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LINEAR MAPS PRESERVING IDEMPOTENT OPERATORS

Ali Taghavi and Roja Hosseinzadeh

Abstract. Let A and B be some standard operator algebras on complex Banach spaces X and Y, respectively. We give the concrete forms of linear idempotence preserving maps $\Phi : A \longrightarrow B$ on finite-rank operators.

1. Introduction

The study of linear maps on operator algebras preserving certain properties or subsets is a topic which attracts much attention of many authors. Let $B(X)$ and $B(Y)$ be the algebras of all bounded linear operators on complex Banach spaces X and Y, respectively. In [5], the additive idempotence preserving surjections $\Phi : B(X) \longrightarrow B(Y)$ which have another several properties, are considered. Also in [3] and [7] the concrete forms of surjective maps on $B(X)$ which preserves the nonzero idempotency of products of operators have been given.

Recall that a standard operator algebra on X is a norm closed subalgebra of $B(X)$ which contains the identity and all finite-rank operators. In this paper, we consider linear surjections between standard operator algebras such that preserve idempotent operators. Let $P(X) = \{P \in B(X) : P^2 = P\}$ be the set of all idempotent operators and $N(X) = \{N \in B(X) : N^k = 0 \text{ for some positive integer} k\}$ be the set of all nilpotent operators. We denote by $P_1(X)$ and $N_1(X)$ the set of all rank-1 idempotent operators and the set of all rank-1 nilpotent operators in $B(X)$, respectively. If X has dimension n with $2 \leq n < \infty$, $B(X)$ is identified with the algebra M_n of $n \times n$ complex matrices. Let X' denote the dual space of X and $\dim X$ denote the dimension of X. For an operator $T \in B(X)$, $R(T)$ and $\operatorname{rank} T$ denote the range and rank of T, respectively. Let $F(X)$ and $F_1(X)$ denote the set of all finite-rank operators and the set of all rank-1 operators in $B(X)$, respectively. For every nonzero $x \in X$ and $f \in X'$, the symbol $x \otimes f$ stands for the rank-1 linear operator on X defined by $(x \otimes f)y = f(y)x$ for any $y \in X$. Note that every rank-1 operator in $B(X)$ can be written in this way. The rank-1 operator $x \otimes f$ is an idempotent if and only if $f(x) = 1$ and $x \otimes f$ is an nilpotent if and only if $f(x) = 0$. Given

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$P, Q \in P(X)$, we say $P < Q$ if $PQ = QP = P$ and $P \neq Q$. In addition, we say that P and Q are orthogonal if $PQ = QP = 0$.

The aim of this paper is to prove the following theorems.

Theorem 1.1. *Let A and B be some standard algebras on complex infinite-dimensional Banach spaces X and Y, respectively. Also let $\Phi : A \longrightarrow B$ be a linear surjective map. If Φ preserves idempotent operators in both direction, then one of the following forms hold.*

(1) *There exists a bijective bounded linear or conjugate linear operator $A : X \longrightarrow Y$ such that*

$$\Phi(T) = ATA^{-1}$$

for all $T \in F(X)$, or

(2) *there exists a bijective bounded linear or conjugate linear operator $A : X' \longrightarrow Y$ such that*

$$\Phi(T) = AT'A^{-1}$$

for all $T \in F(X)$, where T' denote the dual operator of T.

Note that the following theorem was proved in [5]. But in this note we show it is a consequence of above the theorem.

Theorem 1.2. *Let Φ be a linear map on M_n. Then Φ preserves idempotent operators in both direction if and only if there exists an invertible $A \in M_n$ such that one of the following forms hold.*

(1) $\Phi(T) = ATA^{-1}$ *for all $T \in M_n$;*

(2) $n = 2$ *and $\Phi(T) = AT^tA^{-1}$ for all $T \in M_n$, where A^t denotes the transpose of A.*

Note that, linearity of $\Phi\mid_{F(X)}$ in main theorems is enough.

2. Proofs

Assume that X and Y are complex infinite dimensional Banach spaces, A and B are some standard algebras and $\Phi : A \longrightarrow B$ is an additive map which preserves idempotent operators in both directions.

First we prove some elementary results which are useful in the proofs of main theorems.

Lemma 2.1. Φ *is injective.*

Proof. Assume on the contrary that there exists a nonzero operator $T \in A$ such that $\Phi(T) = 0$. Then T is idempotent and it is clear that there exists an idempotent operator S such that $T + S$ is not idempotent. By the hypothesis, since Φ preserves idempotent operators in both directions, $\Phi(T + S) = \Phi(S)$ is not idempotent. This is a contradiction and hence Φ is injective. □

Lemma 2.2. *Let $N \in A$ be of finite-rank and $N^2 = 0$. Then $\Phi(N)^2 = 0$.*

Proof. We know that $R(N)$ is finite dimensional. So we have $X = R(N) \oplus M$ for some closed subspace M of X. Thus by this decomposition N has the following operator matrix

$$N = \begin{pmatrix} 0 & N_1 \\ 0 & 0 \end{pmatrix}.$$

Put

$$P = \begin{pmatrix} I & 0 \\ 0 & 0 \end{pmatrix}.$$

It is clear that $P + nN \in P(X)$ for all $n \in N$. It's clear that $\dim R(P) = \dim R(N)$. So $P \in A$ and hence $P + nN \in A$ for all $n \in N$. Thus $(\Phi(P) + n\Phi(N)) \in P(Y)$ for all $n \in N$. That is,

$$\Phi(P) + n\Phi(N) = (\Phi(P))^2 + n(\Phi(P)\Phi(N) + \Phi(N)\Phi(P)) + n^2(\Phi(N))^2$$

for all $n \in N$. Setting $n = 1$ and $n = 2$ in last the equality yield

$$\Phi(N) = \Phi(P)\Phi(N) + \Phi(N)\Phi(P) + (\Phi(N))^2,$$

$$2\Phi(N) = 2(\Phi(P)\Phi(N) + \Phi(N)\Phi(P)) + 4(\Phi(N))^2$$

that imply $\Phi(N)^2 = 0$ and hence the proof is complete. □

We next assume that Φ is surjective.

Lemma 2.3. $\Phi(I) = I$.

Proof. Since $I \in B$, by surjectivity of Φ, there exists $T \in A$ such that $\Phi(T) = I$. Assume on the contrary that $T \neq I$. Then T is idempotent and there exists an idempotent operator S such that $T - S$ is not idempotent. By the hypothesis, since Φ preserves idempotent operators in both directions, $\Phi(T-S) = I - \Phi(S)$ is not idempotent. This is a contradiction and hence $\Phi(I) = I$. □

Lemma 2.4. *Φ preserves the orthogonality of idempotents in both directions.*

Proof. If $P, Q \in A$ are the idempotent operators such that $P \perp Q$, then $P + Q \in P(X)$ and hence $\Phi(P) + \Phi(Q) \in P(Y)$. Since $\Phi(P), \Phi(Q) \in P(Y)$, we obtain

$$\Phi(P)\Phi(Q) + \Phi(Q)\Phi(P) = 0.$$

Thus we have

$$\Phi(P)\Phi(Q)\Phi(P) + \Phi(Q)\Phi(P) = 0$$

and

$$\Phi(P)\Phi(Q) + \Phi(P)\Phi(Q)\Phi(P) = 0.$$

These together imply that

$$\Phi(P)\Phi(Q)\Phi(P) = 0$$

and hence

$$\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = 0$$

that implies $\Phi(P) \perp \Phi(Q)$. The converse is similar. The proof is complete. □

Lemma 2.5. *Φ preserves the order of idempotents in both directions.*

Proof. If $P, Q \in A$ are the idempotent operators such that $P < Q$, then

$$(Q - I)P = P(Q - I) = 0$$

and so $P \perp (Q - I)$. By Lemmas 2.4 and 2.3 we obtain

$$\Phi(P) \perp (\Phi(Q) - I)$$

that is,

$$\Phi(P)(\Phi(Q) - I) = (\Phi(Q) - I)\Phi(P) = 0.$$

This implies

$$\Phi(P)\Phi(Q) = \Phi(Q)\Phi(P) = \Phi(P)$$

that is,

$$\Phi(P) < \Phi(Q).$$

The converse is similar. The proof is complete. □

Lemma 2.6. $\Phi(P_1(X)) = P_1(Y)$.

Proof. If $P \in P_1(X)$ and rank$\Phi(P) \geq 2$, then there exists $R \in P_1(Y)$ such that $R < \Phi(P)$. Since $R \in B$, there exists $R_1 \in A$ such that $R = \Phi(R_1)$. By Lemma 2.5, from $\Phi(R_1) < \Phi(P)$ we obtain $R_1 < P$. Since P is rank-1, so $\operatorname{rank} R_1 = 0$, that implies $R_1 = 0$ and then $R = 0$. This contradiction show $\operatorname{rank} \Phi(P) = 1$. Hence $\Phi(P_1(X)) \subseteq P_1(Y)$. Moreover, Φ^{-1} has the same property of Φ. Therefore $\Phi(P_1(X)) = P_1(Y)$. □

Lemma 2.7. $\Phi(N_1(X)) = N_1(Y)$.

Proof. Let $N = x \otimes f \in N_1(X)$ for some nonzero $x \in X$ and nonzero $f \in X'$ such that $f(x) = 0$. Then $\Phi(N) \in N(X)$ by Lemma 2.2. Take $f_1 \in X'$ such that $f_1(x) = 1$. If $Q = x \otimes f_1$, then Q and $Q+N$ are in $P_1(X)$. So by Lemma 2.6 we can write $\Phi(Q) = y_1 \otimes g_1$ and $\Phi(Q + N) = y_2 \otimes g_2$ for some $y_1, y_2 \in X$ and $g_1, g_2 \in X'$ such that $g_1(y_1) = g_2(y_2) = 1$. Put $P = \frac{1}{2}((Q + N) + Q)$. It's clear that $P \in P_1(X)$ and then by Lemma 2.6, $\Phi(P) \in P_1(X)$. We Know

$$\Phi(P) = \frac{1}{2}(\Phi(Q + N) + \Phi(Q)) = \frac{1}{2}(y_1 \otimes g_1 + y_2 \otimes g_2).$$

It follows that either y_1 and y_2 or g_1 and g_2 are linearly dependent. If y_1 and y_2 are linearly dependent, then we may assume that $y_1 = y_2$. Thus

$$\Phi(P) = \frac{1}{2} y_1 \otimes (g_1 + g_2)$$

that implies $g_1(y_1) + g_2(y_1) = 2$. Since $g_1(y_1) = 1$, we obtain $g_2(y_1) = 1$. Thus $\Phi(N) = \Phi(Q + N) - \Phi(Q) = y_1 \otimes (g_1 - g_2)$ and $(g_1 - g_2)(y_1) = 0$. Hence $\Phi(N) \in N_1(X)$. By similar discussion we obtain the same result if g_1 and g_2 are linearly dependent. The proof is complete. □

We next assume that ϕ is linear.

Lemma 2.8. $\Phi(F_1(X)) = F_1(Y)$ *and* $\Phi(F(X)) = F(Y)$.

Proof. Since every non-nilpotent rank-1 operator is a nonzero scalar multiple of rank-1 idempotent operator, we know that $\Phi(F_1(X)) = F_1(Y)$ by Lemmas 2.6 and 2.7 and also by linearity of Φ. Moreover every finite-rank operator can be written as a linear combination of finitely many rank-1 operators. It follows from the linear property of Φ that $\Phi(F(X)) = F(Y)$. □

Proof of Theorem 1.1. Now by the mentioned properties of Φ, Theorem 1.2 in [4] assures us that either

(i) there exist linear transformations $A : X \longrightarrow Y$ and $C : X' \longrightarrow Y'$ such that $\Phi(x \otimes f) = Ax \otimes Cf$ for all $x \in X$ and $f \in X'$; or

(ii) there exist linear transformations $A : X' \longrightarrow Y$ and $C : X \longrightarrow Y'$ such that $\Phi(x \otimes f) = Af \otimes Cx$ for all $x \in X$ and $f \in X'$.

Since $\Phi \mid_{F(X)}$ is bijective, in both cases, A and C are bijective. Suppose (1) holds. By Lemmas 2.6 and 2.7 and the linearity of Φ, it can be shown that $Cf(Ax) = f(x)$ for all $x \in X$ and $f \in X'$, which implies that C is the adjoint of A^{-1}, and hence C is bounded. Thus, A^{-1} and A are bounded too. Furthermore, for any $y \in X$,

$$\Phi(x \otimes f)y = (Ax \otimes Cf)y = (Cf)(y)Ax = f(A^{-1}y)Ax = A(x \otimes f)A^{-1}y.$$

Thus $\Phi(T) = ATA^{-1}$ for all rank-1 operator T and so the assertion follows easily by linearity of Φ.

Suppose (ii) holds. Then by a similar argument, we can show that $\Phi(T) = AT'A^{-1}$ for all rank-1 operator T and hence $\Phi(T) = AT'A^{-1}$ for all $T \in F(X)$. The proof is complete. □

Proof of Theorem 1.2. The sufficient part is obvious. Let Φ be a linear map on M_n preserving idempotent operators in both directions. It is easy to check that all of the above lemmas are true when X be a Banach space with $\dim X \geq 2$. Then by Lemma 2.1, Φ is injective and thus surjective. So by these and another properties of Φ that have been mentioned by above lemmas, again Theorem 1.2 in [4] assures us the same mentioned two cases in the above proof occur. If (i) holds, then we easily have that $\Phi(T) = ATB$ for all $T \in M_n$. It is clear that $B = A^{-1}$. We have the form (1).

If (ii) holds, then we similarly have that $\Phi(T) = AT^tA^{-1}$ for all $T \in M_n$. However this can not occur if $n \geq 3$. Thus in this case we have $n = 2$. The proof is complete. □

Since $B(X)$ is a standard operator algebra, so we can have following result.

Theorem 2.9. *Let X and Y be complex infinite-dimensional Banach spaces and $\Phi : B(X) \longrightarrow B(Y)$ be a linear map. If Φ preserves idempotent operators in both directions, then one of the following forms hold.*

(1) *There exists a bijective bounded linear or conjugate linear operator $A : X \longrightarrow Y$ such that*

$$\Phi(T) = ATA^{-1}$$

for all $T \in F(X)$, *or*

(2) *there exists a bijective bounded linear or conjugate linear operator* $A : X' \longrightarrow Y$ *such that*

$$\Phi(T) = AT'A^{-1}$$

for all $T \in F(X)$.

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Ali Taghavi
Department of Mathematics
Faculty of Basic Sciences
University of Mazandaran
P. O. Box 47416-1468, Babolsar, Iran
E-mail address: taghavi@umz.ac.ir

Roja Hosseinzadeh
Department of Mathematics
Faculty of Basic Sciences
University of Mazandaran
P. O. Box 47416-1468, Babolsar, Iran
E-mail address: ro.hosseinzadeh@umz.ac.ir