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THE ALTERNATIVE DUNFORD-PETTIS PROPERTY IN SUBSPACES OF OPERATOR IDEALS

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Abstract. For several Banach spaces X and Y and operator ideal $\mathcal{U}$, if $\mathcal{U}(X,Y)$ denotes the component of operator ideal $\mathcal{U}$; according to Freedman's definitions, it is shown that a necessary and sufficient condition for a closed subspace $\mathcal{M}$ of $\mathcal{U}(X,Y)$ to have the alternative Dunford-Pettis property is that all evaluation operators $\phi_x : \mathcal{M} \to Y$ and $\psi_{y^*} : \mathcal{M} \to X^*$ are DP1 operators, where $\phi_x(T) = Tx$ and $\psi_{y^*}(T) = T^*y^*$ for $x \in X$, $y^* \in Y^*$ and $T \in \mathcal{M}$.

1. Introduction

A Banach space X has the Dunford-Pettis property (DP) if for each weakly convergent sequences $x_n \to x$ in X and $x_n^* \to 0$ in X^*, we have $x_n^*(x_n) \to 0$ as $n \to \infty$. But if under the additional condition $\|x_n\| = \|x\| = 1$ for all integer n, the conclusion $x_n^*(x_n) \to 0$ is obtained, we say that X has the alternative Dunford-Pettis property (DP1).

As an easy consequence of definition, the Banach space X has the DP1 if and only if for each weakly null sequences (x_n) in X and (x_n^*) in X^* and each $x \in X$ with $\|x_n + x\| = \|x\| = 1$, we have $x_n^*(x_n) \to 0$. An straightforward computation also shows that one can replace the condition $\|x_n\| = \|x\| = 1$, in the definition, by the weaker condition $\|x_n\| \to \|x\|$.

For example, the standard sequence spaces c_0, l_1, l_∞ and all $L^1(\mu)$ and $C(K)$ spaces, for each compact Hausdorff K, have the DP and so DP1 property [10].

The concept of DP1 which has introduced by Freedman in [12], is in general weaker than the DP; for example every (infinite dimensional) Hilbert space has DP1, but does not have the DP [10, 12]. Also there are Banach spaces such as von Neumann algebras, that the DP1 and the DP on them are coincide [12].

Another concept which has introduced by Freedman is the concept of DP1 operators, that is weaker than the concept of completely continuous operators. A bounded linear operator $T : X \to Y$ between Banach spaces X and Y is called

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completely continuous or Dunford-Pettis operator, if T maps weakly convergent sequences to norm convergent sequences, and the operator T is said to be a DP1 operator if T carries weakly convergent sequences on the unit sphere of X to norm convergent sequences. This means that for every weakly convergent sequence $x_n \to x$ in X with $\|x_n\| = \|x\| = 1$, we have $\|Tx_n - Tx\| \to 0$. We refer the reader to [5], [6] and [15] for valuable results on DP1.

In [14], the authors have obtained some characterizations of arbitrary Banach space X which contains no copy of l_1 and has the DP (or equivalently, the dual X^* of X has the Schur property, i.e., weak and norm convergence of sequences in X^* are coincide), with respect to compactness of every weakly compact operator T from X into arbitrary Banach space Y. A similar result, among other things, will be found about Banach spaces with the DP or DP1 property in [10, Theorem 1] and [12, Theorem 1.4]. Specially, a Banach space X has the DP property if and only if every weakly compact operator $T : X \to Y$ is completely continuous; while X has the DP1 property if and only if every weakly compact operator $T : X \to Y$ is DP1. If $\mathcal{M}$ is a closed subspace of some operator ideals, there is a well known refinement of it about the Dunford-Pettis property.

If $\mathcal{U}$ is a (Banach) operator ideal, by meaning of [9] or [16], let $\mathcal{U}(X, Y)$ be any its component and $\mathcal{M}$ be a closed subspace of it. For each $x \in X$ and $y^* \in Y^*$, we denote the evaluation operators at x and y^* respectively, by $\phi_x : \mathcal{M} \to Y$ and $\psi_{y^*} : \mathcal{M} \to X^*$ where, $\phi_x(T) = Tx$, $\psi_{y^*}(T) = T^*y^*$ and $T \in \mathcal{M}$. We also use the standard notations $K_{w^*}(X^*, Y)$ and $K(X, Y)$ for the Banach spaces of all compact weak*-weak continuous operators and all compact operators between related Banach spaces. $K(X)$ is an abbreviation of $K(X, X)$; $\langle x, x^* \rangle$ denotes the duality between $x \in X$ and $x^* \in X^*$ and T^* refers to the adjoint of the operator T.

In [18], A. Ülger proved that for any Hilbert space H, if $\mathcal{M} \subseteq K(H)$ is a closed subspace, then $\mathcal{M}$ has the DP (or equivalently $\mathcal{M}^*$ has the Schur property) if and only if all evaluation operators $\phi_x : \mathcal{M} \to H$ and $\psi_x : \mathcal{M} \to H$ are compact operators if and only if all evaluation operators are completely continuous. The same conclusion has obtained by E. Saksman and H. O. Tylli in [17] for closed subspaces of $K(l_p)$ with $1 < p < \infty$.

In [14], the authors extend these conclusions to closed subspaces of several operator ideals. They proved that for a large class of Banach spaces X and Y, the Schur property of the dual $\mathcal{M}^*$ of closed subspace $\mathcal{M}$ of arbitrary operator ideal $\mathcal{U}(X, Y)$, is a sufficient condition for compactness and so complete continuity of all evaluation operators ϕ_x and ψ_{y^*}.

On the opposite direction, they have shown that for several Banach spaces X and Y with Schauder decompositions, if $\mathcal{M}$ is a closed subspace of $K(X, Y)$ or $K_{w^*}(X^*, Y)$, then the Schur property of $\mathcal{M}^*$ is a necessary condition for compactness of all point evaluations.

Also, in [1], the authors study the DP1 property for closed subspaces of $K(X, Y)$, where X and Y admit Schauder basis and the basis of X is shrinking;

and they proved some necessary and sufficient conditions for the DP1 property of suitable subspaces of $K(X, Y)$.

Here, we will show that similar consequences of [14], that extend some results of [1], remain valid for the DP1 property and a suitable class of closed subspaces of some operator ideals, where in this case the evaluation operators must be assumed DP1 operators.

2. Main results

Recall that, by Freedman's Theorem [12], an arbitrary Banach space X has the DP1 property if and only if every weakly compact operator T from X into arbitrary Banach space Y is DP1. So, in order to prove a key result of this article, one can give a necessary and sufficient condition among Banach spaces containing no copy of l_1 to have the DP1.

Theorem 2.1. *A Banach space X containing no copy of l_1 has the DP1 property if and only if for every weakly sequentially complete (wsc) Banach space Y, every operator $T : X \to Y$ is DP1.*

Proof. Suppose that X has the DP1 and $T : X \to Y$ is an operator into wsc Banach space Y. If (x_n) is an arbitrary sequence in the unit ball of X, then by Rosenthal's l_1-Theorem [11], (x_n) has a weakly Cauchy subsequence (x_{n_k}). This shows that (Tx_{n_k}) is weakly Cauchy and so is weakly convergent. Therefore the operator T is weakly compact and the hypothesis of DP1 property of X implies that T is DP1.

On the other hand, since by Davis-Figiel-Johnson-Pelczynski's Theorem [11], every weakly compact operator factors through a reflexive (and so wsc) Banach space, the opposite implication is also clear. □

Theorem 2.2. *Suppose that X^* and Y are wsc and $\mathcal{M} \subseteq \mathcal{U}(X, Y)$ is a closed subspace containing no copy of l_1. If $\mathcal{M}$ has the DP1, then all of the evaluation operators ϕ_x and ψ_{y^*} are DP1 operators.*

Proof. Since X^* and Y are wsc, by Theorem 2.1, the bounded linear operators ϕ_x and ψ_{y^*} are DP1. □

Notice that if X and Y are two reflexive Banach spaces, a referring to Freedman's Theorem imply the same conclusion, without any assumption on non containment of l_1 by $\mathcal{M}$. This assertion also treated in [1].

If X and Y are Banach lattices, X contains no complemented copy of l_1 and Y contains no copy of c_0, then X^* and Y are wsc [13, V.II] and we can apply Theorem 2.2 for any closed subspace $\mathcal{M} \subseteq \mathcal{U}(X, Y)$. As another corollary, if instead of X and Y, the closed subspace $\mathcal{M} \subseteq \mathcal{U}(X, Y)$ is a Banach lattice, we have the following corollary which can be proved by the same method applied in the proof of Corollary 2.4 of [14]:

Corollary 2.3. *Suppose that X contains no complemented copy of l_1 and Y contains no copy of c_0. If $\mathcal{M} \subseteq \mathcal{U}(X, Y)$ is a Banach lattice containing no copy* *of l_1 and satisfying the DP1, then all of the evaluation operators ϕ_x and ψ_{y^*} are DP1 operators.*

Here we use similar techniques to those in [1] and [14] to obtain some characterizations of the DP1 property for suitable closed subspaces of some compact operator ideals between Banach spaces that extend some results of [1]. We need some notations.

If V is a complemented subspace of a Banach space X, the projection of X onto V is denoted by P_V and $P_{V'} = I - P_V$ is the projection onto complementary subspace V' of V. As mentioned in [14], if $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ are Schauder decompositions of X and Y respectively, and $\mathcal{M} \subseteq \mathcal{U}(X,Y)$ is a closed subspace, we say that $\mathcal{M}$ has the $\mathcal{P}$-property if for all integers m_0 and n_0 and every operators $T, S \in \mathcal{M}$,

$$\|P_W T P_V + P_{W'} S P_{V'}\| \leq \max\{\|P_W T P_V\|, \|P_{W'} S P_{V'}\|\},$$

where $V = X_1 \oplus \cdots \oplus X_{m_0}$ and $W = Y_1 \oplus \cdots \oplus Y_{n_0}$. Finally, if $(X_n)_{n=1}^{\infty}$ is a shrinking Schauder decomposition for X [13], we denote the corresponding Schauder decomposition of X^* by $(X_n^*)_{n=1}^{\infty}$.

Theorem 2.4. *Let X and Y have monotone finite dimensional Schauder decompositions* (*abb. FDD*) *such that the decomposition of X is shrinking. Let $\mathcal{M}$ be a closed subspace of $K_{w^*}(X^*, Y)$ which has the $\mathcal{P}$-property. If all of the evaluation operators ϕ_{x^*} and ψ_{y^*} are DP1 operators, then $\mathcal{M}$ has the DP1 property.*

Proof. Suppose that $(X_n)_{n=1}^{\infty}$ and $(Y_n)_{n=1}^{\infty}$ are finite dimensional Schauder decompositions of X and Y respectively. Since the decompositions of X^* and Y are monotone $\|P_V\| = \|P_W\| = 1$, and $\|P_{W'}\| \leq 2$ for all $V = X_1^* \oplus \cdots \oplus X_{m_0}^*$ and $W = Y_1 \oplus \cdots \oplus Y_{n_0}$.

Let $(K_n) \subseteq \mathcal{M}$ and $(\Gamma_n) \subseteq \mathcal{M}^*$ be weakly null sequences in $\mathcal{M}$ and $\mathcal{M}^*$ respectively and $K \in \mathcal{M}$ such that $\|K\| = \|K_n + K\| = 1$ and there exists $r > 0$ such that for all integer n, $|\langle K_n, \Gamma_n \rangle| \geq r$. Let (ε_n) be a sequence of positive numbers such that $\sum n\varepsilon_n < \infty$.

We shall construct by induction, subsequences (Λ_n) of (Γ_n) and (S_n) of (K_n) such that for all n, there exist (finite dimensional) subspaces V and W of X^* and Y respectively, satisfying the following properties:

$$(*)\quad \begin{aligned} &\|S_i P_{V'}\| \leq \varepsilon_{n+1} \text{ and } \|P_{W'} S_i\| \leq \varepsilon_{n+1} \text{ for all } i = 1, 2, \ldots, n, \\ &|\langle S_i, \Lambda_{n+1} \rangle| < r2^{-(n+1)},\ i = 1, 2, \ldots, n, \\ &|\langle S_{n+1}, \Lambda_{n+1} \rangle| > r \text{ and } |\langle S_{n+1}, \Lambda_i \rangle| \leq r2^{-(n+1)},\ i = 1, 2, \ldots, n, \\ &\|S_{n+1} P_V\| \leq \varepsilon_{n+1} \text{ and } \|P_W S_{n+1}\| \leq \varepsilon_{n+1}. \end{aligned}$$

Suppose that $\Lambda_1 = \Gamma_1$, and $S_1 = K_1$ and inductively, suppose that $\Lambda_1, \ldots, \Lambda_n \in (\Gamma_i)$ and $S_1, \ldots, S_n \in (K_i)$ have been chosen. To obtain Λ_{n+1} and S_{n+1}, by Lemma 3.2 of [14] we find finite dimensional subspaces $V = X_1^* \oplus \cdots \oplus X_{m_0}^*$

and $W = Y_1 \oplus \cdots \oplus Y_{n_0}$ of X^* and Y respectively, such that

$$\|S_i P_{V'}\| \le \varepsilon_{n+1} \text{ and } \|P_{W'} S_i\| \le \varepsilon_{n+1} \text{ for all } i = 1, 2, \ldots, n.$$

Since P_V and P_W are finite rank operators, it is easy to check that the operators $K \mapsto KP_V$ and $K \mapsto P_W K$ from $\mathcal{M}$ into $K_{w^*}(X^*, Y)$ are DP1 (see for instance, Remark 2.3 of [1]). Thus by hypothesis on (K_n) we have

$$\|K_n P_V\| \to 0 \text{ and } \|P_W K_n\| \to 0 \text{ as } n \to \infty.$$

So there exists an integer $N_1 > 0$ such that for all $j \ge N_1$:

$$\|K_j P_V\| \le \varepsilon_{n+1} \text{ and } \|P_W K_j\| \le \varepsilon_{n+1}.$$

On the other hand, the weak nullity of the sequences (K_n) and (Γ_n) imply the existence of two integers N_2 and N_3 such that

$$|\langle K_j, \Lambda_i\rangle| < r2^{-(n+1)} \text{ for all } i = 1, 2, \ldots, n, \text{ and all } j \ge N_2,$$
$$|\langle S_i, \Gamma_j\rangle| < r2^{-(n+1)} \text{ for all } i = 1, 2, \ldots, n, \text{ and all } j \ge N_3.$$

Now select an integer j_0 bigger than N_1, N_2 and N_3 and set $\Lambda_{n+1} = \Gamma_{j_0}$ and $S_{n+1} = K_{j_0}$. This finishes the induction process. We have constructed a subsequence (Λ_n) of (Γ_n) and a subsequence (S_n) of (K_n) such that for all integer n, there exist finite dimensional subspaces V and W of X^* and Y respectively, that satisfy all conditions of $(\star)$. These properties, as shown in [14], yield that

$$\left\| P_W \sum_{i=1}^{n} S_i P_V - \sum_{i=1}^{n} S_i \right\| \le 4n\varepsilon_{n+1} \text{ and } \|P_{W'} S_{n+1} P_{V'} - S_{n+1}\| \le 5\varepsilon_{n+1}.$$

Hence

$$\begin{aligned}\left\|\sum_{i=1}^{n+1} S_i\right\| \le\ & \left\|\sum_{i=1}^{n} S_i - P_W \sum_{i=1}^{n} S_i P_V\right\| + \|S_{n+1} - P_{W'} S_{n+1} P_{V'}\| \\ & + \left\| P_W \sum_{i=1}^{n} S_i P_V + P_{W'} S_{n+1} P_{V'} \right\| \\ \le\ & (4n+5)\varepsilon_{n+1} + \max\left\{ \left\|\sum_{i=1}^{n} S_i\right\|, 8 \right\}.\end{aligned}$$

Note that the last inequality holds by the $\mathcal{P}$-property of $\mathcal{M}$ and that $\|S_{n+1}\| \le 2$. This shows that the sequence $T_n = \sum_{i=1}^{n} S_i$ is bounded and so has a weak*-cluster point $T \in \mathcal{M}^{**}$. For each j, choose an integer $n > j$ such that $|\langle T - T_n, \Lambda_j\rangle| < r2^{-j}$. Therefore

$$\begin{aligned}|\langle T, \Lambda_j\rangle| &\ge |\langle T_n, \Lambda_j\rangle| - |\langle T - T_n, \Lambda_j\rangle| \\ &\ge |\sum_{i=1}^{n} \langle S_i, \Lambda_j\rangle| - \frac{r}{2^j}\end{aligned}$$

$$\geq |\langle S_j, \Lambda_j\rangle| - \sum_{i=1}^{j-1} |\langle S_i, \Lambda_j\rangle| - \sum_{i=j+1}^{n} |\langle S_i, \Lambda_j\rangle| - \frac{r}{2^j}$$
$$\geq r - \sum_{i=1}^{j-1} \frac{r}{2^j} - \sum_{i=j+1}^{n} \frac{r}{2^i} - \frac{r}{2^j}$$
$$\geq r - r(\sum_{i=1}^{j-1} \frac{1}{2^{i+1}} + \sum_{i=j+1}^{n} \frac{1}{2^i}) - \frac{r}{2^j}$$
$$\geq r - r(\sum_{i=2}^{\infty} \frac{1}{2^i}) - \frac{r}{2^j} = \frac{r}{2} - \frac{r}{2^j} > \frac{r}{3} > 0$$

for sufficiently large j. Hence $\langle T, \Lambda_j\rangle$ and so $\langle T, \Gamma_j\rangle$ does not tend to zero. Thus the sequence (Γ_j) does not converge weakly to zero, which gives a contradiction. □

Remark 2.5. Note that the proof of Lemma 3.2 of [14] is based on the fact that for each bounded and weak*-weak continuous operator $K : X^* \to Y$, the adjoint operator K^* maps elements of Y^* into X. So we need $\mathcal{M} \subseteq K_{w^*}(X^*, Y)$. In fact, under the same assumptions of Theorem 2.4, if $\mathcal{M} \subseteq K(X^*, Y)$, the conclusion of Lemma 3.2 of [14] is false. However, under the same assumptions on X and Y, a similar result can be inferred for closed subspaces of $K(X, Y)$:

Theorem 2.6. *Let X and Y have monotone FDDs, such that the decomposition of X is shrinking. Let $\mathcal{M}$ be a closed subspace of $K(X, Y)$ which has the $\mathcal{P}$-property. If all of the evaluation operators ϕ_x and ψ_{y^*} are DP1 operators, then $\mathcal{M}$ has the DP1 property.*

If X is an l_p-direct sum and Y is an l_q-direct sum of Banach spaces with $1 < p \leq q < \infty$, or X has a Schauder decomposition and Y is a c_0-direct sum of Banach spaces, then the proof of Corollaries 3.5 and 3.6 of [14] shows that $K(X, Y)$ (resp. $K_{w^*}(X^*, Y)$) and so its closed subspace $\mathcal{M}$ has the $\mathcal{P}$-property. So we have the following two corollaries:

Corollary 2.7. *Let X be an l_p-direct sum and Y be an l_q-direct sum of finite dimensional Banach spaces and $1 < p \leq q < \infty$. If $\mathcal{M}$ is a closed subspace of $K(X, Y)$ such that all evaluation operators ϕ_x and ψ_{y^*} are DP1 operators, then $\mathcal{M}$ has the DP1 property.*

Corollary 2.8. *Let X have a monotone shrinking FDD and Y be a c_0-direct sum of finite dimensional Banach spaces. If $\mathcal{M}$ is either a closed subspace of $K(X, Y)$ or $K_{w^*}(X^*, Y)$ such that all of the corresponding evaluation operators are DP1, then $\mathcal{M}$ has the DP1 property.*

Remark 2.9. The proof of Theorem 2.4 is based on the facts that for each two Banach spaces X and Y with monotone (shrinking) FDDs, suitable closed subspaces of them are complemented and Lemma 3.2 of [14] is valid. By [3], in the Hilbert space setting, a lemma similar to that lemma is valid; every closed subspace of a Hilbert space is complemented and an inequality similar to that of the definition of $\mathcal{P}$-property holds for operators between two Hilbert spaces. So by a proof similar to Theorem 2.4 one can prove the following theorem:

Theorem 2.10. *Let H_1 and H_2 be two Hilbert spaces and $\mathcal{M}$ be a closed subspace of $K(H_1, H_2)$. Then $\mathcal{M}$ has the DP1 property if and only if all of the evaluation operators ϕ_x and ψ_y are DP1 operators.*

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