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ON HARMONICITY IN A DISC AND n-HARMONICITY

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Abstract. Let $\tau \neq \delta_0$ be either a power bounded radial measure with compact support on the unit disc D with $\tau(D) = 1$ such that there is a $\delta > 0$ so that $|\hat{\tau}(s)| \neq 1$ for every $s \in \Sigma(\delta) \setminus \{0, 1\}$, or just a radial probability measure on D. Here, we provide a decomposition of the set $\mathbf{X} = \{h \in L^\infty(D) \mid \lim_{n\to\infty} h * \tau^n \text{ exists}\}$. Let $\tau_1, \ldots, \tau_n$ be measures on D with above mentioned properties. Here, we prove that if $f \in L^\infty(D^n)$ satisfies an invariant volume mean value property with respect to $\tau_1, \ldots, \tau_n$, then f is n-harmonic.

1. Introduction

Let D be the open unit disc of $\mathbb{C}$, ν be the Lebesgue measure on $\mathbb{C}$ normalized to $\nu(D) = 1$ and let μ be the conformally invariant measure on D defined by $d\mu(z) = (1 - |z|^2)^{-2}\, d\nu(z)$, which satisfies

$$\int_D u\, d\mu = \int_D u \circ \varphi\, d\mu$$

for every $u \in L^1(\mu)$ and for every $\varphi \in \mathrm{Aut}(D)$. And then let us denote $L^p_R(\mu)$ to be the subspace of $L^p(\mu)$ which consists of radial functions.

It is known that (see [3], [4]) $L^1_R(\mu)$ is a commutative Banach algebra under the convolution

$$(u * v)(z) = \int_D u\big(\varphi_z(x)\big)\, v(x)\, d\mu(x), \quad \text{where } \varphi_z(x) = \frac{z - x}{1 - \bar{z}x}. \tag{1.1}$$

Likewise if τ is a radial measure with $\tau(D) = 1$, we define the convolution $u * \tau$ on D by

$$(u * \tau)(z) = \int_D u\big(\varphi_z(x)\big)\, d\tau(x).$$

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For $f \in L^1_R(\mu)$, we define its Gelfand transform by

$$\hat{f}(s) = \int_D f(z)\Big(\frac{1-|z|^2}{|1-z|^2}\Big)^s \, d\mu(z), \quad 0 \le \Re s \le 1.$$

Likewise if τ is a radial measure with $\tau(D) = 1$, its Gelfand transform is defined by

$$\hat{\tau}(s) = \int_D \Big(\frac{1-|z|^2}{|1-z|^2}\Big)^s \, d\tau(z).$$

We know that if $u \in L^1(D, \tau)$ is harmonic, then $u \circ \varphi$ is also harmonic for every $\varphi \in \mathrm{Aut}(D)$.

Thus $u \circ \varphi$ satisfies a volume mean value property;

$$u\big(\varphi(0)\big) = \int_D u \circ \varphi \, d\tau \quad \text{for every } \varphi \in \mathrm{Aut}(D),$$

which is equivalent to $u * \tau = u$.

One of the main results of [1] is that if ν is a normalized Lebesgue measure and $u \in L^1(D, \nu)$ satisfies $u * \nu = u$, then u is harmonic. Much earlier, Furstenberg [5], [6] proved that if m is a radial probability measure on D and $u \in L^\infty(D)$ satisfies $u * m = u$, then u is harmonic. Indeed, his result says much more is true: *On any dimensional symmetric domain, a bounded function which satisfies a certain convolution type of mean value property is harmonic with respect to the intrinsic metric.* For example, applying Furstenberg's result to the polydisc D^n, we get that if $m_1, \ldots, m_n$ are radial probability measures on D, and if $f \in L^\infty(D^n)$ satisfies

$$f\big(\psi(0, \ldots, 0)\big) = \int_D \cdots \int_D f \circ \psi \, dm_1 \cdots dm_n \quad \text{for every } \psi \in \mathrm{Aut}(D^n),$$

then f is n-harmonic (It means $\Delta_1 f = \cdots = \Delta_n f = 0$).

In 1992, Benyamini and Weit [3] introduced another type of non-positive measure on D and got results analogous to those of Furstenberg. Here we state two theorems of [3].

Theorem 1.1 ([3, Theorems 2.1 and 2.3])**.** *Let $\tau (\tau \ne \delta_0)$ be a power bounded radial measure with compact support on D with $\tau(D) = 1$, and there is a $\delta > 0$ so that $|\hat{\tau}(s)| \ne 1$ for every $s \in \Sigma(\delta) \setminus \{0, 1\}$, where $\Sigma(\delta)$ is the strip $-\delta < \Re s < 1 + \delta$. Or let τ be a radial probability measure on D. If $u \in L^1_R(\mu)$ satisfies $\int_D u \, d\mu = 0$, then $u * \tau^n \to 0$ in the norm of $L^1(\mu)$.*

Theorem 1.2 ([3, Theorem 3.1])**.** *Let $\tau (\tau \ne \delta_0)$ be a power bounded radial measure with compact support on D with $\tau(D) = 1$, and there is a $\delta > 0$ so that $|\hat{\tau}(s)| \ne 1$ for every $s \in \Sigma(\delta) \setminus \{0, 1\}$, where $\Sigma(\delta)$ is the strip $-\delta < \Re s < 1 + \delta$. Or let τ be a radial probability measure on D. If $v \in L^\infty(D)$ satisfies $v * \tau = v$, then v is harmonic.*

This paper, in Theorem 1.3, we use the results of Benyamini and Weit (Theorems 1.1 and 1.2) to investigate the existence of the limit of $h * \tau^n$ for

$h \in L^\infty(D)$. Indeed, we decompose the space $\mathbf{X} = \{h \in L^\infty(D) \mid \lim_{n\to\infty} h * \tau^n \text{ exists}\}$ as a direct sum of the space of bounded harmonic functions and the closure of the space $A = \{h - h*\tau \mid h \in L^\infty(D)\}$. And then, in Theorem 1.4, we prove that Theorem 1.2 can extend to the n dimensional polydisc D^n. Indeed, one can easily see that if f is an integrable n-harmonic function on D^n and $\tau_1, \ldots, \tau_n$ are radial measures on D with $\tau_1(D) = \cdots = \tau_n(D) = 1$ ($\tau_1, \ldots, \tau_n$ need not be positive measures.), then for every $z_1, \ldots, z_n \in D$, we have

$$f(z_1, \ldots, z_n) = \int_D \cdots \int_D f\big(\varphi_{z_1}(x_1), \cdots, \varphi_{z_n}(x_n)\big)\, d\tau_1(x_1) \cdots d\tau_n(x_n). \tag{1.2}$$

Recently, the author [8] proved that in case $n \geq 2$, $f \in L^p(D^n, \tau_1 \times \cdots \times \tau_n)$ for $1 \leq p < \infty$, satisfying (1.2) does not imply n-harmonicity even if $\tau_1, \ldots, \tau_n$ are radial probability measures. This paper, Theorem 1.4 asserts that, in case $f \in L^\infty(D^n)$, if $\tau_1, \ldots, \tau_n$ satisfy the conditions of the above mentioned Benyamini and Weit's theorems, then satisfying (1.2) implies n-harmonicity of f. Here are our main results.

Theorem 1.3. *Let $\tau \neq \delta_0$ be a power bounded radial measure with compact support on D with $\tau(D) = 1$, and there is a $\delta > 0$ so that $|\hat{\tau}(s)| \neq 1$ for every $s \in \Sigma(\delta) \setminus \{0, 1\}$. Or let τ be a radial probability measure on D. Suppose we denote $\mathbf{X} = \{h \in L^\infty(D) \mid \lim_{n\to\infty} h * \tau^n \text{ exists}\}$, H the set of all bounded harmonic functions in D and $A = \{h - h * \tau \mid h \in L^\infty(D)\}$. Then $\mathbf{X}$ can be decomposed as $\mathbf{X} = H \oplus \overline{A}$, where $\overline{A}$ is the closure of A. Also, if there is $C > 0$ such that $|\tau| \leq C\mu$, then $\mathbf{X}$ is a proper subset of $L^\infty(D)$.*

Theorem 1.4. *Let $\tau_1, \ldots, \tau_n$ be radial measures on D with compact support and $\tau_i(D) = 1, \tau_i \neq \delta_0$ for $1 \leq i \leq n$. And suppose, for each $1 \leq i \leq n$ there is a $\delta_i > 0$ so that $|\hat{\tau}_i(s)| \neq 1$ for every $s \in \Sigma(\delta_i) \setminus \{0, 1\}$. Or let $\tau_1, \ldots, \tau_n (\tau_i \neq \delta_0$ for $1 \leq i \leq n)$ be radial probability measures on D. If $f \in L^\infty(D^n)$ satisfies*

$$f(z_1, \ldots, z_n) = \int_D \cdots \int_D f\big(\varphi_{z_1}(x_1), \ldots, \varphi_{z_n}(x_n)\big)\, d\tau_1(x_1) \cdots d\tau_n(x_n)$$

for every $z_1, \ldots, z_n \in D$, then f is n-harmonic.

We provide the proof of Theorem 1.3 in Section 2 and proof of Theorem 1.4 in Section 3.

2. Iterates of convolutions

Now we will prove Theorem 1.3 and then, in Proposition 2.1, we will express the subspace $A = \{h - h*\tau \mid h \in L^\infty(D)\}$ of $L^\infty(D)$ in Theorem 1.3 in terms of iterates of convolutions. In the proof of Theorem 1.3, we will do the case when $\tau \neq \delta_0$ is a power bounded radial measure with compact support on D with $\tau(D) = 1$, and there is a $\delta > 0$ so that $|\hat{\tau}(s)| \neq 1$ for every $s \in \Sigma(\delta) \setminus \{0, 1\}$. In the case when τ is a radial probability measure, the proof is identical.

Proof of Theorem 1.3. First, we'll show that $\mathbf{X} \subset H + \overline{A}$.

For an $h \in \mathbf{X}$, let $Ph \in L^\infty(D)$ satisfy $\lim_{n\to\infty} \| h * \tau^n - Ph \|_\infty = 0$. Then Ph is continuous on D and the convergence is also pointwise almost everywhere, thus by the dominated convergence theorem, we have

$$(Ph) * \tau = P(h * \tau) = \lim_{n\to\infty} h * \tau^{n+1} = Ph.$$

Thus by Theorem 1.2, Ph is harmonic in D. Now we denote $h_1 = Ph$ and $h_2 = h - Ph$ and then we get $h = h_1 + h_2$. Now we'll prove that $h_2 \in \overline{A}$.

For $g \in L^1(\nu)$ we define $B_\tau g = g * \tau$. Then we see that $A = (I - B_\tau)L^\infty(D)$. Now let $d \in L^\infty(D)^*$ satisfy $\langle d\ ,\ g - B_\tau g\rangle = 0$ for every $g \in L^\infty(D)$, then we get $\langle B_\tau^* d - d,\ g\rangle = 0$ for every $g \in L^\infty(D)$. This means that $B_\tau^* d = d$. Hence we have

$$\langle d\ ,\ h - Ph\rangle = \big\langle (B_\tau^*)^k d\ ,\ h - Ph\big\rangle = \big\langle d\ ,\ B_\tau^k (h - Ph)\big\rangle \text{ for all } k. \tag{2.1}$$

But from the definition of the operator P, we see that

$$\lim_{k\to\infty} \| B_\tau^k\ (h - Ph) \|_\infty = 0.$$

Thus, by taking the limit $k \to \infty$ in (2.1), we get $\langle d\ ,\ h - Ph\rangle = 0$. Hence by the Hahn-Banach theorem, $h_2 = h - Ph$ is in the closure of $(I - B_\tau)L^\infty(D)$ and this proves that $\mathbf{X} \subset H + \overline{A}$.

On the other hand, for $\psi \in \mathrm{Aut}(D)$ and $z \in D$, $\varphi_{\psi(z)} \circ \psi \circ \varphi_z$ takes 0 to 0 thus is $e^{i\theta}$ for some θ. Hence by rotation-invariance of τ,

$$B_\tau(g \circ \psi)(z) = \int_D g\big((\psi(\varphi_z(x))\big)\, d\tau(x) = \int_D g\big(\varphi_{\psi(z)} e^{i\theta} x\big)\, d\tau(x) = (B_\tau g)(\psi(z)).$$

Also if $u \in L_R^1(\mu)$ and $v \in L_R^\infty(D)$, then we get

$$\int_D u \cdot (B_\tau v)\, d\mu = (u * v * \tau)(0) = (u * \tau * v)(0) = \int_D (B_\tau u) \cdot v\, d\mu.$$

This means that the operator B_τ on $L_R^\infty(D)$ is the adjoint of B_τ on $L_R^1(\mu)$. And, since $L_R^\infty(D)$ is the dual space of $L_R^1(\mu)$, we see that the spectrum of B_τ on $L_R^\infty(D)$ is the same as the spectrum of B_τ on $L_R^1(\mu)$. Now let λ be in the spectrum of B_τ on $L^\infty(D)$, then there exists a sequence $\{g_k\}$ in $L^\infty(B_n)$ with $\|g_k\|_\infty = 1$ satisfying $\lim_{k\to\infty} \| B_\tau g_k - \lambda g_k \|_\infty = 0$.

Let $\phi_k \in \mathrm{Aut}(D)$ satisfy $\| R(g_k \circ \phi_k) \|_\infty = 1$ where $Rg(z) = \frac{1}{2\pi}\int_0^{2\pi} g(ze^{i\theta})d\theta$ denotes the radialization of g. Since B_τ and R are bounded on $L^\infty(B_n)$, we have

$$\begin{aligned}\| B_\tau\big(R(g_k \circ \phi_k)\big) - \lambda R(g_k \circ \phi_k) \|_\infty &= \| R\ \big(B_\tau(g_k \circ \phi_k)\big) - R(\lambda g_k \circ \phi_k) \|_\infty \\ &\le \| B_\tau(g_k \circ \phi_k) - \lambda g_k \circ \phi_k \|_\infty \\ &= \| (B_\tau g_k) \circ \phi_k - \lambda g_k \circ \phi_k \|_\infty \\ &= \| B_\tau g_k - \lambda g_k \|_\infty \to 0 \text{ as } k \to \infty.\end{aligned}$$

Hence λ is in the spectrum of B_τ on $L_R^\infty(D)$.

In the proof of Theorem 2.3 of [3], it is shown that the spectrum of B_τ, as an operator on $L^1_R(\mu)$, intersects the unit circle at most at the single point 1. Hence, by the above argument, the spectrum of B_τ on $L^\infty(D)$ also intersects the unit circle at most at the single point 1. By the theorem of Katznelson and Tzafriri [7, Theorem 1], we see that

$$\lim_{k\to\infty} \|B_\tau^k(I - B_\tau)\| = 0 \quad \text{on } L^\infty(D). \tag{2.2}$$

By (2.2) we see that if $h \in \overline{A}$, then $h * \tau^n \to 0$ in $L^\infty(D)$ and thus $H \cap \overline{A} = \{0\}$. Therefore, we get $\mathbf{X} = H \oplus \overline{A}$ and to complete the proof, it remains to show that $\mathbf{X}$ is a proper subset of $L^\infty(D)$ when we assume that there is $C > 0$ such that $|\tau| \le C\mu$. Theorem 6.1 of [2] deals with a similar case.

Suppose we assume that $\mathbf{X} = L^\infty(D)$. Then $\lim_{n\to\infty} h * \tau^n$ exists for every $h \in L^\infty(D)$.

Now we choose $u \in L^1_R(\mu)$ with $\int_D u \, d\mu \ne 0$. Then for every $\ell \in L^\infty_R(D)$,

$$\lim_{n\to\infty} \int_D \ell \cdot (u * \tau^n) \, d\mu = \lim_{n\to\infty} \int_D u \cdot (\ell * \tau^n) \, d\mu \;\text{ exists.} \tag{2.3}$$

Since $L^1_R(\mu)$ is weak complete, $u * \tau^n$ converges weakly to some $v \in L^1_R(\mu)$. And then $u * \tau^{n+1}$ converges to $v * \tau$, which implies that $v * \tau = v$. For each $z \in D$, we have

$$|v(z)| = |\,(v*\tau)(z)\,| \le \int_D |v\circ\varphi_z| \, d|\tau| \le C\int_D |v\circ\varphi_z| \, d\mu = C\int_D |v| \, d\mu = \|v\|_{L^1(\mu)}.$$

Thus v is bounded and by Furstenberg's theorem, v is harmonic in D. Since a constant is the only radial harmonic function, and since 0 is the only constant that belongs to $L^1(\mu)$, we conclude that v is the constant zero. Now putting $\ell = 1$ to the integral in (2.3), we get

$$\int_D u \, d\mu = \int_D u \cdot (1 * \tau^n) \, d\mu = \int_D u * \tau^n \, d\mu,$$

which tends to 0 as $n \to \infty$ to conclude that $\int_D u \, d\mu = 0$ contradicting our assumption.

Therefore, $\mathbf{X}$ is a proper subset of $L^\infty(D)$ when we assume that there is $C > 0$ such that $|\tau| \le C\mu$ and this completes the proof of the theorem. □

In the proof of Theorem 1.3, we've shown that

$$\overline{A} = \{h \in L^\infty(D) \mid \lim_{n\to\infty} h * \tau^n = 0\}.$$

Now in the following proposition, we will show that the space A can be expressed in terms of iterates of convolutions.

Proposition 2.1. *With the same assumptions and notations as Theorem 1.3, we have*

$$A = \{h \in L^\infty(D) \mid \limsup_{n\to\infty} \| \sum_0^n h * \tau^k \|_\infty < \infty \}. \tag{2.4}$$

Proof. Let $f = h - h * \tau$ for some $h \in L^\infty(D)$ and let $K = \sup \|\tau^n\|$. Then $\| \sum_0^n f * \tau^k \|_\infty = \| h - h * \tau^{n+1} \|_\infty \le (1+K)\|h\|_\infty$. Thus we can see that

$$A \subset \left\{ h \in L^\infty(D) \mid \limsup_{n\to\infty} \| \sum_0^n h * \tau^k \|_\infty < \infty \right\}.$$

On the other hand, pick $h \in L^\infty(D)$ such that $\limsup \| \sum_0^n h * \tau^k \|_\infty = M < \infty$.

Now if we denote $h_k = \sum_{j=0}^k h * \tau^j$, then $h_k - h_k * \tau = h - h * \tau^{k+1}$. Hence if we let $H_n = \frac{1}{n+1} \sum_{k=0}^n h_k$, then we get $\|F_n\|_\infty \le M$ and we also have

$$\begin{aligned} H_n - H_n * \tau &= \frac{1}{n+1} \sum_{k=0}^n (h_k - h_k * \tau) \\ &= \frac{1}{n+1} \sum_{k=0}^n (h - h * \tau^{k+1}) \\ &= h - \frac{1}{n+1} \sum_{k=0}^n h * \tau^{k+1}. \end{aligned}$$

Hence

$$\| H_n - H_n * \tau - h \|_\infty \le \frac{1}{n+1} M \to 0.$$

But a norm bounded sequence H_n has a subsequence H_{n_j} that converges weak* to some $g \in L^\infty(D)$ and as in the proof of Theorem 1.3, the operator $I - B_\tau$ is self-adjoint in $L^1(\mu)$, which makes $(I - B_\tau)H_{n_j}$ converge to $(I - B_\tau)g$ weak* in $L^\infty(D)$. Since $(I - B_\tau)H_n = H_n - H_n * \tau$ converge to h in norm, h is the unique weak* limit of $(I - B_\tau)H_n$. Hence we have $h = (I - B_\tau)g = g - g * \tau \in A$. This completes the proof of the proposition. $\square$

3. *n*-harmonicity

Here we prove Theorem 1.4. Even though the theorem is true for every $n \in \mathbb{N}$, for the notational simplicity, in the proof we restrict ourselves to the case of $n = 2$.

Proof of Theorem 1.4. Let $f \in L^\infty(D^2)$ and let us denote

$$(Tf)(z,w) = \int\int_{D^2} f\big(\varphi_z(x), \varphi_w(y)\big)\, d\tau_1(x)\, d\tau_2(y)$$

and then assume that f satisfies $Tf = f$.

First we prove the case that f is 2-radial, i.e., $f(z,w) = f(|z|,|w|)$ for all $z, w \in D$.

Since $\tau_1(D) = \tau_2(D) = 1$ for $v \in L^\infty_R(D)$, we can write by induction,

$$(v*\tau_1^n)(z) = \int_D v(x)\, P_n(z,x)\, d\tau_1(x) \text{ and } (v*\tau_2^n)(z) = \int_D v(x)\, Q_n(z,x)\, d\tau_2(x)$$

for some $P_n(z,x)$ and $Q_n(z,x)$ which satisfy

$$\int_D P_n(z,x)\ d\tau_1(x) = \int_D Q_n(z,x)\ d\tau_2(x) = 1 \ \text{ for all } n \geq 1.$$

Since τ_1, τ_2 are power bounded, there is $M > 0$ such that, for every $n \geq 1$,

$$\| \ v * \tau_1^n \ \|_\infty \leq M\|v\|_\infty \ \text{ and } \ \| \ v * \tau_2^n \ \|_\infty \leq M\|v\|_\infty.$$

Now fix $w \in D$ and define $(f_w)_n \in L_R^\infty(D)$ as

$$(f_w)_n(z) = \int_D f(z,y)\ Q_n(w,y)\ d\tau_2(y).$$

Then we get $\|(f_w)_n\|_\infty \leq M\|f\|_\infty$ for every $n \geq 1$. And we also get

$$\begin{aligned}
\big((f_w)_n * \tau_1^n\big)(z) &= \int_D (f_w)_n(x)\ P_n(z,x)\ d\tau_1(x) \\
&= \int\int_{D^2} f(x,y)\ P_n(z,x)\ Q_n(z,y)\ d\tau_1(x)\ d\tau_2(y) \\
&= (T^n f)(z,w).
\end{aligned}$$

Now let $u \in L_R^1(\mu)$ satisfy $\int_D \ u \ d\mu = 0$. Then for each fixed $w \in D$, we have

$$\begin{aligned}
\int_D u(z)\ f(z,w)\ d\mu(z) &= \int_D u(z)(T^n f)(z,w)\ d\mu(z) \\
&= \int_D u(z)\big((f_w)_n * \tau_1^n\ \big)\ d\mu(z) \\
&= \big(u * (f_w)_n * \tau_1^n\ \big)(0) \\
&= \big((u * \tau_1^n) * (f_w)_n\ \big)(0) \\
&= \int_D (u * \tau_1^n)(z)\ (f_w)_n(z)\ d\mu(z),
\end{aligned}$$

since the convolution is commutative and associative because $u, \tau, (f_w)_n$ are radial. Hence we get,

$$\left| \int_D u(z)\ f(z,w)\ d\mu(z) \right| \ \leq \ \|(f_w)_n\|_\infty \ \| \ u * \tau_1^n \ \|_{L^1(\tau)}.$$

Here we have $\|(f_w)_n\|_\infty \ \leq \ M\|f\|_\infty$ and by Theorem 1.1 ([3, Theorem 2.3]), we get

$$\| \ u * \tau_1^n \ \|_{L^1(\tau)} \ \to \ 0 \qquad \text{as} \qquad n \to \infty.$$

Hence we conclude that

$$\int_D u(z) f(z,w)\ d\mu(z) = 0,$$

which means $f(z,w)$ is a constant for every fixed w; i.e., f is a function of w variable only.

Let $f(z,w) = g(w)$. Then $Tf = f$ implies that $g * \tau_2 = g$. Now applying Theorem 1.2 ([3, Theorem 3.1]), we conclude that g is a constant (since g is radial and harmonic). Hence f is a constant.

The rest of the proof follows a well-known traditional method (See, for example, proofs of Theorem 3.1 of [3] or Corollary 19 of [4]). Let $f \in L^\infty(D^2)$ satisfies $Tf = f$. Then we consider the 2-radialization Rf of f defined by

$$(Rf)(z,w) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(ze^{i\theta}, we^{i\eta})\ d\theta\ d\eta,$$

which satisfies, by Fubini's theorem, that $T(Rf) = R(Tf) = Rf$. Hence by previous argument Rf is a constant, which means

$$f(0,0) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f(ze^{i\theta}, we^{i\eta})\ d\theta\ d\eta \quad \text{for all} \quad z, w \in D. \tag{3.1}$$

Now pick $(z,w) \in D^2$ and let $\psi \in \mathrm{Aut}(D^2)$ be defined by $\psi(x,y) = (\varphi_z(x), \varphi_w(y))$. Then by the rotation invariance of τ_1 and τ_2, we can easily get $T(f \circ \psi) = Tf \circ \psi = f \circ \psi$. Thus we can replace f by $f \circ \psi$ in (3.1) to get

$$f(z,w) = \frac{1}{(2\pi)^2} \int_0^{2\pi} \int_0^{2\pi} f\big(\varphi_z(xe^{i\theta}), \varphi_w(ye^{i\eta})\ \big)\ d\theta\ d\eta \quad \text{for all} \quad x, y \ \in \ D.$$

Put $y = 0$ in the above equation then use 4.2.4 of [9] to get $\Delta_1 f = 0$, then we put $x = 0$ to get $\Delta_2 f = 0$. Hence f is 2-harmonic, which completes the proof of the theorem. □

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References

[1] P. Ahern, M. Flores, and W. Rudin, *An invariant volume-mean-value property*, J. Funct. Anal. **111** (1993), no. 2, 380–397.

[2] J. Arazy and M. Engliš, *Iterates and the boundary behavior of the Berezin transform*, Ann. Inst. Fourier (Grenoble) **51** (2001), no. 4, 1101–1133.

[3] Y. Benyamini and Y. Weit, *Harmonic analysis of spherical functions on* SU(1, 1), Ann. Inst. Fourier (Grenoble) **42** (1992), no. 3, 671–694.

[4] M. Engliš, *Functions invariant under the Berezin transform*, J. Funct. Anal. **121** (1994), no. 1, 233–254.

[5] H. Furstenberg, *A Poisson formula for semi-simple Lie groups*, Ann. of Math. (2) **77** (1963), 335–386.

[6] ———, *Boundaries of Riemannian symmetric spaces*, Symmetric spaces (Short Courses, Washington Univ., St. Louis, Mo., 1969–1970), pp. 359–377. Pure and Appl. Math., Vol. 8, Dekker, New York, 1972.

[7] Y. Katznelson and L. Tzafriri, *On power bounded operators*, J. Funct. Anal. **68** (1986), no. 3, 313–328.

[8] J. Lee, *Weighted Berezin transform in the polydisc*, J. Math. Anal. Appl. **338** (2008), no. 2, 1489–1493.

[9] W. Rudin, *Function Theory in the Unit Ball of* $\mathbb{C}^n$, Springer-Verlag, New York Inc., 1980.

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