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CONTINUOUS CHARACTERIZATION OF THE TRIEBEL-LIZORKIN SPACES AND FOURIER MULTIPLIERS

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Abstract. We give a set of continuous characterizations for the homogeneous Triebel-Lizorkin spaces and use them to study boundedness properties of Fourier multiplier operators whose symbols satisfy a generalization of Hörmander's condition. As an application, we give new direct proofs of the imbedding theorems of the Sobolev type.

1. Introduction

In the present article we aim at establishing a set of new characterizations for the homogeneous Triebel-Lizorkin spaces on $\mathbb{R}^n$ and studying a class of Fourier multipliers by making use of our characterizations.

For a systematic approach, let $\mathcal{A}$ denote the class of Schwartz function φ on $\mathbb{R}^n$ such that its Fourier transform $\widehat{\varphi}$ has support in $\{1/2 \le |\xi| \le 2\}$ and $|\widehat{\varphi}(\xi)| \ge c > 0$ for $3/5 \le |\xi| \le 5/3$. Given a triple of parameters $\alpha \in \mathbb{R}$ and $0 < p < \infty$, $0 < r \le \infty$, we recall ([14], [18]) that a tempered distribution f belongs to the homogeneous Triebel-Lizorkin space $\dot{F}^{\alpha}_{p,\,r}$, modulo polynomials, if the quasi-norm

$$\|f\|_{\dot{F}^{\alpha}_{p,\,r}} = \left\| \left(\sum_{j\in\mathbb{Z}} \left(2^{j\alpha}\, |f * \varphi_{2^{-j}}|\right)^r \right)^{1/r} \right\|_{L^p} \tag{1.1}$$

is finite, where $\varphi \in \mathcal{A}$ and $\varphi_{2^{-j}}(x) = 2^{jn}\varphi(2^j x)$, with the usual interpretation for $r = \infty$. An extension to the case $p = \infty$ reads as

$$\|f\|_{\dot{F}^{\alpha}_{\infty,\,r}} = \sup_Q \left(\frac{1}{|Q|} \int_Q \sum_{2^{-j} \le l(Q)} \left(2^{j\alpha}\, |f * \varphi_{2^{-j}}(x)|\right)^r dx \right)^{1/r}, \tag{1.2}$$

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where the supremum is taken over all dyadic cubes Q (see [9]). A different choice of φ in both definitions yields equivalent quasi-norms as long as it is taken from the class $\mathcal{A}$.

In contrast to these discrete definitions involving sums, there are continuous counterparts involving integrals. As an example, it is shown by H. Bui, M. Paluszyński, and M. Taibleson ([1]) that

$$\|f\|_{\dot{F}^{\alpha}_{p,\,r}} \approx^{1} \left\| \left(\int_0^\infty \left(t^{-\alpha} \left| f * \varphi_t \right| \right)^r \frac{dt}{t} \right)^{1/r} \right\|_{L^p} \tag{1.3}$$

for the same range of parameters as in the definition (1.1), where $\varphi \in \mathcal{A}$ and $\varphi_t(x) = t^{-n}\varphi(x/t)$ for each $t > 0$. An inspection shows that the characterizing means defined on the right side of (1.3) are nothing but simple variants of the classical Littlewood-Paley g-functions which arise in the theory of H^p spaces, the real-variable Hardy spaces on $\mathbb{R}^n$. In fact, setting $v(x,t) = t^{-\alpha}(f * \varphi_t)(x)$, we have

$$\left(\int_0^\infty \left(t^{-\alpha} \left| f * \varphi_t(x) \right| \right)^r \frac{dt}{t} \right)^{1/r} = \left[g\left(v^{r/2} \right)(x) \right]^{2/r}, \tag{1.4}$$

where g denotes the Littlewood-Paley operator

$$g(u)(x) = \left(\int_0^\infty |u(x,t)|^2 \frac{dt}{t} \right)^{1/2}$$

defined for each continuous function $u(x,t)$ on $\mathbb{R}^{n+1}_+$ ([15], [11]).

In the light of this motivational example and observation, we shall characterize Triebel-Lizorkin spaces in terms of variants of the characterizing means for H^p or BMO spaces. As we shall describe the details in Section 3, those variants in question will be obtained from the Littlewood-Paley g_λ-functions, Lusin S-functions and certain kinds of maximal functions by applying the same rule explained as in (1.4).

In dealing with Triebel-Lizorkin spaces, it is important to understand the basic structure determined by the values of parameters. In the first place, for fixed $p, r,$ the spaces $\dot{F}^{\alpha}_{p,\,r}$ with parameters $\alpha \in \mathbb{R}$ can be viewed as a scale of potential spaces in the following sense. Let I_α denote the Riesz potentials defined by

$$(I_\alpha f)\widehat{\ }(\xi) = |\xi|^{-\alpha}\, \hat{f}(\xi) \tag{1.5}$$

with the agreement that I_0 corresponds to the identity. It is plain to observe that $\dot{F}^{\alpha}_{p,\,r} = I_\alpha(\dot{F}^{0}_{p,\,r})$, the image spaces of $\dot{F}^{0}_{p,\,r}$ under the transformations I_α. With the aid of certain singular integrals, it is in fact possible to identify each $\dot{F}^{\alpha}_{p,\,r}$ as the $\dot{F}^{0}_{p,\,r}$-Sobolev space of order α when $\alpha > 0$. While these facts are more or less known, we shall give a precise description in the next section.

[1]This notation will mean the quasi-norm equivalence throughtout this paper

On the other hand, regarding the roles of parameters $p, r,$ we have

$$\dot F^0_{p,\,2} = H^p \quad (0 < p < \infty), \quad \dot F^0_{\infty,\,2} = \mathrm{BMO}. \tag{1.6}$$

In view of the monotone imbedding[2] property

$$\dot F^\alpha_{p,\,r_1} \hookrightarrow \dot F^\alpha_{p,\,r_2}, \quad 0 < r_1 \le r_2 \le \infty, \tag{1.7}$$

hence, we may interpret that the $\dot F^0_{p,\,r}$ are monotone family of refinements when $0 < r < 2$ or extensions when $2 < r \le \infty$ of H^p and BMO spaces.

From this structure point of view, it is evident that the homogeneous Triebel-Lizorkin spaces provide an ideal framework for developing an extensive Sobolev theory. As a matter of fact, many authors have studied the problems of Sobolev-type inequalities, traces, pointwise multipliers, restrictions or extensions related with Lipschitz-type domains, and the like on these function spaces.

At present we are interested in studying Fourier multipliers whose symbols satisfy a generalization of Hörmander's condition. To be specific, we shall consider a family of Fourier multiplier operator T_m, defined as $(T_m f)\hat{} = m\hat f$, with the following condition on m.

Given a positive integer ℓ and $\alpha \in \mathbb{R}$, $m \in C^\ell(\mathbb{R}^n \setminus \{0\})$ and

$$\sup_{R>0}\left[R^{-n+2\alpha+2|\sigma|}\int_{R<|\xi|<2R}\left|\partial_\xi^\sigma m(\xi)\right|^2 d\xi\right] \le A_\sigma, \quad |\sigma| \le \ell. \tag{1.8}$$

When $\alpha = 0$, it is known as the Hörmander condition (see [11], [15]). Typical examples are given by the symbols of singular integrals R_j which will be defined in the next section. When $\alpha \ne 0$, a typical example is given by $m(\xi) = |\xi|^{-\alpha}$ of I_α which satisfies the condition (1.8) for every positive integer ℓ. Another example is the symbol of a differential operator ∂^σ of order $|\sigma| = \alpha$ when $\alpha > 0$.

It turns out that our characterizing means are effective in investigating mapping properties of T_m on the homogeneous Triebel-Lizorkin spaces. As a particular instance of our results, we shall obtain the Sobolev imbedding theorems in the full range. While we put our emphasis on the homogeneous Tribel-Lizorkin spaces, we shall also obtain mapping properties of T_m on the homogeneous Besov-Lipschitz spaces by a slight modification of our methods.

In what follows, the letter C will denote a positive constant which may differ in each occurrence and may depend on the parameters but not on the variable quantities involved. As usual, the Fourier transform of an integrable function ϕ on $\mathbb{R}^n$ will be defined as

$$\widehat{\phi}(\xi) = \int_{\mathbb{R}^n} e^{-i\xi\cdot x}\,\phi(x)\,dx, \quad \xi \in \mathbb{R}^n.$$

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[2]Given two quasi-norm spaces A, B, the notation $A \hookrightarrow B$ will mean the continuous imbedding of A into B, that is, $A \subset B$ and $\|\cdot\|_B \le c\|\cdot\|_A$ for a uniform constant c.

2. A basic structure of Triebel-Lizorkin spaces

In general, given a quasi-norm space $(B, \|\cdot\|_B)$ of tempered distributions, the image spaces $I_\alpha(B)$ are said to be the homogeneous B-Sobolev spaces provided the following conditions hold:

(i) For a positive integer ℓ, $f \in I_\ell(B)$ if and only if $\partial^\sigma f \in B$ for all σ of order $|\sigma| = \ell$ and

$$\|f\|_{I_\ell(B)} \approx \sum_{|\sigma|=\ell} \|\partial^\sigma f\|_B \,.$$

(ii) For a non-integral value of α, $I_\alpha(B)$ can be identified as an interpolation space between two neighboring spaces of integral order.

As for the spaces $\dot F^\alpha_{p,\,r}$, we state the following elementary facts which must be well known. We shall write $\dot F^0_{p,\,r} = F_{p,\,r}$ for simplicity.

Proposition 2.1. *Let* $0 < p,\, r \le \infty$.

(1) *For* $\alpha \in \mathbb{R}$, $\dot F^\alpha_{p,\,r} = I_\alpha\big(F_{p,\,r}\big)$ *so that* $f \in \dot F^\alpha_{p,\,r}$ *if and only if there exists a unique* $g \in F_{p,\,r}$ *such that*

$$f = I_\alpha(g) \quad \textit{and} \quad \|f\|_{\dot F^\alpha_{p,\,r}} \approx \|g\|_{F_{p,\,r}}.$$

(2) *For a positive integer* ℓ, $f \in \dot F^\ell_{p,\,r}$ *if and only if* $\partial^\sigma f \in F_{p,\,r}$ *for all* σ *of order* $|\sigma| = \ell$ *and*

$$\|f\|_{\dot F^\ell_{p,\,r}} \approx \sum_{|\sigma|=\ell} \|\partial^\sigma f\|_{F_{p,\,r}} \,.$$

Proof. Clearly I_α preserves the class $\mathcal{A}$. In view of the identities

$$I_\alpha(f) * \varphi_{2^{-j}} = 2^{-j\alpha}\left[f * (I_\alpha\varphi)_{2^{-j}}\right], \quad j \in \mathbb{Z},$$

it maps $F_{p,\,r}$ boundedly into $\dot F^\alpha_{p,\,r}$. Since its inverse is given by $I_{-\alpha}$, the transformation $I_\alpha : F_{p,\,r} \to \dot F^\alpha_{p,\,r}$ is indeed an isometric isomorphism up to equivalence of quasi-norms. This proves the statement (1).

As for (2), we consider Riesz singular integrals (R_j) defined as

$$(R_j f)\,\hat{}\,(\xi) = -i\,\frac{\xi_j}{|\xi|}\,\hat f(\xi),\ j = 1, \dots, n. \tag{2.1}$$

By the same reasoning as above, each R_j is easily seen to be a bounded mapping from $F_{p,\,r}$ into itself. Put

$$R = (R_1, \dots, R_n), \quad \nabla = (\partial_1, \dots, \partial_n).$$

A simple manipulation of symbols shows that for each $|\sigma| = \ell$,

$$\partial^\sigma = (-1)^\ell R^\sigma I_{-\ell}, \quad I_{-\ell} = (\nabla \cdot R)^\ell,$$

which yields the assertion (2). $\square$

According to H. Triebel [18], M. Frazier and B. Jawerth [9], for any real numbers $\alpha < \gamma$, if $\beta = (1-\theta)\alpha + \theta\gamma$ for $0 < \theta < 1$, then $\dot{F}^{\beta}_{p,\,r}$ can be identified as the interpolation space

$$\left(\dot{F}^{\alpha}_{p,\,r},\ \dot{F}^{\gamma}_{p,\,r}\right)_{\theta} = \dot{F}^{\beta}_{p,\,r}.$$

It follows that we may interpret the spaces $\dot{F}^{\alpha}_{p,\,r}$ as the homogeneous $F_{p,\,r}$-Sobolev spaces in the sense described as above. This important point of view on Triebel-Lizorkin spaces may be justified from the fact

$$\dot{F}^{\alpha}_{p,\,2} = \dot{H}^{p}_{\alpha} \quad (0 < p < \infty), \quad \dot{F}^{\alpha}_{\infty,\,2} = I_{\alpha}(\mathrm{BMO}), \tag{2.2}$$

which have been investigated by R. Strichartz (see [16], [17] and also [7]).

Remark 2.1. In a similar manner, the inhomogeneous B-Sobolev spaces can be defined in general as the image spaces $J_\alpha(B)$, where the J_α denote the Bessel potentials defined as

$$(J_\alpha f)\widehat{\ }(\xi) = \left(1 + |\xi|^2\right)^{-\alpha/2} \hat{f}(\xi) \tag{2.3}$$

(see A. P. Calderón [4]). With $B = F_{p,\,r}$, however, the spaces $J_\alpha(F_{p,\,r})$ do not coincide with the inhomogeneous Triebel-Lizorkin spaces $F^{\alpha}_{p,\,r}$ when $0 < p \le 1$. In particular, the inhomogeneous Hardy-Sobolev spaces $J_\alpha(F_{p,\,2}) = H^p_\alpha$ are not part of scales of Triebel-Lizorkin spaces when $0 < p \le 1$. Indeed, it is known that $F^{\alpha}_{p,\,2} = h^p_\alpha$, where the h^p stand for the local Hardy spaces.

3. New characterizations

The purpose of this section is to obtain a set of new characterizations for the homogeneous Triebel-Lizorkin spaces. Based on the work ([1], [2], [3]) of H. Bui, M. Paluszyński and M. Taibleson, our characterizing means are defined in terms of integrals rather than sums.

In accordance with [1], given $\alpha \in \mathbb{R}$, we denote by $\mathcal{O}_\alpha$ the class of Schwartz function φ on $\mathbb{R}^n$ such that

(i) $\sup_{t>0} |\widehat{\varphi}(t\xi)| > 0$ for each $\xi \neq 0$ and
(ii) $(\partial^\sigma \widehat{\varphi})(0) = 0$ for all $|\sigma| \le [\alpha]$ when $\alpha \ge 0$.

Evidently, $\mathcal{A} \subset \mathcal{O}_\alpha$ for any α. Each $\mathcal{O}_\alpha$ will serve as a *minimal* admissible class of Schwartz functions in characterizing $\dot{F}^{\alpha}_{p,\,r}$.

Often referred to as the Tauberian condition, the main reason of considering the condition (i) lies in the following ([6], [5]).

Lemma 3.1 (Calderon's reproducing formula)**.** *A Schwartz function φ satisfies the condition* (i) *if and only if there exists a Schwartz function ζ such that $\widehat{\zeta}$ has compact support away from the origin and*

$$\int_0^\infty \widehat{\varphi}(s\xi)\, \widehat{\zeta}(s\xi)\, \frac{ds}{s} = 1 \quad (\xi \neq 0).$$

On the other hand, as it is shown in [1], the moment vanishing condition (ii) turns out to be a minimal requirement in passing from one choice of Schwartz functions to another for the proofs of equivalence properties of characterization.

3.1. Basic characterizing means

In principle all of our characterizing means are simple variants of those that arise in characterization of H^p or BMO spaces. Following A. P. Calderón and A. Torchinsky [5], C. Fefferman and E. M. Stein [8], we recall:

Definition 3.1. Given a continuous function $u(x,t)$ on $\mathbb{R}_+^{n+1}$ and real numbers $b>0, \lambda>0$, we define six types of functions associated with $u(x,t)$ as follows.

(1) The Littlewood-Paley g-function

$$g(u)(x) = \left(\int_0^\infty |u(x,t)|^2 \frac{dt}{t}\right)^{1/2}.$$

(2) The Littlewood-Paley g_λ-function

$$G_\lambda(u)(x) = \left[\int_{\mathbb{R}_+^{n+1}} |u(y,t)|^2 \left(1+\frac{|y-x|}{t}\right)^{-2\lambda} t^{-n}\, dy \frac{dt}{t}\right]^{1/2}.$$

(3) The Lusin S-function or area integral

$$S_b(u)(x) = \left(\int_{\Gamma_b(x)} |u(y,t)|^2 (bt)^{-n} dy \frac{dt}{t}\right)^{1/2},$$

where $\Gamma_b(x)$ denotes the cone $\{\,(y,t) : |y-x|<bt\,\}$.

(4) The maximal function

$$M_+(u)(x) = \sup_{t>0} |u(x,t)|.$$

(5) The non-tangential maximal function

$$\mathcal{M}_b(u)(x) = \sup_{\Gamma_b(x)} |u(y,t)|.$$

(6) The Carleson maximal function

$$\mathcal{N}(u)(x) = \sup_{t>0}\left[\frac{1}{|B(x,t)|}\int_{B(x,t)}\int_0^t |u(y,s)|^2 \frac{ds}{s} dy\right]^{1/2}.$$

There is an additional maximal function which will be useful later on.

Definition 3.2. Given a continuous function $u(x,t)$ on $\mathbb{R}_+^{n+1}$ and $\lambda>0$, the Peetre maximal function associated with $u(x,t)$ is defined as

$$u_\lambda^*(x,t) = \sup_{y\in\mathbb{R}^n} |u(y,t)| \left(1+\frac{|y-x|}{t}\right)^{-\lambda}.$$

3.2. The case $0 < p < \infty$, $0 < r < \infty$

In this case we characterize the space $\dot{F}^{\alpha}_{p,\,r}$ by means of variants of Littlewood-Paley functions (1), (2) and Lusin S-functions (3) given as in Definition 3.1. As a general rule, our variants in question are obtained from replacing $u(x,t)$ by $[t^{-\alpha}u(x,t)]^{r/2}$ and then taking $2/r$ power.

Definition 3.3. Let $u(x,t)$ be a continuous function on $\mathbb{R}^{n+1}_{+}$. Given $\alpha \in \mathbb{R}$ and $0 < r < \infty$, we define

$$\begin{aligned}
g^{\alpha}_{r}(u)(x) \quad &= \left(\int_0^\infty \left(t^{-\alpha}|u(x,t)|\right)^r \frac{dt}{t}\right)^{1/r}, \\
G^{\alpha}_{\lambda,\,r}(u)(x) \quad &= \left[\int_{\mathbb{R}^{n+1}_{+}} \left(t^{-\alpha}|u(y,t)|\right)^r \left(1+\frac{|y-x|}{t}\right)^{-\lambda r} t^{-n}\, dy\frac{dt}{t}\right]^{1/r}, \\
S^{\alpha}_{b,\,r}(u)(x) \quad &= \left(\int_{\Gamma_b(x)} \left(t^{-\alpha}|u(y,t)|\right)^r (bt)^{-n} dy\frac{dt}{t}\right)^{1/r}.
\end{aligned}$$

In practice we associate these functions with a tempered distribution f and a Schwartz function φ on $\mathbb{R}^n$ through the specific function

$$u(x,t) = (f*\varphi_t)(x), \quad x \in \mathbb{R}^n,\, t>0.$$

A characterization of Triebel-Lizorkin spaces is established by H. Bui, M. Paluszyński, and M. Taibleson in terms of the functions $g^{\alpha}_{r}(f*\varphi_t)$.

Theorem 3.1 ([1], [2])**.** *Let $\alpha \in \mathbb{R}$ and $0 < p,\, r < \infty$. For a tempered distribution f on $\mathbb{R}^n$ and $\varphi \in \mathcal{O}_\alpha$, put $u(x,t) = (f*\varphi_t)(x)$. Then $f \in \dot{F}^{\alpha}_{p,\,r}$ if and only if $g^{\alpha}_{r}(u) \in L^p$ or $g^{\alpha}_{r}(u^*_\lambda) \in L^p$ and*

$$\|f\|_{\dot{F}^{\alpha}_{p,\,r}} \approx \|g^{\alpha}_{r}(u)\|_p \approx \|g^{\alpha}_{r}\,(u^*_\lambda)\|_p\,, \tag{3.1}$$

where the second equivalence holds for $\lambda > \max\,(n/p,\, n/r)$. A different choice of φ from the class $\mathcal{O}_\alpha$ yields equivalent quasi-norms in (3.1).

Our aim is to characterize $\dot{F}^{\alpha}_{p,\,r}$ in terms of $S^{\alpha}_{b,\,r}$ and $G^{\alpha}_{\lambda,\,r}$. It will be a simple consequence of the following general properties.

Lemma 3.2. *Let $u(x,t)$ be continuous on $\mathbb{R}^{n+1}_{+}$, $\alpha \in \mathbb{R}$ and $0 < r < \infty$.*

(1) *For any $b, d > 0$, if $S^{\alpha}_{d,\,r}(u) \in L^p$, then $S^{\alpha}_{b,\,r}(u) \in L^p$ with*

$$\left\|S^{\alpha}_{b,\,r}(u)\right\|_p \le C_{n,p} \begin{cases} (1+b/d)^{n(1/p-1/r)} \left\|S^{\alpha}_{d,\,r}(u)\right\|_p & (0<p\le r), \\ (1+d/b)^{n/2} \left\|S^{\alpha}_{d,\,r}(u)\right\|_p & (r<p<\infty). \end{cases}$$

(2) *For any $\lambda > 0$, with $\omega_n = |B(0,1)|$,*

$$S^{\alpha}_{1,\,r}(u)(x) \le \left(2^\lambda\, \omega_n^{1/r}\right) g^{\alpha}_{r}\,(u^*_\lambda)\,(x) \quad \textit{and}$$

$$g_r^\alpha(u)(x) \le \left(\frac{1}{\omega_n}\right)^{1/r} \liminf_{b\to 0} \left[S_{b,r}^\alpha(u)(x)\right].$$

(3) *For any* $b>0$ *and* $0<p<\infty$, *if* $\lambda > \max(n/p,\, n/r)$, *then*

$$\left\|S_{b,r}^\alpha(u)\right\|_p \approx \left\|G_{\lambda,r}^\alpha(u)\right\|_p.$$

Proof. Upon exploiting the relation

$$S_{b,r}^\alpha(u)(x) = \left[S_b(v^{r/2})(x)\right]^{2/r} \quad \text{with} \quad v(x,t) = t^{-\alpha}u(x,t),$$

the property (1) follows from Theorem 3.4 of [5].

By definition, the first inequality of (2) is a consequence of the estimate

$$|u(y,t)| \le 2^\lambda u_\lambda^*(x,t) \quad \text{for} \quad |y-x|<t.$$

As $t^{-\alpha}u(x,t)$ is continuous in $t>0$, we have

$$\left(t^{-\alpha}|u(x,t)|\right)^r = \lim_{b\to 0}\frac{1}{|B(x,bt)|}\int_{B(x,bt)}\left(t^{-\alpha}|u(y,t)|\right)^r dy.$$

It follows from Fatou's lemma that

$$\begin{aligned}\left[g_r^\alpha(u)(x)\right]^r &= \frac{1}{\omega_n}\int_0^\infty \left[\lim_{b\to 0}\int_{B(x,bt)} (t^{-\alpha}|u(y,t)|)^r (bt)^{-n}\,dy\right]\frac{dt}{t}\\ &\le \frac{1}{\omega_n}\liminf_{b\to 0}\int_0^\infty\int_{B(x,bt)} (t^{-\alpha}|u(y,t)|)^r (bt)^{-n}\,dy\,\frac{dt}{t}\\ &= \frac{1}{\omega_n}\liminf_{b\to 0}\left[S_{b,r}^\alpha(u)(x)\right]^r,\end{aligned}$$

which proves the second inequality of (2) (see Theorem 6.8 of [5]).

Finally, the equivalence (3) can be proved directly by a minor modification of the proof of Theorem 3.5, [5]. □

Our main characterization result is the following.

Theorem 3.2. *Let* $\alpha\in\mathbb{R}$ *and* $0<p,\, r<\infty$. *For a tempered distribution* f *on* $\mathbb{R}^n$ *and* $\varphi\in\mathcal{O}_\alpha$, *put* $u(x,t) = (f*\varphi_t)(x)$. *Then* $f\in\dot{F}_{p,r}^\alpha$ *if and only if* $S_{b,r}^\alpha(u)\in L^p$ *for any* $b>0$ *and*

$$\|f\|_{\dot{F}_{p,r}^\alpha} \approx \left\|S_{b,r}^\alpha(u)\right\|_p. \tag{3.2}$$

A different choice of φ *from* $\mathcal{O}_\alpha$ *or* $b>0$ *yields equivalent quasi-norms in* (3.2). *Moreover, the same conclusion holds if we replace* $S_{b,r}^\alpha(u)$ *by* $G_{\lambda,r}^\alpha(u)$ *with* $\lambda>\max(n/p,\, n/r)$.

Proof. In view of (1) of Lemma 3.2, it suffices to deal with the case $b=1$. The properties (1), (2) of Lemma 3.2 show that

$$\|g_r^\alpha(u)\|_p \le C\left\|S_{1,r}^\alpha(u)\right\|_p \le C\,\|g_r^\alpha(u_\lambda^*)\|_p$$

from which we get (3.2) in view of Theorem 3.1. The statement about $G_{\lambda,r}^\alpha$ results from the property (3) of Lemma 3.2. □

3.3. The case $0 < p \leq \infty$, $r = \infty$

In this case we characterize the space $\dot{F}^{\alpha}_{p,\infty}$ by means of variants of maximal functions (4), (5) of Definition 3.1 which are obtained from replacing $u(x,t)$ by $t^{-\alpha}u(x,t)$.

Definition 3.4. Let $u(x,t)$ be a continuous function on $\mathbb{R}^{n+1}_{+}$. Given $\alpha \in \mathbb{R}$ and $b > 0$, we define

$$M^{\alpha}_{+}(u)(x) = \sup_{t>0} t^{-\alpha}\,|u(x,t)|, \quad \mathcal{M}^{\alpha}_{b}(u)(x) = \sup_{\Gamma_b(x)} t^{-\alpha}\,|u(y,t)|.$$

As before, we associate these functions with a tempered distribution f and a Schwartz function φ on $\mathbb{R}^n$ through $u(x,t) = (f * \varphi_t)(x)$. It is shown ([1], [2], [3]) that for any choice $\varphi \in \mathcal{O}_\alpha$,

$$\|f\|_{\dot{F}^{\alpha}_{p,\infty}} \approx \left\|M^{\alpha}_{+}(u)\right\|_p \approx \left\|M^{\alpha}_{+}(u^*_\lambda)\right\|_p, \tag{3.3}$$

where the last equivalence holds for $\lambda > n/p$.

Theorem 3.3. *Let $\alpha \in \mathbb{R}$ and $0 < p \leq \infty$. For a tempered distribution f on $\mathbb{R}^n$ and $\varphi \in \mathcal{O}_\alpha$, put $u(x,t) = (f * \varphi_t)(x)$. Then $f \in \dot{F}^{\alpha}_{p,\infty}$ if and only if $\mathcal{M}^{\alpha}_{b}(u) \in L^p$ for any $b > 0$ and*

$$\|f\|_{\dot{F}^{\alpha}_{p,\infty}} \approx \|\mathcal{M}^{\alpha}_{b}(u)\|_p\,. \tag{3.4}$$

A different choice of φ from $\mathcal{O}_\alpha$ or $b > 0$ yields equivalent quasi-norms.

Proof. In view of the relation

$$\mathcal{M}^{\alpha}_{b}(u)(x) = \mathcal{M}_b\left(t^{-\alpha}u\right)(x),$$

Theorem 2.3 of [5] shows that

$$\|\mathcal{M}^{\alpha}_{b}(u)\|_p \leq C_{n,p}\left(1 + \frac{b}{d}\right)^{n/p} \|\mathcal{M}^{\alpha}_{d}(u)\|_p\,, \quad b \geq d > 0. \tag{3.5}$$

Writing the maximal function $M^{\alpha}_{+}(u^*_\lambda)(x)$ in full, we have

$$\begin{aligned} M^{\alpha}_{+}(u^*_\lambda)(x) &= \sup_{t>0,\, y\in\mathbb{R}^n} \left(t^{-\alpha}|u(y,t)|\right)\left(1 + \frac{|y-x|}{t}\right)^{-\lambda} \\ &= N_\lambda(t^{-\alpha}u)(x), \end{aligned}$$

where N_λ corresponds to the maximal function considered in [5]. It follows from Theorem 2.4 of [5] that

$$\|\mathcal{M}^{\alpha}_{b}(u)\|_p \approx \left\|M^{\alpha}_{+}(u^*_\lambda)\right\|_p\,, \quad \lambda > n/p. \tag{3.6}$$

Owing to the known result (3.3), the desired properties are simple consequence of the estimates (3.5), (3.6). $\square$

3.4. The case $p = \infty$, $0 < r < \infty$

In this case we do not give a new characterization of $\dot{F}^{\alpha}_{\infty,\,r}$ but we just recall the work of H. Bui and M. Taibleson. They considered variants of Carleson maximal functions (6) of Definition 3.1 obtained from replacing $u(x,t)$ by $t^{-\alpha}u(x,t)$.

Definition 3.5. Let $u(x,t)$ be a continuous function on $\mathbb{R}^{n+1}_{+}$. Given $\alpha \in \mathbb{R}$ and $0 < r < \infty$, we define

$$\mathcal{N}^{\alpha}_{r}(u)(x) = \sup_{t>0}\left[\frac{1}{|B(x,t)|}\int_{B(x,t)}\int_0^t \left(s^{-\alpha}|u(y,s)|\right)^r \frac{ds}{s}dy\right]^{1/r}.$$

Theorem 3.4 ([3])**.** *Let $\alpha \in \mathbb{R}$ and $0 < r < \infty$. For a tempered distribution f on $\mathbb{R}^n$ and $\varphi \in \mathcal{O}_\alpha$, put $u(x,t) = (f * \varphi_t)(x)$. Then $f \in \dot{F}^{\alpha}_{\infty,\,r}$ if and only if $\mathcal{N}^{\alpha}_{r}(u) \in L^\infty$ or $\mathcal{N}^{\alpha}_{r}(u^*_\lambda) \in L^p$ and*

$$\|f\|_{\dot{F}^{\alpha}_{\infty,\,r}} \approx \|\mathcal{N}^{\alpha}_{r}(u)\|_\infty \approx \|\mathcal{N}^{\alpha}_{r}(u^*_\lambda)\|_\infty\,, \tag{3.7}$$

where the second equivalence holds for a sufficiently large λ (as large as stated in p. 544, [3]*). A different choice of φ from $\mathcal{O}_\alpha$ yields equivalent quasi-norms in* (3.7)*.*

4. Basic estimates

Making use of our characterizations, we now proceed to investigate the mapping properties of T_m under the assumption (1.8) on m. In this section we set up a few basic estimates that will be useful later on. Let K denote the distribution whose Fourier transform is m.

Lemma 4.1. *Let ψ, ζ be Schwartz functions on $\mathbb{R}^n$ such that $\widehat{\psi}$, $\widehat{\zeta}$ have compact support away from the origin. Assume that m satisfies* (1.8)*.*

(1) *If $\lambda > 0$ and $\ell > \lambda + n/2$, then for $t > 0$,*

$$\int_{\mathbb{R}^n}\left(1+\frac{|z|}{t}\right)^{\lambda}|(K*\psi_t)(z)|\,dz \le C\,t^{\alpha}.$$

(2) *If $\ell \ge \lambda > 0$, then for $s, t > 0$,*

$$\int_{\mathbb{R}^n}\left(1+\frac{|z|}{s}\right)^{2\lambda}|(K*\zeta_s*\psi_t)(z)|^2\,dz \le C_{k,N}\,s^{-n+2\alpha}\left(\frac{t}{s}\right)^{2k}\left(1+\frac{t}{s}\right)^{-2N}$$

for any pair of positive integers k, N.

Proof. Dilating the functions $\widehat{\psi}$, $\widehat{\zeta}$ appropriately, we may assume both have support in $\{1/2 \le |\xi| \le 2\}$. To prove (1), we choose μ so that $\mu > n/2$ and $\lambda + \mu \le \ell$. By the Cauchy-Schwartz inequality,

$$\begin{aligned}
&\left[\int_{\mathbb{R}^n} \left(1+\frac{|z|}{t}\right)^{\lambda} |(K*\psi_t)(z)|\, dz\right]^2 \\
&\le \int_{\mathbb{R}^n} \left(1+\frac{|z|}{t}\right)^{-2\mu} dz \int_{\mathbb{R}^n} \left(1+\frac{|z|}{t}\right)^{2(\lambda+\mu)} |(K*\psi_t)(z)|^2\, dz \\
&\le C\, t^n \int_{\mathbb{R}^n} \left(1+\frac{|z|}{t}\right)^{2\ell} |(K*\psi_t)(z)|^2\, dz \\
&= C\, t^{2n} \int_{\mathbb{R}^n} (1+|z|)^{2\ell} |(K*\psi_t)(tz)|^2\, dz. \qquad (4.1)
\end{aligned}$$

Applying the binomial theorem and the Plancherel theorem, the integral in (4.1) is easily seen to be bounded by

$$\begin{aligned}
&t^{-2n} \sum_{|\sigma|\le\ell} C_\sigma \int_{\mathbb{R}^n} \left|\partial_\xi^\sigma \left[m\left(\frac{\xi}{t}\right)\widehat{\psi}(\xi)\right]\right|^2 d\xi \\
&\le C\, t^{-2n} \sum_{|\sigma|\le\ell} t^{n-2|\sigma|} \int_{1/t\le|\xi|\le 2/t} \left|\left(\partial_\xi^\sigma m\right)(\xi)\right|^2 d\xi \\
&\le C\, t^{-2n+2\alpha},
\end{aligned}$$

where the last inequality is due to the hypothesis (1.8) on m. Inserting this estimate into (4.1), we obtain the desired estimate of (1).

To prove (2), we change variables $z \mapsto sz$ and apply the Plancherel theorem to observe

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(1+\frac{|z|}{s}\right)^{2\lambda} |(K*\zeta_s*\psi_t)(z)|^2\, dz \\
&\le C\, s^{-n} \sum_{|\sigma|\le\ell} \int_{\mathbb{R}^n} \left|\partial_\xi^\sigma \left[m\left(\frac{\xi}{s}\right)\widehat{\psi}\left(\frac{t\xi}{s}\right)\widehat{\zeta}(\xi)\right]\right|^2 d\xi. \qquad (4.2)
\end{aligned}$$

By Leibniz's rule, the last integral in (4.2) is bounded by

$$\begin{aligned}
&\sum_{|\sigma|\le\ell} \sum_{\sigma_1+\sigma_2+\sigma_3=\sigma} \frac{\sigma!}{\sigma_1!\sigma_2!\sigma_3!}\, s^{-2|\sigma_1|} \left(\frac{t}{s}\right)^{2|\sigma_2|} \\
&\times \int_{\mathbb{R}^n} \left|\left(\partial_\xi^{\sigma_1} m\right)\left(\frac{\xi}{s}\right)\left(\partial_\xi^{\sigma_2}\widehat{\psi}\right)\left(\frac{t\xi}{s}\right)\left(\partial_\xi^{\sigma_3}\widehat{\zeta}\right)(\xi)\right|^2 d\xi \\
&\le C \left(\frac{t}{s}\right)^{2k} \left(1+\frac{t}{s}\right)^{-2N} \sum_{|\sigma_1|\le\ell} s^{n-2|\sigma_1|} \int_{1/s\le|\xi|\le 2/s} \left|\left(\partial_\xi^{\sigma_1} m\right)(\xi)\right|^2 d\xi
\end{aligned}$$

$$\le C\, s^{2\alpha} \left(\frac{t}{s}\right)^{2k} \left(1+\frac{t}{s}\right)^{-2N},$$

where we have used the fact that $\widehat{\psi}$ has a zero of infinite order at the origin and the hypothesis (1.8) on m. Inserting this estimate into (4.2), we obtain the stated estimate of (2). □

We now deduce the following which will play crucial roles in our subsequent development. For simplicity, we shall write $G^0_{\lambda,\,r} = G_{\lambda,\,r}$.

Lemma 4.2. *Given $\alpha \in \mathbb{R}$ and a positive integer ℓ, assume that m satisfies the condition* (1.8). *Let $\lambda > 0$, $2 \le r < \infty$ and let $\varphi, \psi \in \mathcal{A}$. For a tempered distribution f, set $u(x,t) = (f * \varphi_t)(x)$.*

(1) *If $\ell > \lambda + n/2$ and $\Phi = \varphi * \psi$, then for all $x, y \in \mathbb{R}^n$, $t > 0$,*

$$\left|(T_m f * \Phi_t)\,(y)\right| \le C\, t^{\alpha} \left(1 + \frac{|y-x|}{t}\right)^{\lambda} u^*_\lambda(x,t).$$

(2) *If $\ell > \lambda + n(1/2 - 1/r)$, then for $|y-x| < t$,*

$$\left|(T_m f * \psi_t)\,(y)\right| \le C\, t^{\alpha}\, G_{\lambda,\,r}(u)(x).$$

(3) *If $\ell > \lambda + n/2$, then for $|y-x| < t$,*

$$\left|(T_m f * \psi_t)\,(y)\right| \le C\, t^{\alpha}\, N_\lambda(u)(x),$$

where $N_\lambda(u) = M_+(u^*_\lambda)$, that is,

$$N_\lambda(u)(x) = \sup_{t>0,\, y\in\mathbb{R}^n} |u(y,t)| \left(1 + \frac{|y-x|}{t}\right)^{-\lambda}.$$

Proof. In view of the representation

$$(T_m f * \Phi_t)\,(y) = \int_{\mathbb{R}^n} u(y-z)(K * \psi_t)(z)\, dz,$$

the first estimate is an immediate consequence of (1) of Lemma 4.1.

According to Lemma 3.1, there exists a Schwartz function ζ such that $\widehat{\zeta}$ has compact support away from the origin and

$$\int_0^\infty \widehat{\varphi}(s\xi)\, \widehat{\zeta}(s\xi)\, \frac{ds}{s} = 1, \quad \xi \ne 0.$$

It follows that

$$\begin{aligned}(T_m f * \psi_t)(y) &= \int_0^\infty (f * \varphi_s * K * \zeta_s * \psi_t)(y) \frac{ds}{s}\\ &= \int_0^\infty \int_{\mathbb{R}^n} u(y-z,s)(K * \zeta_s * \psi_t)(z)\, dz \frac{ds}{s}.\end{aligned}$$

For $|y-x| < t$, Hölder's inequality gives, with $1/r + 1/q = 1$,

$$\left|(T_m f * \psi_t)(y)\right| \le G_{\lambda,\,r}(u)(x) \left[\int_0^\infty \left(1 + \frac{t}{s}\right)^{\lambda q} s^{n(q-1)}\right.$$

$$\int_{\mathbb{R}^n} \left(1+\frac{|z|}{s}\right)^{\lambda q} |(K*\zeta_s*\psi_t)(z)|^q \, dz \frac{ds}{s}\Bigg]^{1/q}. \tag{4.3}$$

Choose μ so that $\mu > n(1/q-1/2)$ and $\ell \ge \lambda+\mu$. By Hölder's inequality again and (2) of Lemma 4.1, we estimate

$$\begin{aligned}
&\int_{\mathbb{R}^n} \left(1+\frac{|z|}{s}\right)^{\lambda q} |(K*\zeta_s*\psi_t)(z)|^q \, dz \frac{ds}{s} \\
&\le C\, s^{n(1-q/2)} \left[\int_{\mathbb{R}^n} \left(1+\frac{|z|}{s}\right)^{2(\lambda+\mu)} |(K*\zeta_s*\psi_t)(z)|^2 \, dz \frac{ds}{s}\right]^{q/2} \\
&\le C\, s^{n(1-q)+\alpha q} \left(\frac{t}{s}\right)^{kq} \left(1+\frac{t}{s}\right)^{-Nq},
\end{aligned}$$

where k, N are arbitrary positive integers. Upon selecting k, N so that $k > \alpha$ and $N > \lambda + k - \alpha$, it is easy to see that

$$\int_0^\infty s^{\alpha q} \left(\frac{t}{s}\right)^{kq} \left(1+\frac{t}{s}\right)^{(-N+\lambda)q} \frac{ds}{s} \le C\, t^{\alpha q}.$$

Inserting these two estimates into (4.3), we obtain the estimate in (2).

The proof of the estimate in (3) is similar. □

5. Main results on Fourier multipliers

Owing to the characterization results of Triebel-Lizorkin spaces established in Section 3, the basic estimates of Lemma 4.2 enable us to deduce mapping properties of T_m on Triebel-Lizorkin spaces rather easily. As it is standard in deriving Sobolev-type inequalities, our results will be obtained from estimating the characterizing means relevant for the target spaces in terms of those relevant for the domain spaces.

To begin with, we have the following result, a special case of which is often referred to as the lifting property.

Theorem 5.1. *Let* $\alpha, \gamma \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < r \le \infty$. *Suppose that* m *satisfies the condition* (1.8) *with* $\ell > \max(n/p, n/r) + n/2$. *Then*

$$\|T_m f\|_{\dot F^{\alpha+\gamma}_{p,r}} \le C\, \|f\|_{\dot F^{\gamma}_{p,r}}. \tag{5.1}$$

If $\ell > \lambda + n/2$ *and* λ *is sufficiently large* (*as in Theorem 3.4*), *then*

$$\|T_m f\|_{\dot F^{\alpha+\gamma}_{\infty,r}} \le C\, \|f\|_{\dot F^{\gamma}_{\infty,r}}. \tag{5.2}$$

Proof. Under the same setting as in Lemma 4.2, the first estimate (1) shows that with $U(x,t) = (T_m f * \Phi_t)(x)$,

$$t^{-(\alpha+\gamma)}\, U^*_\lambda(x,t) \le C\, t^{-\gamma}\, u^*_\lambda(x,t),$$

which yields the stated results in view of theorems in Section 3. □

Our principal result is the following.

Theorem 5.2. *Given* $\alpha \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < r \le \infty$, *let* β *be any real with* $\beta < \alpha$ *and let* p_* *be determined by*

$$\beta - n/p_* = \alpha - n/p, \qquad 0 < p_* \le \infty. \tag{5.3}$$

Assume that m *satisfies the condition* (1.8) *with*

$$\ell > \begin{cases} \max(n/p,\, n/r) + n(1/2 - 1/r) & \text{if} \quad 2 \le r \le \infty, \\ \max(n/p,\, n/2) & \text{if} \quad 0 < r < 2. \end{cases} \tag{5.4}$$

Then T_m *maps* $F_{p,\,r}$ *boundedly into* $\dot{F}^{\beta}_{p_*,\,q}$ *for any* $0 < q \le \infty$ *with*

$$\|T_m f\|_{\dot{F}^{\beta}_{p_*,\,q}} \le C\, \|f\|_{F_{p,\,r}}\,. \tag{5.5}$$

Proof. In view of the monotone imbedding property, it suffices to consider the case $0 < q < \infty$ (see Remark 5.1 below). Fix q with $0 < q < \infty$. As in Lemma 4.2, we choose $\varphi,\, \psi \in \mathcal{A}$ and put

$$U(x,t) = (T_m f * \psi_t)(x), \quad u(x,t) = (f * \varphi_t)(x).$$

Part I. Assume first $f \in F_{p,\,r}$ with $2 \le r < \infty$. Due to different characterizing means, there are two cases for which we need to treat separately.

Case 1. $0 < p_* < \infty$, that is, $\alpha - \beta - n/p < 0$.

We choose λ so that $\lambda > \max(n/p,\, n/r)$ and $\lambda + n(1/2 - 1/r) < \ell$. The second estimate of Lemma 4.2 states that

$$|U(y,t)| \le C\, t^{\alpha}\, G_{\lambda,\,r}(u)(x) \quad \text{for} \quad |y-x| < t. \tag{5.6}$$

Raising this inequality to the power of p and then integrating over the ball $B(y,t)$ with respect to dx, we get

$$|U(y,t)| \le C\, t^{\alpha - n/p}\, \|G_{\lambda,\,r}(u)\|_p \quad \text{for} \quad |y-x| < t. \tag{5.7}$$

For a constant $A > 0$, we use these two inequalities to estimate

$$\begin{aligned} \left[S^{\beta}_{1,\,q}(U)(x)\right]^q &= \left(\int_0^A + \int_A^\infty\right) \int_{|y-x|<t} |U(y,t)|^q t^{-n-q\beta}\, dy \frac{dt}{t} \\ &\le C \left\{ A^{(\alpha-\beta)q} \left[G_{\lambda,\,r}(u)(x)\right]^q + A^{(\alpha-\beta-n/p)}\, \|G_{\lambda,\,r}(u)\|_p^q \right\}. \end{aligned}$$

Optimizing this inequality over A, that is, taking A so as to

$$A^{n/p} = \|G_{\lambda,\,r}(u)\|_p \,/\, \left[G_{\lambda,\,r}(u)(x)\right],$$

we obtain the estimate

$$S^{\beta}_{1,\,q}(U)(x) \le C\, \left[G_{\lambda,\,r}(u)(x)\right]^{p/p_*}\, \|G_{\lambda,\,r}(u)\|_p^{1-p/p_*}\,.$$

This implies instantly that

$$\left\|S^{\beta}_{1,\,q}(U)\right\|_{p_*} \le C\, \|G_{\lambda,\,r}(u)\|_p\,, \tag{5.8}$$

which yields the claim (5.5) in view of Theorem 3.2.

Case 2. $p_* = \infty$, that is, $\alpha - \beta - n/p = 0$.

The estimate (5.6) shows that for $|y-x|<t$,

$$\int_0^t \left(s^{-\beta}|U(y,s)|\right)^q \frac{ds}{s} \le C\,t^{(\alpha-\beta)q}\,[G_{\lambda,r}(u)(x)]^q .$$

Raising this inequality to the power of p/q and then integrating over the ball $B(y,t)$ with respect to dx, we get

$$\int_0^t \left(s^{-\beta}|U(y,s)|\right)^q \frac{ds}{s} \le C\,t^{(\alpha-\beta-n/p)q}\,\|G_{\lambda,r}(u)\|_p^q = C\,\|G_{\lambda,r}(u)\|_p^q .$$

It follows that for all $t>0$,

$$\frac{1}{|B(x,t)|}\int_{B(x,t)}\int_0^t \left(s^{-\beta}|U(y,s)|\right)^q \frac{ds}{s} \le C\,\|G_{\lambda,r}(u)\|_p^q ,$$

which implies that

$$\mathcal{N}_q^\beta(U)(x) \le C\,\|G_{\lambda,r}(u)\|_p , \tag{5.9}$$

whence we get (5.5) in view of Theorem 3.4.

Part II. Suppose now $f\in F_{p,\infty}$. Replacing the estimate (5.6) by

$$|U(y,t)| \le C\,t^\alpha\,N_\lambda(u)(x) \quad \text{for} \quad |y-x|<t,$$

which follows from (3) of Lemma 4.2, the same arguments yield

$$\left\|S_{1,q}^\beta(U)\right\|_{p_*} \le C\,\|N_\lambda(u)\|_p \quad \text{when} \quad 0<p_*<\infty, \tag{5.10}$$

$$\mathcal{N}_q^\beta(U)(x) \le C\,\|N_\lambda(u)\|_p \quad \text{when} \quad p_*=\infty. \tag{5.11}$$

Part III. We finally consider the case $f\in F_{p,r}$ with $0<r<2$. Owing to the imbedding $F_{p,r}\hookrightarrow F_{p,2}$, the claimed statement and inequality (5.5) are direct consequences of Part I. □

Remark 5.1. In the case $q=\infty$, a direct proof of (5.5) can be obtained in a similar manner from estimating the non-tangential maximal functions. To be precise, noticing

$$\mathcal{M}_1^\beta(U)(x) \le \left(\sup_{0<t\le A} + \sup_{t>A}\right)\left[\sup_{y\in B(x,t)} t^{-\beta}|U(y,t)|\right],$$

with $A>0$ to be optimized, and proceeding as above, we obtain

$$\left\|\mathcal{M}_1^\beta(U)\right\|_{p_*} \le C\,\|G_{\lambda,r}(u)\|_p . \tag{5.12}$$

On account of Theorem 3.3, this proves the claim (5.5).

While we restricted our domains to the spaces $F_{p,r}$, it is straightforward to extend Theorem 5.2 to arbitrary domains.

Corollary 5.1. *Given* $\alpha, \gamma \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < r \le \infty$, *let* β *be any real with* $\beta < \alpha + \gamma$ *and let* p_* *be determined by*

$$\beta - n/p_* = \alpha + \gamma - n/p, \qquad 0 < p_* \le \infty. \tag{5.13}$$

Assume that m *satisfies the condition* (1.8) *with* (5.4). *Then* T_m *maps* $\dot{F}^{\gamma}_{p,\,r}$ *boundedly into* $\dot{F}^{\beta}_{p_*,\,q}$ *for any* $0 < q \le \infty$ *with*

$$\|T_m f\|_{\dot{F}^{\beta}_{p_*,\,q}} \le C\, \|f\|_{\dot{F}^{\gamma}_{p,\,r}}\,. \tag{5.14}$$

Proof. The result follows from either a minor modification of the proof of Theorem 5.2 or considering the symbols $\widetilde{m}(\xi) = m(\xi)\,|\xi|^{-\gamma}$ and Proposition 2.1. We shall omit the details. □

6. Application to Sobolev imbedding

An immediate consequence of Theorem 5.2 is the following Sobolev imbedding theorem (refer to B. Jawerth [12], H. Triebel [19], J. Johnsen and W. Sickel [13], where the last paper contains a brief history).

Theorem 6.1. *Given reals* $\alpha > \beta$ *and* $0 < p < \infty$, $0 < r \le \infty$, *let* $0 < p_* \le \infty$ *be determined from* $\beta - n/p_* = \alpha - n/p$. *Then*

$$\dot{F}^{\alpha}_{p,\,r} \subset \bigcap_{q>0} \dot{F}^{\beta}_{p_*,\,q}$$

with the continuous imbedding $\dot{F}^{\alpha}_{p,\,r} \hookrightarrow \dot{F}^{\beta}_{p_*,\,q}$ *for each* $0 < q \le \infty$.

Proof. Take $m(\xi) = |\xi|^{-\alpha}$ in Theorem 5.2 and apply Proposition 2.1. □

Remark 6.1. In view of the homogeneity (see [10])

$$\|f_t\|_{\dot{F}^{\alpha}_{p,\,r}} = t^{\,n(1/p-1)-\alpha}\, \|f\|_{\dot{F}^{\alpha}_{p,\,r}}\,,$$

where f_t denotes the distribution defined by means of Fourier transform $\widehat{f_t}(\xi) = \hat{f}(t\xi)$ for each $t > 0$, it is easy to observe that the condition $\beta - n/p_* = \alpha - n/p$ arises rather necessarily.

In addition, we should point out that our result includes the case $p_* = \infty$ for which we do not know if it is known yet. Another point is that our proof deals with an arbitrary r although it suffices to deal with the case $r = \infty$ due to the monotone imbedding property (1.7).

Theorem 6.1 gives the best possible improvements of Sobolev-type inequalities in the scales of Triebel-Lizorkin spaces. To pick out some of special cases, we state the following.

Corollary 6.1. *Let* $\alpha > 0$, $0 < p < \infty$ *and* $0 < q \le 2 \le r \le \infty$.

(1) *If* $0 < p < \alpha/n$, *then with* $1/p_* = 1/p - \alpha/n$,

$$\dot{H}^p_\alpha \hookrightarrow \dot{F}^{\alpha}_{p,\,r} \hookrightarrow F_{p_*,\,q} \hookrightarrow H^{p_*}.$$

(2) *If* $p = \alpha/n$, *then*

$$\dot{H}^p_\alpha \hookrightarrow \dot{F}^\alpha_{p,\,r} \hookrightarrow F_{\infty,\,q} \hookrightarrow \mathrm{BMO}.$$

(3) *If* $\alpha/n < p < \infty$, *then*

$$\dot{H}^p_\alpha \hookrightarrow \dot{F}^\alpha_{p,\,r} \hookrightarrow \dot{F}^{\alpha-n/p}_{\infty,\,q} \hookrightarrow I_{\alpha-n/p}(\mathrm{BMO}).$$

Proof. Take $\beta = 0$ if $0 < p_* < \infty$ and $\beta = \alpha - n/p$ if $p_* = \infty$. □

7. Further results on Besov-Lipschitz spaces

Given any $\varphi \in \mathcal{A}$, the homogeneous Besov-Lipschitz spaces are the spaces of tempered distribution f on $\mathbb{R}^n$ such that the quasi-norms

$$\|f\|_{\dot{B}^\alpha_{p,\,r}} = \left(\sum_{j\in\mathbb{Z}} \left(2^{j\alpha}\,\|f * \varphi_{2^{-j}}\|_p\right)^r\right)^{1/r} \qquad (\alpha \in \mathbb{R},\, 0 < p,\, r \le \infty) \tag{7.1}$$

are finite (defined modulo polynomials but with no need to separate the case $p = \infty$). By the same reasoning as in Section 2, we may view

$$\dot{B}^\alpha_{p,\,r} = I_\alpha(B_{p,\,r}), \quad \text{where} \quad B_{p,r} = \dot{B}^0_{p,\,r}. \tag{7.2}$$

As a continuous version of (7.1), it is proved in [1] that

$$\|f\|_{\dot{B}^\alpha_{p,\,r}} \approx \left(\int_0^\infty \left(t^{-\alpha}\|u^*_\lambda(\cdot,t)\|_p\right)^r \frac{dt}{t}\right)^{1/r}, \quad \lambda > n/p, \tag{7.3}$$

where $u(x,t) = (f * \varphi_t)(x)$, $\varphi \in \mathcal{O}_\alpha$ and u^*_λ denotes the Peetre maximal function of u defined as in Definition 3.2.

Owing to the characterization (7.3), it is plain to adopt the same methods and ideas as in the preceding sections to obtain the mapping properties of T_m on Besov-Lipschitz spaces.

Theorem 7.1. *Given* $\alpha \in \mathbb{R}$, $0 < p < \infty$ *and* $0 < r \le \infty$, *let* β *be any real with* $\beta < \alpha$ *and let* p_* *be determined by*

$$\beta - n/p_* = \alpha - n/p, \qquad 0 < p_* \le \infty.$$

If m *satisfies the condition* (1.8) *with* $\ell > n(1/p + 1/2)$, *then*

$$\|T_m f\|_{\dot{B}^\beta_{p_*,\,r}} \le C\, \|f\|_{B_{p,\,r}}. \tag{7.4}$$

Proof. Choose $\varphi, \psi \in \mathcal{A}$. Let $\Phi = \varphi * \psi$ and

$$u(x,t) = (f * \varphi_t)(x), \quad U(x,t) = (T_m f * \Phi_t)(x).$$

It follows easily from the estimate (1) of Lemma 4.2 that

$$U^*_\lambda(x,t) \le C\, t^\alpha\, u^*_\lambda(x,t), \quad U^*_\lambda(x,t) \le C\, t^{\alpha-n/p}\, \|u^*_\lambda(\cdot,t)\|_p. \tag{7.5}$$

Fix $t > 0$. Normalizing if necessary, we may assume $C = 1$ in both estimates of (7.5). With $A = \|u^*_\lambda(\cdot,t)\|_p$, it follows that for any $q > p$,

$$\int_{\mathbb{R}^n} [U^*_\lambda(x,t)]^q\, dx = q\int_0^{A\,t^{\alpha-n/p}} \left|\left\{U^*_\lambda(\cdot,t) > s\right\}\right|\, s^{q-1}\, ds$$

$$\le C\, t^{\alpha q} \int_0^{A\, t^{-n/p}} \left|\left\{u^*_\lambda(\cdot,t) > s\right\}\right| \, s^{q-1}\, ds$$
$$\le C\, t^{\alpha q}\, \|u^*_\lambda(\cdot,t)\|_p^p \int_0^{A\, t^{-n/p}} s^{q-p-1}\, ds$$
$$= C\, t^{q(\alpha - n/p + n/q)}\, \|u^*_\lambda(\cdot,t)\|_p^q\,,$$

where the second inequality follows from Chebychev's inequality. Thus

$$t^{-(\alpha-n/p+n/q)}\, \left\|U^*_\lambda(\cdot,t)\right\|_q \le C\, \left\|u^*_\lambda(\cdot,t)\right\|_p. \tag{7.6}$$

Upon setting $\beta = \alpha - n/p + n/q$, $q = p_*$, (7.6) gives

$$\int_0^\infty \left(t^{-\beta}\left\|U^*_\lambda(\cdot,t)\right\|_{p_*}\right)^r \frac{dt}{t} \le \int_0^\infty \left(\left\|u^*_\lambda(\cdot,t)\right\|_p\right)^r \frac{dt}{t},$$

which yields the desired result in view of (7.3). □

Upon taking $m(\xi) = |\xi|^{-\alpha}$, we have (see e.g., [9]):

Corollary 7.1. *Given reals $\alpha > \beta$ and $0 < p < \infty$, $0 < r \le \infty$, let $0 < p_* \le \infty$ be determined from $\beta - n/p_* = \alpha - n/p$. Then*

$$\dot{B}^\alpha_{p,\,r} \hookrightarrow \dot{B}^\beta_{p_*,\,r}.$$

In the last place, we record an interesting imbedding result of Besov-Lipschitz spaces into Triebel-Lizorkin spaces that follows from Theorem 6.1 and elementary imbedding properties.

Corollary 7.2. *Given reals $\alpha > \beta$ and $0 < p < \infty$, let $0 < p_* \le \infty$ be determined from $\beta - n/p_* = \alpha - n/p$. If $0 < r \le p$, then*

$$\dot{B}^\alpha_{p,\,r} \subset \bigcap_{q>0} \dot{F}^\beta_{p_*,\,q}$$

with the continuous imbedding $\dot{B}^\alpha_{p,\,r} \hookrightarrow \dot{F}^\beta_{p_*,\,q}$ for each $0 < q \le \infty$.

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