g''}^2-3{g'}^2{g''}^2+g'g'''+{g'}^3g''')u^6\}. \end{aligned}$$

Suppose that the Gauss map G of M is of pointwise 1-type, that is, (1.1) holds on M. With the help of (4.3) and (4.4), it implies that the function F is independent of the parameter v, that is, F depends only on u. Moreover, the first two components of constant vector C are zero, that is, $C=(0,0,c)$ for some constant c. In other words, the constant vector C is parallel to the axis.

By a similar argument as above, we have the same results in case of spacelike helicoidal surfaces with spacelike axis.

Case 2. Suppose that M is a helicoidal surface with timelike axis parameterized by (2.3) for some smooth functions f and g.

If f is constant, then M is part of an ordinary circular cylinder parameterized by

$$x(u,v)=(g(u)+hv,\ a\cos v,\ a\sin v),\ h\in\mathbb{R} \tag{4.8}$$

for a nonzero constant a. Then, its Gauss map G satisfies $\Delta G=\frac{1}{a^2}G$ and thus it is of non-proper pointwise 1-type of the first kind.

We now consider that f is not constant. Putting $f(u)=u$, the parametrization of M can be written as

$$x(u,v)=(g(u)+hv,\ u\cos v,\ u\sin v),\ u>0,\ h\in\mathbb{R}. \tag{4.9}$$

If M is timelike, we derive the Gauss map G and its Laplacian ΔG as follows:

$$G=\frac{1}{\sqrt{-u^2+u^2{g'}^2+h^2}}(-u,-ug'\cos v+h\sin v,-ug'\sin v-h\cos v) \tag{4.10}$$

and
(4.11)
$$\Delta G=-\frac{1}{(-u^2+u^2{g'}^2+h^2)^{\frac{7}{2}}}(D_2,A_2\sin v+B_2\cos v,-A_2\cos v+B_2\sin v),$$

where A_2, B_2 and D_2 are some functions of u similarly defined by (4.5), (4.6) and (4.7).

If M satisfies (1.1), we see that the function F depends only on u by using (4.10) and (4.11) similarly as we developed in Case 1 and the vector C is parallel to the axis.

The same conclusion can be made in case of spacelike helicoidal surfaces with timelike axis.

Case 3. Suppose that M is a helicoidal surface with null axis parameterized by

$$x(u,v) = (f(u) + \frac{v^2}{2}p(u) + hv,\ g(u) + \frac{v^2}{2}p(u) + hv,\ p(u)v),\ h \in \mathbb{R},$$

where $p(u) = f(u) - g(u) \neq 0$.

Since the induced metric on M is nondegenerate, $(f(u) - g(u))^2(f'^2(u) - g'^2(u)) + h^2(f'(u) - g'(u))^2$ never vanishes and so $f'(u) - g'(u) \neq 0$ everywhere. Thus, we may change the variable in such a way that $p(u) = f(u) - g(u) = -2u$.

Let $k(u) = f(u) + u$. Then, the functions f and g in the profile curve γ look like

$$f(u) = k(u) - u \quad \text{and} \quad g(u) = k(u) + u.$$

Thus, the parametrization of M becomes

$$x(u,v) = (k(u) - u - uv^2 + hv,\ k(u) + u - uv^2 + hv,\ -2uv). \tag{4.12}$$

Since M is nondegenerate, $4u^2k'(u) - h^2 \neq 0$ everywhere it is defined. So, we get the Gauss map G and the Laplacian ΔG of G as follows:

(4.13)

$$G = \frac{1}{\sqrt{|4u^2k'(u) - h^2|}}(uk'(u) + u + uv^2 - vh,\ uk'(u) - u + uv^2 - vh,\ 2uv - h)$$

and

$$\Delta G = -\frac{1}{|4u^2k'(u) - h^2|^{\frac{7}{2}}}\big(2uX + Y,\ -2uX + Y,\ 2(2uv - h)X\big), \tag{4.14}$$

where we have put

(4.15)

$$X = X(u) = h^4 + 4h^2k'u^2 + 9h^2k''u^3 + h^2k'''u^4 - 4k'k''u^5 + 8k''^2u^6 - 4k'k'''u^6$$

and

(4.16)

$$\begin{aligned} Y &= Y(u,v) \\ &= 10h^4k'u + 7h^4k''u^2 - 32h^2k'^2u^3 + h^4k'''u^3 - 14h^2k'k''u^4 + 32k'^3u^5 \\ &\quad + 6h^2k''^2u^5 - 6h^2k'k'''u^5 + 8k'^2k''u^6 - 8k'k''^2u^7 + 8k'^2k'''u^7 - 2h^5v \\ &\quad - 8h^3k'u^2v - 18h^3k''u^3v - 2h^3k'''u^4v + 8hk'k''u^5v - 16hk''^2u^6v \\ &\quad + 8hk'k'''u^6v + 2h^4uv^2 + 8h^2k'u^3v^2 + 18h^2k''u^4v^2 + 2h^2k'''u^5v^2 \\ &\quad - 8k'k''u^6v^2 + 16k''^2u^7v^2 - 8k'k'''u^7v^2. \end{aligned}$$

We now suppose that M is spacelike, that is, $4u^2k'(u) - h^2 > 0$ and the Gauss map G is of pointwise 1-type.

Let $(\Delta G)_i$ be the i-th component of ΔG for $i = 1, 2, 3$. Then, we have

$$(\Delta G)_1 = F(u,v)\big(\frac{uk' + u + uv^2 - vh}{\sqrt{4u^2k' - h^2}} + c_1\big), \tag{4.17}$$

$$(\Delta G)_2 = F(u,v)\Big(\frac{uk' - u + uv^2 - vh}{\sqrt{4u^2k' - h^2}} + c_2\Big), \tag{4.18}$$

$$(\Delta G)_3 = F(u,v)\Big(\frac{2uv - h}{\sqrt{4u^2k' - h^2}} + c_3\Big), \tag{4.19}$$

where $C = (c_1, c_2, c_3)$. Subtracting (4.18) from (4.17), we get

$$-\frac{4uX(u)}{(4u^2k' - h^2)^{\frac{7}{2}}} = F(u,v)\Big(\frac{2u}{\sqrt{4u^2k' - h^2}} + c_1 - c_2\Big). \tag{4.20}$$

Hence, we see that the function F depends only on u, that is, $F(u,v) = F(u)$. Differentiating the equation (4.19) with respect to the parameter v, we obtain

$$-\frac{4uX(u)}{(4u^2k' - h^2)^{\frac{7}{2}}} = F(u)\frac{2u}{\sqrt{4u^2k' - h^2}}. \tag{4.21}$$

With the help of (4.20) and (4.21), we have $c_1 = c_2$ and moreover,

$$F(u) = -\frac{2X(u)}{(4u^2k' - h^2)^3}. \tag{4.22}$$

Putting (4.22) in (4.19), we obtain $c_3 = 0$. This means that the constant vector C is parallel to the axis, that is, $C = (c, c, 0)$ for some constant c.

A similar conclusion is achieved in case that M is timelike, that is, $4u^2k'(u) - h^2 < 0$. Thus, the proof of lemma is completed. □

5. Helicoidal surfaces with spacelike axis in $\mathbb{E}^3_1$

In this section, we study the helicoidal surface with spacelike axis in $\mathbb{E}^3_1$.

Let M be a helicoidal surface with spacelike axis parameterized by (4.2) for some smooth function g, which has pointwise 1-type Gauss map, that is, $\Delta G = F(G + C)$ for a nonzero smooth function F and a constant vector C. By Lemma 4.1, the Laplacian of the Gauss map satisfies $\Delta G = F(u)(G + (0,0,c))$ for some constant c. First, we assume that M is timelike, i.e., $u^2g'^2 + u^2 - h^2 > 0$. Then, from (4.3) and (4.4), we have A_1, B_1 and D_1 as follows

$$A_1 = hF(u^2g'^2 + u^2 - h^2)^3,$$
$$B_1 = -Fug'(u^2g'^2 + u^2 - h^2)^3,$$
$$D_1 = F(u^2g'^2 + u^2 - h^2)^3\Big(u - c\sqrt{u^2g'^2 + u^2 - h^2}\Big).$$

We now assume that M is a genuine helicoidal surface, that is, $h \neq 0$. From the above equations, we have

$$B_1 + ug'\frac{A_1}{h} = 0 \quad \text{and} \quad D_1 = \frac{A_1}{h}\Big(u - c\sqrt{u^2g'^2 + u^2 - h^2}\Big). \tag{5.1}$$

With the help of (4.5), (4.7) and (5.1), we get $cF(u^2g'^2 + u^2 - h^2)^{\frac{7}{2}} = 0$. Since F is a nonzero function and $u^2g'^2 + u^2 - h^2 \neq 0$ everywhere, we obtain $c = 0$.

Therefore, the Gauss map G is of pointwise 1-type of the first kind. Similarly, we can derive the same conclusion for the spacelike case. Thus, we have:

Theorem 5.1. *Let M be a genuine helicoidal surface with spacelike axis in a Minkowski 3-space $\mathbb{E}^3_1$. If the Gauss map G of M is of pointwise 1-type, then it is of the first kind, that is, the Gauss map satisfies the equation $\Delta G = FG$ for some nonzero smooth function F.*

By Lemma 5.1 in [13], if M is a surface in a Minkowski 3-space $\mathbb{E}^3_1$ with pointwise 1-type Gauss map of the first kind, then its mean curvature is constant. Hence, we have:

Corollary 5.2. *Let M be a genuine helicoidal surface with spacelike axis in a Minkowski 3-space $\mathbb{E}^3_1$. Then M has pointwise 1-type Gauss map if and only if M has constant mean curvature.*

We now prove:

Theorem 5.3. *A genuine helicoidal surface M with spacelike axis in a Minkowski 3-space $\mathbb{E}^3_1$ has pointwise 1-type Gauss map if and only if it is an open part of either a hyperbolic cylinder parameterized by* (4.1) *or the surface parameterized by*

$$x(u,v) = (u \sinh v,\ u \cosh v,\ g(u) + hv),\ u > 0,\ h \neq 0,$$

where

$$g(u) = \begin{cases} \pm \int (\alpha u + \frac{a}{u}) \sqrt{\frac{u^2 - h^2}{u^2 - (\alpha u^2 + a)^2}}\, du & \text{if } M \text{ is timelike,} \\ \pm \int (\alpha u + \frac{a}{u}) \sqrt{\frac{-u^2 + h^2}{u^2 + (\alpha u^2 + a)^2}}\, du & \text{if } M \text{ is spacelike} \end{cases}$$

for some constant a and constant mean curvature α.

Proof. Suppose that M is a genuine helicoidal surface with spacelike axis in $\mathbb{E}^3_1$ with pointwise 1-type Gauss map. Then, by Theorem 5.1 and Corollary 5.2, it is of the first kind and mean curvature is constant.

If the function f in the parametrization (2.1) is constant, then M is part of a hyperbolic cylinder as is shown in Lemma 4.1.

We now suppose that f is not constant. By (4.2), the parametrization of M is given by

$$x(u,v) = (u \sinh v,\ u \cosh v,\ g(u) + hv),\ u > 0,\ h \in \mathbb{R}.$$

First, consider the case that M is timelike, that is, $-u^2 + h^2 - u^2 {g'}^2 < 0$. Then, M has constant mean curvature α if and only if $g = g(u)$ is a solution of the following differential equation

$$h^2 u g'' - u^3 g'' + 2h^2 g' - u^2 g' - u^2 {g'}^3 = 2\alpha (u^2 - h^2 + u^2 {g'}^2)^{\frac{3}{2}}. \tag{5.2}$$

If $u^2 - h^2 > 0$, we put $u^2 - h^2 = w^2$. By the change of variables $\frac{u}{w} g' = \tan y$, the equation (5.2) becomes

$$-w^3 \sec^2 y\ y' - \frac{w^3}{u} \tan y \sec^2 y = 2\alpha w^3 \sec^3 y.$$

It follows

$$-u\cos y\ y' - \sin y = 2\alpha u,$$

which yields $\sin y = -\alpha u - \frac{a}{u}$ for some constant a. Thus, the function $g(u)$ is obtained by

$$g(u) = \pm\int (\alpha u + \frac{a}{u})\sqrt{\frac{u^2-h^2}{u^2-(\alpha u^2+a)^2}}\ du.$$

Similarly, if $u^2-h^2<0$, the change of variables $u^2-h^2=-w^2$ and $\frac{u}{w}g' = \sec y$ enables (5.2) to be

$$w^3\sec y\tan y\ y' - \frac{w^3}{u}\sec y\tan^2 y = 2\alpha w^3\tan^3 y.$$

From this, the same result as above is derived. Consequently, if M is a timelike surface with constant mean curvature α, we have

$$g(u) = \pm\int (\alpha u + \frac{a}{u})\sqrt{\frac{u^2-h^2}{u^2-(\alpha u^2+a)^2}}\ du.$$

We now suppose that M is spacelike, that is, $-u^2+h^2-u^2{g'}^2>0$. Then, M has constant mean curvature α if and only if $g=g(u)$ is a solution of the following differential equation

$$(5.3)\qquad -h^2ug''+u^3g''-2h^2g'+u^2g'+u^2{g'}^3 = 2\alpha(-u^2+h^2-u^2{g'}^2)^{\frac{3}{2}}.$$

By the change of variables $-u^2+h^2=w^2$ and $\frac{u}{w}g'=\cos y$, the equation (5.3) yields

$$g(u) = \pm\int (\alpha u + \frac{a}{u})\sqrt{\frac{-u^2+h^2}{u^2+(\alpha u^2+a)^2}}\ du.$$

Thus, we complete the proof. □

Furthermore, if $a=\alpha=0$, then g is constant. In this case, the parametrization of M becomes

$$x(u,v) = (u\sinh v,\ u\cosh v,\ b+hv),\ h\neq 0,\ b\in\mathbb{R}.$$

It is nothing but a right helicoid of type I in $\mathbb{E}^3_1$.
If $\alpha=0$ and $a\neq 0$, then we obtain

$$g(u) = \begin{cases} \pm\int \frac{a}{u}\sqrt{\frac{u^2-h^2}{u^2-a^2}}\ du & \text{if } M \text{ is timelike},\\ \pm\int \frac{a}{u}\sqrt{\frac{h^2-u^2}{u^2+a^2}}\ du & \text{if } M \text{ is spacelike}.\end{cases}$$

If $\alpha\neq 0$ and $a=0$, then we get

$$g(u) = \begin{cases} \pm\int \alpha\sqrt{\frac{u^2-h^2}{1-\alpha^2u^2}}\ du & \text{if } M \text{ is timelike},\\ \pm\int \alpha\sqrt{\frac{h^2-u^2}{1+\alpha^2u^2}}\ du & \text{if } M \text{ is spacelike}.\end{cases}$$

In this case, $g(u)$ cannot be expressed as a rational function unless g is a constant.

If $\alpha a \neq 0$, the function $g(u)$ cannot be expressed as a rational function.

In such a case that $g(u)$ is a rational function, M is said to be of *rational kind.*

Thus, we conclude the following:

Corollary 5.4. *A genuine helicoidal surface of rational kind with spacelike axis in a Minkowski 3-space $\mathbb{E}^3_1$ has pointwise 1-type Gauss map if and only if it is an open part of either a hyperbolic cylinder or a right helicoid of type I in $\mathbb{E}^3_1$.*

Combining the results of the above theorem and a characterization of surfaces of revolution with pointwise 1-type Gauss map in a Minkowski 3-space ([11]), we have a characterization of helicoidal surfaces of rational kind with pointwise 1-type Gauss map in $\mathbb{E}^3_1$.

Corollary 5.5 (Characterization)**.** *A helicoidal surface of rational kind with spacelike axis has pointwise 1-type Gauss map in a Minkowski 3-space if and only if it is part of a hyperbolic cylinder, a hyperbolic cone or a right helicoid of type I in a Minkowski 3-space.*

6. Helicoidal surfaces with timelike axis in $\mathbb{E}^3_1$

In this section, we examine the helicoidal surface with timelike axis in $\mathbb{E}^3_1$.

Let M be a helicoidal surface with timelike axis parameterized by (4.9) for some smooth function g. If M has pointwise 1-type Gauss map, then by the Lemma 4.1, the Laplacian of the Gauss map satisfies

$$\Delta G = F(u)(G + (c, 0, 0))$$

for some constant c. If M is timelike, we have the functions of u, A_2, B_2 and D_2 from (4.10) and (4.11)

$$\begin{aligned} A_2 &= -hF(u^2 g'^2 - u^2 + h^2)^3, \\ B_2 &= Fug'(u^2 g'^2 - u^2 + h^2)^3, \\ D_2 &= F(u^2 g'^2 - u^2 + h^2)^3 \Big(u - c\sqrt{u^2 g'^2 - u^2 + h^2}\Big). \end{aligned}$$

We now assume that M is genuine, that is, $h \neq 0$. From the above equations, we get

$$D_2 = -\frac{A_2}{h}\Big(u - c\sqrt{u^2 g'^2 - u^2 + h^2}\Big). \tag{6.1}$$

Using a simple algebraic calculation, we have

$$D_2 = -\frac{A_2}{h} u. \tag{6.2}$$

Combining (6.1) and (6.2), we have $cF(u^2 g'^2 - u^2 + h^2)^{7/2} = 0$. Since the function F is nonzero and $u^2 g'^2 - u^2 + h^2 \neq 0,\ c = 0$. Similarly, we can deal with the matters if M is spacelike. This means that the constant vector C is zero vector and thus we have:

Theorem 6.1. *If a genuine helicoidal surface with timelike axis in a Minkowski 3-space $\mathbb{E}^3_1$ has pointwise 1-type Gauss map, then it is of the first kind.*

We immediately obtain the following:

Corollary 6.2. *Let M be a genuine helicoidal surface with timelike axis in $\mathbb{E}^3_1$. Then, M has pointwise 1-type Gauss map if and only if M has constant mean curvature.*

We also get the following theorem.

Theorem 6.3. *Let M be a genuine helicoidal surface with timelike axis in a Minkowski 3-space $\mathbb{E}^3_1$. Then, M has pointwise 1-type Gauss map if and only if M is an open part of either a circular cylinder parameterized by (4.8) or the surface parameterized by*

$$x(u,v) = (g(u) + hv,\ u\cos v,\ u\sin v),\ u > 0,\ h \neq 0,$$

where

$$g(u) = \begin{cases} \pm\int(\alpha u + \frac{a}{u})\sqrt{\frac{h^2-u^2}{u^2-(\alpha u^2+a)^2}}\,du & \text{if } M \text{ is timelike,} \\ \pm\int(\alpha u + \frac{a}{u})\sqrt{\frac{u^2-h^2}{u^2+(\alpha u^2+a)^2}}\,du & \text{if } M \text{ is spacelike} \end{cases}$$

for some constant a and constant mean curvature α.

Proof. Suppose that a genuine helicoidal surface M with timelike axis in $\mathbb{E}^3_1$ has pointwise 1-type Gauss map.

If the function f in the parametrization (2.3) is constant, M is an open part of a circular cylinder which is proved in Lemma 4.1.

If f is not constant, we assume M is parameterized by (4.9).

First, we consider the case that M is timelike, that is, $u^2 - h^2 - u^2g'^2 < 0$. Similarly to proof of Theorem 5.3, M has constant mean curvature α if and only if $g = g(u)$ is a solution of the following differential equation

$$h^2ug'' - u^3g'' + 2h^2g' - u^2g' + u^2g'^3 = 2\alpha(-u^2 + h^2 + u^2g'^2)^{\frac{3}{2}}. \tag{6.3}$$

If $h^2 - u^2 > 0$, we put $h^2 - u^2 = w^2$. If we put $\frac{u}{w}g' = \tan y$, the equation (6.3) is reduced to

$$y' + \frac{1}{u}\tan y = 2\alpha\sec y.$$

It yields $\sin y = \alpha u + \frac{a}{u}$ for some constant a. Thus, (6.3) gives rise to

$$g(u) = \pm\int(\alpha u + \frac{a}{u})\sqrt{\frac{h^2-u^2}{u^2-(\alpha u^2+a)^2}}\,du.$$

If $h^2 - u^2 < 0$, we put $h^2 - u^2 = -w^2$. By the change of variables such as $\frac{u}{w}g' = \sec y$, the equation (6.3) becomes

$$-u\csc y\cot y\ y' + \csc y = 2\alpha u.$$

The same result could be derived as the case above.

Consequently, if M is timelike, $g(u)$ is obtained by

$$g(u) = \pm \int (\alpha u + \frac{a}{u}) \sqrt{\frac{h^2 - u^2}{u^2 - (\alpha u^2 + a)^2}}\, du.$$

Next, we consider that M is spacelike, that is, $u^2 - h^2 - u^2 g'^2 > 0$. Like the previous case, M has constant mean curvature α if and only if $g = g(u)$ is a solution of the following differential equation

$$(6.4) \qquad -h^2 u g'' + u^3 g'' - 2h^2 g' + u^2 g' - u^2 g'^3 = 2\alpha(u^2 - h^2 - u^2 g'^2)^{\frac{3}{2}}.$$

Since $u^2 - h^2 > 0$, we put $u^2 - h^2 = w^2$. If we make an appropriate change of variables like $\frac{u}{w} g' = \sin y$, (6.4) becomes

$$u \sec^2 y\ y' + \tan y = 2\alpha u,$$

from which,

$$g(u) = \pm \int (\alpha u + \frac{a}{u}) \sqrt{\frac{u^2 - h^2}{u^2 + (\alpha u^2 + a)^2}}\, du$$

for some constant a. Thus, it completes the proof. □

We now examine the cases upon the constants α and a.
If $a = \alpha = 0$, then g is constant. In this case, the surface M is part of a right helicoid of type II.
If $\alpha = 0$ and $a \neq 0$, $g(u)$ is given by

$$g(u) = \begin{cases} \pm \int \frac{a}{u} \sqrt{\frac{u^2 - h^2}{a^2 - u^2}}\, du & \text{if } M \text{ is timelike,} \\ \pm \int \frac{a}{u} \sqrt{\frac{u^2 - h^2}{u^2 + a^2}}\, du & \text{if } M \text{ is spacelike.} \end{cases}$$

In this case, the function $g(u)$ cannot be a rational function.
If $\alpha \neq 0$ and $a = 0$, $g(u)$ is given by

$$g(u) = \begin{cases} \pm \int \alpha \sqrt{\frac{u^2 - h^2}{\alpha^2 u^2 - 1}}\, du & \text{if } M \text{ is timelike,} \\ \pm \int \alpha \sqrt{\frac{u^2 - h^2}{1 + \alpha^2 u^2}}\, du & \text{if } M \text{ is spacelike.} \end{cases}$$

The function $g(u)$ is rational only if $a = 0$ and $h^2\alpha^2 = 1$. In this case, $g(u) = \pm u + b$ for some constant b. Thus, the parametrization of M turns out to be

$$(6.5) \qquad x(u, v) = (\pm u + b + hv,\ u \cos v,\ u \sin v), \quad h \neq 0,\ b \in \mathbb{R}.$$

In this case, the surface is a helicoidal surface of elliptic type. Moreover, if $\alpha a \neq 0$, g is expressed as some elliptic functions or hypergeometric functions.

Thus, we have the following:

Corollary 6.4. *Let M be a genuine helicoidal surface of rational kind with timelike axis in $\mathbb{E}^3_1$. Then, it has pointwise 1-type Gauss map if and only if it is an open part of either right helicoid of type II or helicoidal surface of elliptic type of the form* (6.5).

Together with Theorem 6.3 and the results in [11], we have:

Corollary 6.5 (Characterization). *Let M be a helicoidal surface of rational kind with timelike axis in $\mathbb{E}^3_1$. Then, M has pointwise 1-type Gauss map if and only if it is an open part of a circular cylinder, a right cone, a right helicoid of type II or a helicoidal surface of elliptic type in $\mathbb{E}^3_1$.*

7. Helicoidal surfaces with null axis in $\mathbb{E}^3_1$

In this section, we study the helicoidal surface with null axis in $\mathbb{E}^3_1$.

Let M be a helicoidal surface with null axis parameterized by (4.12). Suppose that M is a spacelike helicoidal surface of rational kind with pointwise 1-type Gauss map of the second kind. Then, the Gauss map G satisfies the condition $\Delta G = F(u)(G + (c, c, 0))$ for a nonzero constant c by Lemma 4.1. Let $(\Delta G)_i$ be the i-th component of ΔG for $i = 1, 2, 3$. From equation (4.13), we have

$$(\Delta G)_1 = F(u)\big(\frac{uk' + u + uv^2 - vh}{\sqrt{4u^2k' - h^2}} + c\big), \tag{7.1}$$

$$(\Delta G)_2 = F(u)\big(\frac{uk' - u + uv^2 - vh}{\sqrt{4u^2k' - h^2}} + c\big), \tag{7.2}$$

$$(\Delta G)_3 = F(u)\big(\frac{2uv - h}{\sqrt{4u^2k' - h^2}}\big). \tag{7.3}$$

Therefore, from the above equations with (4.14), we easily obtain

$$F(u) = -\frac{2X(u)}{(4u^2k' - h^2)^3},$$

where $X(u)$ is defined by (4.15). Putting this function $F(u)$ in (7.1) or (7.2), with the help of (4.14) (4.15) and (4.16), we obtain

(7.4)
$$\begin{aligned}&u\sqrt{4u^2k' - h^2}(8h^2k' + 7h^2k''u - 8k'^2u^2 + h^2k'''u^2 - 4k'k''u^3 + 6k''^2u^4\\&- 4k'k'''u^4) + 2c(h^4 + 4h^2k'u^2 + 9h^2k''u^3 + h^2k'''u^4 - 4k'k''u^5 + 8k''^2u^6\\&- 4k'k'''u^6) = 0.\end{aligned}$$

Since $k(u)$ is a rational function, $Q(u) = \sqrt{4u^2k' - h^2}$ is also a rational function because of (7.4). If we rearrange (7.4) with respect to Q, we have

$$2uQ^3Q'^2 - uQ^4Q'' - 3Q^4Q' = -2c(2Q^4 - 5uQ^3Q' + 3u^2Q^2Q'^2 - u^2Q^3Q''). \tag{7.5}$$

From now on, we regard the rational function Q as a complex meromorphic function. Let $Q(z) = \frac{q(z)}{p(z)}$, where p and q are relatively prime polynomials.

First, we show that $q(z) = az^m$ for some constant a and a positive integer m. Suppose $q(z_0) = 0$. It implies that $Q(z_0) = 0$. Then

$$Q(z) = \sum_{n=k}^{\infty} a_n (z - z_0)^n$$

for some $k \geq 1$ and $a_k \neq 0$. If $z_0 \neq 0, z = z_0 + (z - z_0)$ and $z^2 = {z_0}^2 + 2z_0(z - z_0) + (z - z_0)^2$. If we compare the lowest degree of both sides of (7.5) after putting z in (7.5) instead of u, we see that the lowest degree of the left hand side of (7.5) is $5k - 2$ and that of the right hand side is $4k - 2$. Hence the coefficient of term of degree $4k - 2$ is zero, that is, $-2c(2k^2 + k){z_0}^2{a_k}^4 = 0$, which is a contradiction. Thus, $q(z) = az^m$ for some constant a and a positive integer m and so

$$Q(z) = \frac{az^m}{p(z)}.$$

Let $p(z) = z^k + a_1 z^{k-1} + a_2 z^{k-2} + \cdots + a_k$. Since p and q are relatively prime polynomials, $a_k \neq 0$. The series expansion of $Q(z)$ at $z = 0$ looks like $Q(z) = az^l + a_1 z^{l+1} + a_2 z^{l+2} + \cdots,\ l \geq 1$. Then, the lowest degree of the right hand side of (7.5) is $5l - 1$ and that of the left hand side is $4l$. If $l > 1$, then the coefficient of term with degree $4l$ must be zero, that is, $-4ca^4(l-1)^2 = 0$, this is also a contradiction. If $l = 1$, then the lowest degree of the left hand side of (7.5) is 5 and that of the right hand side is 4. Hence, the coefficient of term with degree 4 must be zero, that is, $a = 0$. Therefore, $Q(z) = 0$, a contradiction. Thus, $m = 0$ and $Q(z) = \frac{a}{p(z)}$. The polynomial $p(z)$ can be written as $p(z) = (z - \alpha_1)(z - \alpha_2) \cdots (z - \alpha_k)$ for some complex numbers $\alpha_1, \alpha_2, \ldots, \alpha_k$ $(\alpha_k \geq 1)$. Since

$$\frac{1}{z - \alpha_1} = \frac{1}{z} + \frac{\alpha_1}{z^2} + \frac{{\alpha_1}^2}{z^3} + \cdots \ (|z| > |\alpha_1|),$$

the complex meromorphic function $Q(z)$ has the form

$$Q(z) = \frac{a}{p(z)} = \frac{a}{z^k} + \frac{a_1}{z^{k+1}} + \frac{a_2}{z^{k+2}} + \cdots \ (|z| > r) \tag{7.6}$$

for some $r > 0$. Putting (7.6) into (7.5) and comparing the degrees of terms in the both sides, the lowest degree of terms in $1/z$'s of the left hand side is $4k+2$ and that of the right hand side is $4k$. Therefore, the coefficient in the term with degree $4k$ in $1/z$ must be zero. In other words, $-2ca^4(k+1)^2(k+2) = 0$. Hence, $a = 0$ and so $Q(z) = 0$. This is a contradiction. Consequently, if $c \neq 0$, (7.5) does not hold if Q is a rational function. The case of timelike surface with null axis is similarly dealt with.

Thus, we have:

Theorem 7.1. *Let M be a helicoidal surface of rational kind with null axis in a Minkowski 3-space $\mathbb{E}^3_1$. Then, there exists no helicoidal surface of rational kind with pointwise 1-type Gauss map of the second kind. In other words, if the Gauss map is of pointwise 1-type, then it must be of the first kind.*

Next, we prove

Theorem 7.2. *Let M be a genuine helicoidal surface with null axis in $\mathbb{E}^3_1$. Then M has pointwise 1-type Gauss map of the first kind if and only if it is an open part of a helicoidal surface of Enneper type, a helicoidal surface of hyperbolic type, a helicoidal surface of de Sitter type or a helicoidal surface of parabolic type.*

Proof. Let M be a genuine helicoidal surface with null axis in $\mathbb{E}^3_1$ parameterized by (4.12). Then, M has pointwise 1-type Gauss map of the first kind if and only if M has constant mean curvature α.

First, consider M is spacelike, that is, $4u^2k' - h^2 > 0$.

Since M has constant mean curvature α, we get

$$k''u^3 - 2k'u^2 + h^2 + \alpha(4u^2k' - h^2)^{3/2} = 0. \tag{7.7}$$

If $\alpha = 0$, that is, M is minimal, $k(u)$ is obtained by

$$k(u) = au^3 - \frac{h^2}{4u} + b$$

for some constants $a > 0$ and b. Therefore, the parametrization of M can be reduced to

$$x(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} -u+au^3 \\ u+au^3 \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{h^2}{4u} + hv + b \\ -\frac{h^2}{4u} + hv + b \\ 0 \end{pmatrix}.$$

Thus, it is part of a spacelike helicoidal surface of Enneper type.

Suppose $\alpha \neq 0$. The equation (7.7) is a Bernoulli's differential equation and can be solved as

$$k(u) = \frac{1}{4}\int \left(\frac{u^2}{(\alpha u^2 + a)^2} + \frac{h^2}{u^2} \right) du$$

for some constant a. If $a = 0$, then $k(u) = -\frac{1}{4\alpha^2 u} - \frac{h^2}{4u} + b$ for some constant b. Thus, the parametrization of M is reduced to

$$x(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} -u-\frac{1}{4\alpha^2 u} \\ u-\frac{1}{4\alpha^2 u} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{h^2}{4u} + hv + b \\ -\frac{h^2}{4u} + hv + b \\ 0 \end{pmatrix},$$

which is part of a helicoidal surface of hyperbolic type.

If $\alpha a \neq 0$, then $k(u)$ is given by

$$k(u) = -\frac{1}{4}\left(\frac{h^2}{u} + \frac{u}{2\alpha(\alpha u^2 + a)} - \frac{\tan^{-1}(\sqrt{\frac{\alpha}{a}}u)}{2\sqrt{|a|}\,|\alpha|^{3/2}} \right) \quad \text{if} \quad \alpha a > 0$$

or

$$k(u) = -\frac{1}{4}\left(\frac{h^2}{u} + \frac{u}{2\alpha(\alpha u^2 - a)} + \frac{\tanh^{-1}(\sqrt{|\frac{\alpha}{a}|}u)}{2\sqrt{|a|}\,|\alpha|^{3/2}} \right) \quad \text{if} \quad \alpha a < 0.$$

In this case, if $\alpha a > 0$, the parametrization of M can be written as

$$x(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} -u - \frac{u}{8\alpha(\alpha u^2+a)} + \frac{\tan^{-1}(\sqrt{\frac{\alpha}{a}}u)}{8\sqrt{|a|}\ |\alpha|^{3/2}} \\ u - \frac{u}{8\alpha(\alpha u^2+a)} + \frac{\tan^{-1}(\sqrt{\frac{\alpha}{a}}u)}{8\sqrt{|a|}\ |\alpha|^{3/2}} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{h^2}{4u} + hv + b \\ -\frac{h^2}{4u} + hv + b \\ 0 \end{pmatrix},$$

or, if $\alpha a < 0$,

$$x(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} -u - \frac{u}{8\alpha(\alpha u^2-a)} - \frac{\tanh^{-1}(\sqrt{|\frac{\alpha}{a}|}u)}{8\sqrt{|a|}\ |\alpha|^{3/2}} \\ u - \frac{u}{8\alpha(\alpha u^2-a)} - \frac{\tanh^{-1}(\sqrt{|\frac{\alpha}{a}|}u)}{8\sqrt{|a|}\ |\alpha|^{3/2}} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{h^2}{4u} + hv + b \\ -\frac{h^2}{4u} + hv + b \\ 0 \end{pmatrix}.$$

We call such a surface a helicoidal surface of parabolic type I^+ or I^- according to the signature of αa.

Similarly we can deal with the case that M is timelike, that is, $4u^2k' - h^2 < 0$.

If $\alpha = 0$, then M is nothing but part of a timelike helicoidal surface of Enneper type. In this case, $k(u)$ is obtained by $k(u) = au^3 - \frac{h^2}{4u} + b$ for some constants $a < 0$ and b.

If $\alpha \neq 0$, then, $k(u)$ is given by

$$k(u) = \frac{1}{4}\int\left(-\frac{u^2}{(\alpha u^2+a)^2} + \frac{h^2}{u^2}\right) du$$

for some constant a. If $a = 0$, then $k(u) = \frac{1}{4\alpha^2 u} - \frac{h^2}{4u} + b$ for some constant b. Hence, the parametrization of M is reduced to

$$x(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} -u + \frac{1}{4\alpha^2 u} \\ u + \frac{1}{4\alpha^2 u} \\ 0 \end{pmatrix} + \begin{pmatrix} -\frac{h^2}{4u} + hv + b \\ -\frac{h^2}{4u} + hv + b \\ 0 \end{pmatrix},$$

which is part of a helicoidal surface of de Sitter type.

If $\alpha a \neq 0$, then $k(u)$ is given by

$$k(u) = -\frac{1}{4}\left(\frac{h^2}{u} - \frac{u}{2\alpha(\alpha u^2+a)} + \frac{\tan^{-1}(\sqrt{\frac{\alpha}{a}}u)}{2\sqrt{|a|}\ |\alpha|^{3/2}}\right) \quad \text{if} \quad \alpha a > 0$$

or

$$k(u) = -\frac{1}{4}\left(\frac{h^2}{u} - \frac{u}{2\alpha(\alpha u^2-a)} - \frac{\tanh^{-1}(\sqrt{|\frac{\alpha}{a}|}u)}{2\sqrt{|a|}\ |\alpha|^{3/2}}\right) \quad \text{if} \quad \alpha a < 0.$$

Hence, if $\alpha a > 0$, the parametrization of M can be expressed as

$$x(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} -u + \frac{u}{8\alpha(\alpha u^2+a)} - \frac{\tan^{-1}(\sqrt{\frac{\alpha}{a}}u)}{8\sqrt{|a|}\ |\alpha|^{3/2}} \\ u + \frac{u}{8\alpha(\alpha u^2+a)} - \frac{\tan^{-1}(\sqrt{\frac{\alpha}{a}}u)}{8\sqrt{|a|}\ |\alpha|^{3/2}} \\ 0 \end{pmatrix}$$
$$+ \begin{pmatrix} -\frac{h^2}{4u} + hv + b \\ -\frac{h^2}{4u} + hv + b \\ 0 \end{pmatrix}$$

or, if $\alpha a < 0$,

$$x(u,v) = \begin{pmatrix} 1+\frac{v^2}{2} & -\frac{v^2}{2} & v \\ \frac{v^2}{2} & 1-\frac{v^2}{2} & v \\ v & -v & 1 \end{pmatrix} \begin{pmatrix} -u + \frac{u}{8\alpha(\alpha u^2-a)} + \frac{\tanh^{-1}(\sqrt{|\frac{\alpha}{a}|}u)}{8\sqrt{|a|}\ |\alpha|^{3/2}} \\ u + \frac{u}{8\alpha(\alpha u^2-a)} + \frac{\tanh^{-1}(\sqrt{|\frac{\alpha}{a}|}u)}{8\sqrt{|a|}\ |\alpha|^{3/2}} \\ 0 \end{pmatrix}$$
$$+ \begin{pmatrix} -\frac{h^2}{4u} + hv + b \\ -\frac{h^2}{4u} + hv + b \\ 0 \end{pmatrix}.$$

A timelike surface M described above is called a helicoidal surface of parabolic type II^+ or II^- according to the signature of αa.

The converse is very straightforward. It completes the proof. □

In particular, if $\alpha a \neq 0$, then $k(u)$ cannot be expressed as a rational function. Therefore, we have:

Corollary 7.3. *Let M be a genuine rational helicoidal surface with null axis in $\mathbb{E}^3_1$. Then M has pointwise 1-type Gauss map if and only if it is an open part of a helicoidal surface of Enneper type, a helicoidal surface of hyperbolic type or a helicoidal surface of de Sitter type.*

Putting together with the results described above and theorems in [11], we give a following characterization.

Corollary 7.4. (Characterization) *A helicoidal surface of rational kind with null axis in a Minkowski 3-space has pointwise 1-type Gauss map if and only if it is part of Enneper's surface of second kind, a de-Sitter space, a hyperbolic space, a helicoidal surface of Enneper type, a helicoidal surface of hyperbolic type or a helicoidal surface of de Sitter type in $\mathbb{E}^3_1$.*

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Miekyung Choi
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail address: mkchoi@knu.ac.kr

Young Ho Kim
Department of Mathematics
Kyungpook National University
Taegu 702-701, Korea
E-mail address: yhkim@knu.ac.kr

Huili Liu
Department of Mathematics
Northeastern University
Shenyang 110004, P. R. China
E-mail address: liuhl@mail.neu.edu.cn

Dae Won Yoon
Department of Mathematics education and RINS
Gyeongsang National University
Jinju 660-701, Korea
E-mail address: dwyoon@gsnu.ac.kr