THE UNIQUENESS THEOREMS OF MEROMORPHIC FUNCTIONS SHARING THREE VALUES AND ONE PAIR OF POLYNOMIALS

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ABSTRACT. In this paper, we deal with a uniqueness theorem of two non-constant meromorphic functions that share three values and one pair of polynomials. The results in this paper improve those given by G. G. Gundersen, G. Brosch and other authors.

1. Introduction and main results

In this paper, by meromorphic functions we will always mean meromorphic functions in the complex plane. We adopt the standard notations in the Nevanlinna theory of meromorphic functions as explained in [8] and [18]. It will be convenient to let E denote any set of positive real numbers of finite linear measure, not necessarily the same at each occurrence. For a nonconstant meromorphic function h, we denote by $T(r,h)$ the Nevanlinna characteristic of h and by $S(r,h)$ any quantity satisfying $S(r,h) = o\{T(r,h)\}\,(r \to \infty, r \notin E)$.

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. We say that f and g share the value a CM, provided that f and g have the same a-points with the same multiplicities. Similarly, we say that f and g share the value a IM, provided that f and g have the same a-points ignoring multiplicities (see [18]). We say that a is a small function of f, if a is a meromorphic function satisfying $T(r,a) = S(r,f)$ as $r \to \infty$. In addition, we need the following definition.

Definition 1.1 (see [3, Definition 1]). Let p be a positive integer and $a \in C \cup \{\infty\}$. Then by $N_{p)}(r, 1/(f-a))$ we denote the counting function of those zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not greater than p, by $\overline{N}_{p)}(r, 1/(f-a))$ we denote the corresponding reduced

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counting function (ignoring multiplicities). By $N_{(p}(r, 1/(f - a))$ we denote the counting function of those zeros of $f - a$ (counted with proper multiplicities) whose multiplicities are not less than p, by $\overline{N}_{(p}(r, 1/(f - a))$ we denote the corresponding reduced counting function (ignoring multiplicities).

Let f and g be two nonconstant meromorphic functions, and let a be a value in the extended plane. Let S be a subset of distinct elements in the extended plane. Next we define

$$E_f(S) = \bigcup_{a \in S} \{z : f(z) = a\},$$

where each a-point of f with multiplicity m is repeated m times in $E_f(S)$ (see [6]). Similarly, we define

$$\overline{E}_f(S) = \bigcup_{a \in S} \{z : f(z) = a\},$$

where each point in $\overline{E}_f(\{a\})$ is counted only once. We say that f and g share the set S CM, provided $E_f(S) = E_g(S)$. We say that f and g share the set S IM, provided $\overline{E}_f(S) = \overline{E}_g(S)$. Next by the notation $f = a \Longrightarrow g = a$ we denote $\overline{E}_f(\{a\}) \subseteq \overline{E}_g(\{a\})$.

In 1926, R. Nevanlinna proved the following theorem.

Theorem A (see [17]). *If f and g are nonconstant meromorphic functions that share five values IM, then $f = g$.*

Theorem B (see [17]). *If f and g are distinct nonconstant meromorphic functions that share four values a_1, a_2, a_3 and a_4 CM, then f is a Möbius transformation of g, two of the shared values, say a_1 and a_2, are Picard values, and the cross ratio $(a_1, a_2, a_3, a_4) = -1$.*

In 1979, G. G. Gundersen proved the following theorem, which improved Theorem B.

Theorem C (see [7, Theorem 1]). *Let f and g be two distinct nonconstant meromorphic functions such that f and g share three values CM and share a fourth value IM. Then f and g share all four values CM, and hence the conclusion of Theorem B holds.*

In 1989, G. Brosch proved the following theorem, which improved Theorem B and Theorem C.

Theorem D (see [5]). *Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1 and ∞ CM, and let a and b be two distinct complex numbers such that $a, b \notin \{0, 1\}$. If $f - a$ and $g - b$ share 0 IM, then f is a Möbius transformation of g.*

Regarding Theorem D, it is natural to ask the following two questions.

Question 1.1 (see [9]). Is it really possible to relax in any way the nature of sharing any one of 0, 1 and ∞ in Theorem D ?

Question 1.2. What can be said if a and b in Theorem D are replaced with two distinct nonconstant polynomials P_1 and P_2 respectively ?

Recently many mathematicians in the world have done a lot of research works concerning Question 1.1, such as T. C. Alzahary [2, 3], I. Lahiri and P. Sahoo [11], X. M. Li and H. X. Yi [12], etc. In these research works, the notion of weighted sharing of values has been used, which measures how close a shared value is to being shared IM or to being shared CM. The notion is explained in the following definition.

Definition 1.2 (see [10, Definition 4]). Let k be a nonnegative integer or infinity. For any $a \in C \cup \{\infty\}$, we denote by $E_k(a, f)$ the set of all a-points of f, where an a-point of multiplicity m is counted m times if $m \le k$, and $k + 1$ times if $m > k$. If $E_k(a, f) = E_k(a, g)$, we say that f, g share the value a with weight k.

Remark 1.1. Definition 1.2 implies that if f, g share a value a with weight k, then z_0 is a zero of $f - a$ with multiplicity $m\,(\le k)$ if and only if it is a zero of $g - a$ with multiplicity $m\,(\le k)$, and z_0 is a zero of $f - a$ with multiplicity $m\,(> k)$, if and only if it is a zero of $g - a$ with multiplicity $n\,(> k)$, where m is not necessarily equal to n. Throughout this paper, we write f, g share (a, k) to mean that f, g share the value a with weight k. Clearly, if f, g share (a, k), then f, g share (a, p) for all integer p, $0 \le p < k$. Also we note that f, g share a value a IM or CM if and only if f, g share $(a, 0)$ or (a, ∞), respectively.

In this paper, we will prove the following two theorems that deals with Question 1.2.

Theorem 1.1. *Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1, ∞ CM, and let P_1 and P_2 be two nonconstant polynomials such that $P_1 \not\equiv P_2$. If $f - P_1$ and $g - P_2$ share 0 IM, then f and g are transcendental meromorphic functions and satisfy one of the following three relations*: (i) $f + g = 1$ *with* $P_1 + P_2 = 1$; (ii) $f = \frac{P_1}{P_2} \cdot g$; (iii) $f = \frac{P_1 - 1}{P_2 - 1} \cdot g + \frac{P_2 - P_1}{P_2 - 1}$.

Theorem 1.2. *Let f and g be two nonconstant entire functions that share 0 and 1 CM, and let P_1 and P_2 be two nonconstant polynomials such that $P_1 \not\equiv P_2$. If $f - P_1 = 0 \Longrightarrow g - P_2 = 0$, then $f = g$.*

2. Some lemmas

Lemma 2.1 (see [7, Theorem 3]). *Let f and g share 0, 1, ∞ IM. Then*

$$(\frac{1}{3} + o(1))T(r, g) < T(r, f) < (3 + o(1))T(r, g)\,(r \notin E).$$

Lemma 2.2 (see [19, Lemma 2.6]). *Let f and g be two distinct nonconstant meromorphic functions that share $(0, k_1)$, $(1, k_2)$ and (∞, k_3), where k_1, k_2 and k_3 are three positive integers satisfying*

$$(2.1) \qquad\qquad k_1 + k_2 + k_3 > k_1 k_2 k_3 + 2.$$

Then   (i) $\overline{N}_{(2}(r, \frac{1}{f}) + \overline{N}_{(2}(r, \frac{1}{f-1}) + \overline{N}_{(2}(r, f) = S(r, f);$

          (ii) $\overline{N}_{(2}(r, \frac{1}{g}) + \overline{N}_{(2}(r, \frac{1}{g-1}) + \overline{N}_{(2}(r, g) = S(r, f).$

Lemma 2.3. *Let f and g be two nonconstant rational functions that share $(0, k_1)$, $(1, k_2)$ and (∞, k_3), where k_1, k_2, k_3 are three positive integers satisfying (2.1). Then $f = g$.*

Proof. Suppose that $f \not\equiv g$. From the fact that f and g are two nonconstant rational functions we have

$$(2.2) \qquad\qquad T(r, f) \le A_1 \log r, \quad T(r, g) \le A_2 \log r,$$

where A_1 and A_2 are positive numbers. Let

$$(2.3) \qquad\qquad \alpha_1 = \frac{f'}{f-1} - \frac{g'}{g-1}$$

and

$$(2.4) \qquad\qquad \beta_1 = \frac{f'}{f} - \frac{g'}{g}.$$

From (2.3), (2.4) and Lemma 2.2 we get

$$(2.5) \qquad\qquad T(r, \alpha_1) + T(r, \beta_1) = S(r, f),$$

which together with $\lim\limits_{\substack{r \to \infty \\ r \notin E}} S(r, f)/T(r, f) = 0$ implies

$$(2.6) \qquad\qquad \lim_{\substack{r \to \infty \\ r \notin E}} \frac{T(r, \alpha_1) + T(r, \beta_1)}{T(r, f)} = 0.$$

From the fact that f and g are two nonconstant rational functions we see that α_1 and β_1 are rational functions. Thus

$$(2.7) \qquad\qquad T(r, \alpha_1) + T(r, \beta_1) = O(\log r).$$

Suppose that one of α_1 and β_1 is not a constant. Then there exists some positive number A_3 such that

$$(2.8) \qquad\qquad T(r, \alpha_1) + T(r, \beta_1) \ge A_3 \log r.$$

From the left inequality of (2.2) and (2.8) we get

$$\frac{T(r, \alpha_1) + T(r, \beta_1)}{T(r, f)} \ge \frac{A_3}{A_1},$$

which contradicts (2.6). Thus α_1 and β_1 are constants, say $\alpha_1 = c_1$ and $\beta_1 = c_2$. Then (2.3) and (2.4) can be rewritten as

$$(2.9) \qquad \frac{f'}{f-1} - \frac{g'}{g-1} = c_1$$

and

$$(2.10) \qquad \frac{f'}{f} - \frac{g'}{g} = c_2$$

respectively. From (2.9) and (2.10) we get

$$(2.11) \qquad f - 1 = A_4(g - 1)$$

and

$$(2.12) \qquad f = A_5 g,$$

where $A_4 \,(\neq 0)$ and $A_5 \,(\neq 0)$ are two complex numbers. From (2.11) and (2.12) we get $A_5 g - 1 = A_4(g - 1)$, which implies $A_4 = A_5$, and so it follows from (2.11) and (2.12) that $A_4 = A_5 = 1$. Thus $f = g$, which contradicts the above supposition. Lemma 2.3 is thus completely proved. $\qquad\square$

Lemma 2.4 (see [18, Theorem 1.5]). *If f is a transcendental meromorphic function in the complex plane, then $\lim\limits_{r\to\infty} T(r,f)/\log r = \infty$.*

Let f and g be two nonconstant meromorphic functions in the complex plane, and a be a value in the extended plane. Let $\overline{N}_E(r,a)$ "count" those points in $\overline{N}(r, 1/(f-a))$, where a is taken by f and g with the same multiplicity, and each point is counted only once, and let $\overline{N}_0(r,a)$ be the reduced counting function of the common a-points of f and g in $\overline{N}(r, 1/(f-a))$, where $\overline{N}(r, 1/(f-\infty))$ means $\overline{N}(r,f)$. We say that f and g share the value a CM*, if

$$\overline{N}(r, \frac{1}{f-a}) - \overline{N}_E(r,a) = S(r,f)$$

and

$$\overline{N}(r, \frac{1}{g-a}) - \overline{N}_E(r,a) = S(r,g).$$

We say that f and g share the value a IM*, if

$$\overline{N}(r, \frac{1}{f-a}) - \overline{N}_0(r,a) = S(r,f)$$

and

$$\overline{N}(r, \frac{1}{g-a}) - \overline{N}_0(r,a) = S(r,g).$$

The two notions can be found in [13] or [18]. If there exist four small functions α_1, α_2, α_3, α_4 of f and g such that $f = (\alpha_1 g + \alpha_2)/(\alpha_3 g + \alpha_4)$, where $\alpha_1 \alpha_4 - \alpha_2 \alpha_3 \not\equiv 0$, then we say that f is a quasi-Möbius transformation of g (see [13] or [18]).

Lemma 2.5 (see [13, Theorem 3]). *Let f and g be two distinct nonconstant meromorphic functions that share 0, 1, ∞ CM*, and let $a \not\equiv 0, 1, \infty$ be a small function of f and g. If $T(r,f) \neq N(r, 1/(f-a)) + S(r,f)$, then f is a quasi-Möbius transformation of g such that f and g satisfy one of the following three relations:* (i) $f = ag$;   (ii) $f + (a-1)g = a$;   (iii) $(f-a)(g+a-1) = a(1-a)$.

Lemma 2.6 (see [1, Lemma 3]). *Let f and g be two distinct nonconstant meromorphic functions that share 0, 1, ∞ CM. If f is a Möbius transformation of g, then f and g satisfy one of the following six relations.*

(i) $fg = 1$;   (ii) $(f-1)(g-1) = 1$;

(iii) $f + g = 1$;   (iv) $f = cg$;

(v) $f - 1 = c(g-1)$;   (vi) $\{(c-1)f + 1\} \cdot \{(c-1)g - c\} = -c$;

where $c \neq 0, 1$ is a complex number.

Lemma 2.7 (see [18, Theorem 1.62]). *Let $f_1, f_2, \ldots, f_n$ be non-constant meromorphic functions, and let $f_{n+1} (\not\equiv 0)$ be a meromorphic function such that $\sum_{j=1}^{n+1} f_j = 1$. If there exists a subset $I \subseteq R^+$ satisfying $\mathrm{mes}\, I = \infty$ such that*

$$\sum_{i=1}^{n+1} N(r, \frac{1}{f_i}) + n \sum_{\substack{i=1 \\ i \neq j}}^{n+1} \overline{N}(r, f_i) < (\lambda + o(1)) T(r, f_j) \ (r \to \infty, \ r \in I, \ j = 1, 2, \ldots, n),$$

where $\lambda < 1$. Then $f_{n+1} = 1$.

Lemma 2.8. *Let f and g be two transcendental meromorphic functions that share 0, 1, ∞ CM, and let P be a nonconstant polynomial. If*

$$(2.13) \qquad\qquad T(r,f) \neq N(r, \frac{1}{f-P}) + S(r,f),$$

then $f = g$.

Proof. Let f and g be distinct. First, from (2.13) and Lemma 2.5 we see that f and g satisfy one of the three relations: $f = Pg$, $f + (P-1)g = P$ and $(f-P)(g+P-1) = P(1-P)$. By the condition that f and g share 0, 1, ∞ CM we have

$$(2.14) \qquad\qquad \frac{f-1}{g-1} = e^{\alpha}$$

and

$$(2.15) \qquad\qquad \frac{f}{g} = e^{\alpha - \beta},$$

where α and β are entire functions. From the supposition $f \not\equiv g$ we have $e^{\alpha} \not\equiv 1$, $e^{\beta} \not\equiv 1$ and $e^{\alpha - \beta} \not\equiv 1$. Combining (2.14) and (2.15), we get

$$(2.16) \qquad\qquad f = \frac{e^{\alpha} - 1}{e^{\beta} - 1}$$

and

$$(2.17) \qquad g = \frac{e^{-\alpha} - 1}{e^{-\beta} - 1}.$$

If one of e^α, e^β and $e^{\beta-\alpha}$ is a constant, from (2.14) and (2.15) we see that f is a Möbius transformation of g. Thus f and g satisfy one of the six relations (i)-(vi) of Lemma 2.6. From this we see that there exist two distinct Picard exceptional values of f and g. This together with Nevanlinna's three small functions theorem (see [18, Theorem 1.36]) implies

$$(2.18) \qquad T(r, f) = N(r, \frac{1}{f - P}) + S(r, f),$$

which contradicts (2.13). Thus none of e^α, e^β and $e^{\beta-\alpha}$ is a constant. If f and g satisfy $f = Pg$, from (2.15) we have $Pe^{\beta-\alpha} = 1$, which is impossible. If f and g satisfy

$$(2.19) \qquad f + (P - 1)g = P,$$

by substituting (2.16) and (2.17) into (2.19) we get

$$(2.20) \qquad e^\alpha - e^\beta + (1 - P) \cdot e^{\beta-\alpha} = 1 - P.$$

Since none of e^α, e^β and $e^{\beta-\alpha}$ is a constant, from (2.20) and Lemma 2.7 we get a contradiction. If f and g satisfy

$$(2.21) \qquad (f - P)(g + P - 1) = P(1 - P),$$

by substituting (2.16) and (2.17) into (2.21) we get

$$(2.22) \quad Pe^{\alpha+\beta} - e^\beta + (1 - P)e^\alpha - Pe^{2\beta} + Pe^{2\beta-\alpha} + (1 - P)e^{\beta-\alpha} = 1 - P.$$

If $e^{\alpha+\beta}$ is a constant, then $e^{2\beta-\alpha}$ is not a constant. This together with (2.22) and Lemma 2.7 implies $Pe^{\alpha+\beta} = 1 - P$, which is impossible. If $e^{2\beta-\alpha}$ is a constant, then $e^{\alpha+\beta}$ is not a constant. This together with (2.22) and Lemma 2.7 implies $e^{2\beta-\alpha} = 1 - P$, which is impossible.

If $e^{\alpha+\beta}$ and $e^{2\beta-\alpha}$ are not constants, from (2.22) and Lemma 2.7 we also get a contradiction. Lemma 2.8 is thus completely proved. $\qquad \square$

Lemma 2.9 (see [14, Theorem 4.1]). *Let f and g be two distinct nonconstant meromorphic functions such that f and g share $(0, k_1)$, $(1, k_2)$ and (∞, k_3), where k_1, k_2 and k_3 are three positive integers satisfying (2.1), and let $a \not\equiv 0, 1, \infty$ be a nonconstant small meromorphic function of f and g. Then either $N_{(3}(r, 1/(f - a)) + N_{(3}(r, 1/(g - a)) = S(r, f)$ holds, or f and g satisfy one of the three relations* (i)-(iii) *of Lemma 2.5.*

Lemma 2.10 (see [13, Theorem 2]). *Let f and g be two distinct nonconstant meromorphic functions that share 0, 1, ∞ CM*, and let $a\,(\not\equiv 0, 1, \infty)$ and $b\,(\not\equiv 0, 1, \infty)$ be two small functions of f and g such that $a \not\equiv b$. If $f - a$ and $g - b$ share 0 CM*, then f is a quasi-Möbius transformation of g.*

Lemma 2.11 (see [15, Lemma 2.6]). *Let f and g be two distinct nonconstant meromorphic functions such that f and g share 0, 1, ∞ IM. If f is a quasi-Möbius transformation of g, then f and g assume one of the following six relations.*

(i) $f \cdot g = 1$;   (ii) $(f - 1)(g - 1) = 1$;

(iii) $f + g = 1$;   (iv) $f = cg$;

(v) $f - 1 = c(g - 1)$;   (vi) $[(c - 1)f + 1] \cdot [(c - 1)g - c] = -c$,

where $c \not\equiv 0, 1, \infty$ is a small function of f and g.

Lemma 2.12 (see [18, Theorem 2.14]). *Let P_1 and P_2 be two nonconstant polynomials, and let a and b be two distinct finite complex numbers. If P_1 and P_2 share a and b IM, then $P_1 = P_2$.*

Lemma 2.13 (see [20, Lemma 1]). *Let h be a nonconstant entire function. Then $T(r, h') = o(T(r, e^h))\,(r \to \infty, r \notin E)$.*

Lemma 2.14 (see [22, Lemma 6]). *Let f_1 and f_2 be two nonconstant meromorphic functions satisfying $\overline{N}(r, f_j) + \overline{N}(r, 1/f_j) = S(r)\,(j = 1, 2)$. Then either $\overline{N}_0(r, 1; f_1, f_2) = S(r)$ or there exist two integers $s, t\,(|s| + |t| > 0)$ such that $f_1^s f_1^t \equiv 1$, where and in what follows, $\overline{N}_0(r, 1; f_1, f_2)$ denotes the reduced counting function of f_1 and f_2 related to the common 1-points, and $T(r) = T(r, f_1) + T(r, f_2)$, $S(r) = o(T(r))(r \to \infty, r \notin E)$ only depending on f_1 and f_2.*

Lemma 2.15 (see [16]). *Let f be a nonconstant meromorphic function, and let $F = \sum_{k=0}^{p} a_k f^k \Big/ \sum_{j=0}^{q} b_j f^j$ be an irreducible rational function in f, where the coefficients $\{a_k\}$ and $\{b_j\}$ are small functions of f, and $a_p \not\equiv 0$, $b_q \not\equiv 0$. Then $T(r, F) = d\,T(r, f) + S(r, f)$, where $d = \max\{p, q\}$.*

Lemma 2.16 (see [21, Corollary 2.2]). *Let f and g be two distinct nonconstant entire functions that share two distinct values CM. Then*

$$\lim_{\substack{r \to \infty \\ r \notin E}} T(r, f)/T(r, g) = 1.$$

3. Proof of theorems

Proof of Theorem 1.1. Suppose that $f \not\equiv g$. From Lemma 2.1 and the condition that f and g share 0, 1, ∞ CM we see that f is a rational function if and only if g is a rational function, and f is a transcendental meromorphic function if and only if g is a transcendental meromorphic function. Suppose that f and g are two rational functions. Then it follows from Lemma 2.3 that $f = g$, which contradicts above supposition. Next we suppose that f and g are two transcendental meromorphic functions. Then from Lemma 2.4 we have $T(r, P_1) + T(r, P_2) = o(T(r, f))$, and so P_1, P_2 are small functions of f and

g. From $f \not\equiv g$ and the assumptions of Theorem 1.1 we have (2.14)-(2.17). Suppose that

$$(3.1) \qquad T(r, f) \neq N(r, \frac{1}{f - P_1}) + S(r, f).$$

Then from (3.1) and Lemma 2.8 we get $f = g$, which contradicts the above supposition. Similarly, if

$$T(r, g) \neq N(r, \frac{1}{g - P_2}) + S(r, f),$$

then $f = g$, which contradicts the above supposition. Thus

$$(3.2) \qquad T(r, f) = N(r, \frac{1}{f - P_1}) + S(r, f)$$

and

$$(3.3) \qquad T(r, g) = N(r, \frac{1}{g - P_2}) + S(r, f).$$

If one of $f = P_1 g$, $f + (P_1 - 1)g = P_1$, $(f - P_1)(g + P_1 - 1) = P_1(1 - P_1)$, $f = P_2 g$, $f + (P_2 - 1)g = P_2$ and $(f - P_2)(g + P_2 - 1) = P_2(1 - P_2)$ holds, then in the same manner as in the proof of Lemma 2.8 we get a contradiction. Thus from Lemma 2.9 we get

$$(3.4)$$
$$N_{(3}(r, \frac{1}{f - P_1}) + N_{(3}(r, \frac{1}{g - P_1}) + N_{(3}(r, \frac{1}{f - P_2}) + N_{(3}(r, \frac{1}{g - P_2}) = S(r, f).$$

From (3.2), (3.3) and (3.4) we get

$$(3.5) \qquad T(r, f) = N_{2)}(r, \frac{1}{f - P_1}) + S(r, f)$$

and

$$(3.6) \qquad T(r, g) = N_{2)}(r, \frac{1}{g - P_2}) + S(r, f).$$

we discuss the following two cases.

Case 1. Suppose that

$$(3.7) \qquad \overline{N}_{(1,2)}(r, P_1, P_2) + \overline{N}_{(2,1)}(r, P_1, P_2) = S(r, f),$$

where and in what follows, $\overline{N}_{(l,k)}(r, P_1, P_2)$ denotes the reduced counting function of those common zeros of $f - P_1$ and $g - P_2$, and each such common zero of $f - P_1$ and $g - P_2$ is of $f - P_1$ with multiplicity l, and of $g - P_2$ with multiplicity k. Then from (3.7) and the condition that $f - P_1$ and $g - P_2$ share 0 IM we see that $f - P_1$ and $g - P_2$ share 0 CM*. This together with Lemma 2.10 and the condition that f and g share 0, 1, ∞ CM implies that f is a quasi-Möbius transformation of g. By Lemma 2.11, we discuss the following two subcases.

Subcase 1.1. Suppose that f and g satisfy one of the three relations (i), (ii) and (vi) of Lemma 2.11. If f and g satisfy (i) of Lemma 2.11, from (3.5), (3.6) and the fact that $f - P_1$ and $g - P_2$ share 0 CM* we get $P_1 P_2 = 1$,

which is impossible. Similarly, if f and g satisfy (ii) of Lemma 2.11 we get $(P_1 - 1)(P_2 - 1) = 1$, which is impossible.

If f and g satisfy (vi) of Lemma 2.11, in the same manner as above we get

$$(3.8) \qquad \{(c-1)P_1 + 1\} \cdot \{(c-1)P_2 - c\} = -c.$$

From (3.6) we get

$$(3.9) \qquad c = \frac{P_2(P_1 - 1)}{P_1(P_2 - 1)}.$$

By substituting (2.16) and (2.17) into (vi) of Lemma 2.11 we get

$$(3.10) \qquad (1-c)e^{\beta} - e^{\alpha+\beta} + ce^{\alpha} + e^{2\beta} - e^{2\beta-\alpha} + ce^{\beta-\alpha} = c.$$

If one of e^{α}, e^{β} and $e^{\beta-\alpha}$ is a constant, then f is a Möbius transformation of g such that f and g satisfy one of the six relations (i)-(vi) of Lemma 2.6. From the above supposition and in the same manner as above we get (3.9) and c is a nonzero constant. Thus P_1 and P_2 share 0, 1 CM. This together with Lemma 2.12 gives $P_1 = P_2$, which is impossible. Thus none of e^{α}, e^{β} and $e^{\beta-\alpha}$ is a constant. If $e^{\alpha+\beta}$ is a constant, then $e^{2\beta-\alpha}$ is not a constant. This together with (3.8), (3.10) and Lemma 2.7 gives $e^{\alpha+\beta} = -\{P_2(P_1 - 1)\}/\{P_1(P_2 - 1)\}$, which implies that $e^{\alpha+\beta}$ is a nonzero complex number. Thus P_1 and P_2 share 0, 1 CM. This together with Lemma 2.12 implies that $P_1 = P_2$, which is impossible. If $e^{2\beta-\alpha}$ is a constant, then $e^{\alpha+\beta}$ is not a constant. Proceeding as above, we get $e^{2\beta-\alpha} = -\{P_2(P_1 - 1)\}/\{P_1(P_2 - 1)\}$, and so we get $P_1 = P_2$, which is impossible.

Subcase 1.2. Suppose that f and g satisfy one of the three relations (iii), (iv) and (v) of Lemma 2.11. Combining (3.5), (3.6) and the fact that $f - P_1$ and $g - P_2$ share 0 CM*, we get (i)-(iii) of Theorem 1.1.

Case 2. Suppose that

$$(3.11) \qquad \overline{N}_{(1,2)}(r, P_1, P_2) + \overline{N}_{(2,1)}(r, P_1, P_2) \neq S(r, f).$$

From (3.11) we see that at least one of the two inequalities $\overline{N}_{(2,1)}(r, P_1, P_2) \neq S(r, f)$ and $\overline{N}_{(1,2)}(r, P_1, P_2) \neq S(r, f)$, say

$$(3.12) \qquad \overline{N}_{(2,1)}(r, P_1, P_2) \neq S(r, f)$$

holds. From (3.11) and (3.12) we will prove

$$(3.13) \qquad \overline{N}_{(1,2)}(r, P_1, P_2) = \overline{N}_{(2,1)}(r, P_1, P_2) + S(r, f).$$

In fact, from (2.16) and (2.17) we get

$$(3.14) \qquad f - P_1 = \frac{e^{\alpha} - P_1 e^{\beta} + P_1 - 1}{e^{\beta} - 1},$$

$$(3.15) \qquad g - P_2 = \frac{e^{-\alpha} - P_2 e^{-\beta} + P_2 - 1}{e^{-\beta} - 1}$$

and

(3.16) $$T(r, g) + T(r, e^\alpha) + T(r, e^\beta) = O(T(r, f)) \, (r \notin E).$$

From (3.16) and Lemma 2.13 we get

(3.17) $$T(r, \alpha_2) + T(r, \beta_2) = S(r, f),$$

where and in what follows,

(3.18) $$\alpha_2 = \alpha' \quad \text{and} \quad \beta_2 = \beta'.$$

If $P_1\beta_2 - P_1\alpha_2 + P_1' = 0$, from (3.18) we deduce that there exists a nonzero complex number A_6 such that $e^{\alpha-\beta} = A_6 P_1$, which is impossible. Thus $P_1\beta_2 - P_1\alpha_2 + P_1' \not\equiv 0$. Similarly, we get $(P_1 - P_1^2)\beta_2 + P_1' \not\equiv 0$, $P_1\beta_2 - P_1\alpha_2 + P_1' \not\equiv 0$ and $P_1' + (1 - P_1)\alpha_2 \not\equiv 0$. Let z_0 be a zero of $f - P_1$ with multiplicity 2, and of $g - P_2$ with multiplicity 1, such that $z_0 \notin S_1 \cup S_2 \cup S_3 \cup S_4$, where

(3.19) $$S_1 = \{z : P_1(z)\beta_2(z) - P_1(z)\alpha_2(z) + P_1'(z) = 0\},$$

(3.20) $$S_2 = \{z : \{P_1(z) - P_1^2(z)\}\beta_2(z) + P_1'(z) = 0\},$$

(3.21) $$S_3 = \{z : P_1(z)\beta_2(z) - P_1(z)\alpha_2(z) + P_1'(z) = 0\}$$

and

(3.22) $$S_4 = \{z : P_1'(z) + \{1 - P_1(z)\}\alpha_2(z) = 0\}.$$

From (3.14) we get

(3.23) $$e^{\alpha(z_0)} - P_1(z_0)e^{\beta(z_0)} + P_1(z_0) - 1 = 0$$

and

(3.24) $$\alpha_2(z_0)e^{\alpha(z_0)} - e^{\beta(z_0)}\{P_1'(z_0) + P_1(z_0) \cdot \beta_2(z_0)\} + P_1'(z_0) = 0.$$

From (3.23) and (3.24) we get

(3.25) $$e^{\alpha(z_0)} = \frac{\{P_1(z_0) - P_1^2(z_0)\}\beta_2(z_0) + P_1'(z_0)}{P_1(z_0)\beta_2(z_0) - P_1(z_0)\alpha_2(z_0) + P_1'(z_0)}$$

and

(3.26) $$e^{\beta(z_0)} = \frac{P_1'(z_0) + \{1 - P_1(z_0)\}\alpha_2(z_0)}{P_1(z_0)\beta_2(z_0) - P_1(z_0)\alpha_2(z_0) + P_1'(z_0)}.$$

Let

(3.27) $$f_1 = \frac{(P_1\beta_2 - P_1\alpha_2 + P_1')e^\alpha}{(P_1 - P_1^2)\beta_2 + P_1'}, \qquad f_2 = \frac{(P_1\beta_2 - P_1\alpha_2 + P_1')e^\beta}{P_1' + (1 - P_1)\alpha_2}$$

and

(3.28) $$T(r) = T(r, f_1) + T(r, f_2), \quad S(r) = o(T(r)) \, (r \to \infty, \, r \notin E).$$

From (3.16), (3.17), (3.27) and (3.28) we get

(3.29) $$S(r) = S(r, f)$$

and

$$\overline{N}(r, f_j) + \overline{N}(r, \frac{1}{f_j}) = S(r) \, (j = 1, 2). \tag{3.30}$$

From (3.25)-(3.27) we get $f_1(z_0) = f_2(z_0) = 1$, and so

$$\overline{N}_{(2,1)}(r, P_1, P_2) \le N_0(r, 1; f_1, f_2) + S(r). \tag{3.31}$$

From (3.12), (3.29) and (3.31) we get

$$N_0(r, 1; f_1, f_2) \ne S(r). \tag{3.32}$$

From (3.27), (3.29), (3.30), (3.32) and Lemma 2.14 we know that there exist two integers s and $t \, (|s| + |t| > 0)$ such that

$$f_1^s \cdot f_2^t = 1. \tag{3.33}$$

From (2.14), (2.15), (3.16), (3.17), (3.27), (3.33) and Lemma 2.15 we get

$$T(r, f) = T(r, g) + S(r, f). \tag{3.34}$$

Again from (2.14) and (2.15) we get

$$\frac{e^{\beta(z_0)}}{e^{\alpha(z_0)}} = \frac{P_2(z_0)}{P_1(z_0)}, \qquad \frac{1}{e^{\alpha(z_0)}} = \frac{P_2(z_0) - 1}{P_1(z_0) - 1}. \tag{3.35}$$

Since (3.24) can be rewritten as

$$\alpha_2(z_0) - \frac{e^{\beta(z_0)}}{e^{\alpha(z_0)}} \{P_1'(z_0) + P_1(z_0) \cdot \beta_2(z_0)\} + \frac{P_1'(z_0)}{e^{\alpha(z_0)}} = 0. \tag{3.36}$$

From (3.35) and (3.36) we get

$$\alpha_2(z_0) - \frac{P_2(z_0)}{P_1(z_0)} \cdot \{P_1'(z_0) + P_1(z_0) \cdot \beta_2(z_0)\} + P_1'(z_0) \cdot \frac{P_2(z_0) - 1}{P_1(z_0) - 1} = 0. \tag{3.37}$$

From (3.12) and (3.37) we get

$$\alpha_2 - \frac{P_2(P_1' + P_1 \cdot \beta_2)}{P_1} + \frac{(P_2 - 1)P_1'}{P_1 - 1} = 0. \tag{3.38}$$

From (3.5), (3.6), (3.34) and the condition that $f - P_1$ and $g - P_2$ share 0 IM we get

$$
\begin{aligned}
T(r, f) - T(r, g) &= N_{2)}(r, \frac{1}{f - P_1}) - N_{2)}(r, \frac{1}{g - P_2}) + S(r, f) \\
&= \overline{N}_{(2,1)}(r, P_1, P_2) - \overline{N}_{(1,2)}(r, P_1, P_2) = S(r, f),
\end{aligned}
$$

namely

$$\overline{N}_{(2,1)}(r, P_1, P_2) - \overline{N}_{(1,2)}(r, P_1, P_2) = S(r, f),$$

which together with (3.12) implies

$$\overline{N}_{(2,1)}(r, P_1, P_2) \ne S(r, f) \quad \text{and} \quad \overline{N}_{(1,2)}(r, P_1, P_2) \ne S(r, f). \tag{3.39}$$

From (3.15), the right inequality of (3.39) and in the same manner as in the proof of (3.38) we get

$$(3.40) \qquad -\alpha_2 - \frac{P_1(P_2' - P_2\beta_2)}{P_2} + \frac{(P_1 - 1)P_2'}{P_2 - 1} = 0.$$

By rewriting (3.38) and (3.40) we get

$$(3.41) \qquad \alpha_2 - P_2\beta_2 = \frac{(1 - P_2)P_1'}{P_1 - 1} + \frac{P_2 P_1'}{P_1}$$

and

$$(3.42) \qquad -\alpha_2 + P_1\beta_2 = \frac{(1 - P_1)P_2'}{P_2 - 1} + \frac{P_1 P_2'}{P_2}$$

respectively. From (3.18) we see that α_2 and β_2 are entire functions. Thus from (3.41) and (3.42) we see that P_1 and P_2 share 0 and 1 IM. This together with Lemma 2.12 implies $P_1 = P_2$, which is impossible. Theorem 1.1 is thus completely proved. $\qquad\square$

Proof of Theorem 1.2. Proceeding as in the beginning of the proof of Theorem 1.1 we see that if f and g are polynomials, then the conclusion of Theorem 1.2 holds. Next we suppose that f and g are two distinct transcendental entire functions. Then we have (2.14)-(2.17). Proceeding as in the proof of Theorem 1.1 we have (3.5) and (3.6). From Lemma 2.1 and Lemma 2.16 we get

$$(3.43) \qquad T(r, f) = T(r, g) + S(r, f).$$

From (3.5), (3.6), (3.43) and the condition $f - P_1 = 0 \implies g - P_2 = 0$ we deduce that $f - P_1$ and $g - P_2$ share 0 IM*. Next in the same manner as in Case 1 and Case 2 in the proof of Theorem 1.1 we get (i)-(iii) of Theorem 1.1.

If f and g satisfy (i) of Theorem 1.1, then 0 is a Picard exceptional value of f and g. Thus $f = e^{\alpha_3}$ and $g = e^{\beta_3}$, where α_3 and β_3 are entire functions. Thus (i) of Theorem 1.1 can be rewritten as $e^{\alpha_3} + e^{\beta_3} = 1$. From this and Lemma 2.7 we get a contradiction.

If f and g satisfy (ii) of Theorem 1.1, then

$$(3.44) \qquad \overline{N}(r, \frac{1}{f - 1}) + \overline{N}(r, \frac{1}{f - P_1/P_2}) = S(r, f).$$

From (3.44) and Nevanlinna's three small functions theorem we get

$$T(r, f) \leq \overline{N}(r, \frac{1}{f - 1}) + \overline{N}(r, \frac{1}{f - P_1/P_2}) = S(r, f),$$

which is impossible.

If f and g satisfy (iii) of Theorem 1.1, then $N(r, 1/f) = O(\log r)$. Thus

$$(3.45) \qquad f = P_3 e^{\alpha_4} \quad \text{and} \quad g = P_4 e^{\beta_4},$$

where P_3 and P_4 are nonzero polynomials, α_4 and β_4 are nonconstant entire functions. By substituting (3.45) into (iii) of Theorem 1.1 we get

$$(3.46) \qquad P_3 e^{\alpha_4} - \frac{P_4(P_1 - 1)}{P_2 - 1} \cdot e^{\beta_4} = \frac{P_2 - P_1}{P_2 - 1}.$$

From (3.46) and Lemma 2.7 we get a contradiction. Theorem 1.2 is thus completely proved. $\qquad\square$

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