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CONVERGENCE OF RELAXED TWO-STAGE MULTISPLITTING METHOD USING AN OUTER SPLITTING

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Abstract. In this paper, we study the convergence of relaxed two-stage multisplitting method using H-compatible splittings or SOR multisplitting as inner splittings and an outer splitting for solving a linear system whose coefficient matrix is an H-matrix. We also provide numerical experiments for the convergence of the relaxed two-stage multisplitting method.

1. Introduction

In this paper, we consider relaxed two-stage multisplitting method for solving a linear system of the form

$$Ax = b, \quad x, b \in \mathbb{R}^n, \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$ is a large sparse H-matrix. Multisplitting method was introduced by O'Leary and White [5] and was further studied by many authors [3, 4, 6, 8, 9]. The multisplitting method can be thought of as an extension and parallel generalization of the classical block Jacobi method [2].

A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called *monotone* if $A^{-1} \geq 0$. A matrix $A = (a_{ij}) \in \mathbb{R}^{n\times n}$ is called an *M-matrix* if A is monotone and $a_{ij} \leq 0$ for $i \neq j$. The *comparison matrix* $\langle A\rangle = (\alpha_{ij})$ of a matrix $A = (a_{ij})$ is defined by

$$\alpha_{ij} = \begin{cases} \;\;|a_{ij}| & \text{if } i = j \\ -|a_{ij}| & \text{if } i \neq j. \end{cases}$$

A matrix A is called an *H-matrix* if $\langle A\rangle$ is an M-matrix. A representation $A = M - N$ is called a *splitting* of A when M is nonsingular. A splitting $A = M - N$ is called *regular* if $M^{-1} \geq 0$ and $N \geq 0$, *weak regular* if $M^{-1} \geq 0$ and $M^{-1}N \geq 0$, *M-splitting* of A if M is an M-matrix and $N \geq 0$, and *H-compatible splitting* of A if $\langle A\rangle = \langle M\rangle - |N|$.

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A collection of triples (M_k, N_k, E_k), $k = 1, 2, \ldots, \ell$, is called a *multisplitting* of A if $A = M_k - N_k$ is a splitting of A for $k = 1, 2, \ldots, \ell$, and E_k's, called weighting matrices, are nonnegative diagonal matrices such that $\sum_{k=1}^{\ell} E_k = I$. The *relaxed two-stage multisplitting method* with a relaxation parameter $\beta > 0$ using $A = M_k - N_k$ as outer splittings and $M_k = B_k - C_k$ as inner splittings is as follows.

ALGORITHM 1: RELAXED TWO-STAGE MULTISPLITTING METHOD

Given an initial vector x_0
For $i = 1, 2, \ldots,$ until convergence
  For $k = 1$ to ℓ
    $y_{k,0} = x_{i-1}$
    For $j = 1$ to s
      $y_{k,j} = \beta {B_k}^{-1}(C_k y_{k,j-1} + N_k x_{i-1} + b) + (1-\beta) y_{k,j-1}$
  $x_i = \displaystyle\sum_{k=1}^{\ell} E_k y_{k,s}$

In Algorithm 1, it is assumed to be $s \geq 1$. Bru et al. [2] showed that if $0 < \beta \leq 1$, then Algorithm 1 converges for an H-matrix A under the assumption that both the outer splittings $A = M_k - N_k$ and the inner splittings $M_k = B_k - C_k$ are H-compatible splittings.

In 1991, Wang [8] studied the convergence of relaxed multisplitting method associated with AOR multisplitting for solving the linear system (1). In this paper, we study the convergence of relaxed two-stage multisplitting method using H-compatible splittings or SOR multisplitting as inner splittings and an outer splitting for solving the linear system (1). This paper is organized as follows. In Section 2, we present some notation and well-known results. In Section 3, we provide convergence results of relaxed two-stage multisplitting method using H-compatible splittings or SOR multisplitting as inner splittings and an outer splitting. In Section 4, we also provide numerical experiments for the convergence of the relaxed two-stage multisplitting method.

2. Preliminaries

For a vector $x \in \mathbb{R}^n$, $x \geq 0$ $(x > 0)$ denotes that all components of x are nonnegative (positive). For two vectors $x, y \in \mathbb{R}^n$, $x \geq y$ $(x > y)$ means that $x - y \geq 0$ $(x - y > 0)$. For a vector $x \in \mathbb{R}^n$, $|x|$ denotes the vector whose components are the absolute values of the corresponding components of x. These definitions carry immediately over to matrices. It follows that $|A| \geq 0$ for any matrix A and $|AB| \leq |A||B|$ for any two matrices A and B of compatible size. Let $\mathrm{diag}(A)$ denote a diagonal matrix whose diagonal part coincides with the diagonal part of A, and let $\rho(A)$ denote the *spectral radius* of a square matrix A. Varga [7] showed that for any square matrices A and B, $|A| \leq B$ implies $\rho(A) \leq \rho(B)$. It is well-known that if $A = M - N$ is a weak regular splitting, then $A^{-1} \geq 0$ if and only if $\rho(M^{-1}N) < 1$ [1, 7]. It was shown that if $A \geq 0$ and there exists a vector $x > 0$ and an $\alpha \geq 0$ such that $Ax \leq \alpha x$, then $\rho(A) \leq \alpha$ [1]. Frommer and Mayer [3] showed that $|A^{-1}| \leq \langle A \rangle^{-1}$ when A is an H-matrix.

Theorem 2.1 ([2]). *Let $A \in \mathbb{R}^{n\times n}$ be a monotone matrix. Assume that the outer splittings $A = M_k - N_k$ are regular and the inner splittings $M_k = B_k - C_k$ are weak regular. If $0 < \beta \leq 1$, then the relaxed two-stage multisplitting method converges to the exact solution of $Ax = b$ for any initial vector x_0.*

Theorem 2.2 ([2]). *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix. Assume that the outer splittings $A = M_k - N_k$ are H-compatible splittings and the inner splittings $M_k = B_k - C_k$ are H-compatible splittings. If $0 < \beta \leq 1$, then the relaxed two-stage multisplitting method converges to the exact solution of $Ax = b$ for any initial vector x_0.*

The SOR multisplitting to be used in this paper is defined as follows.

Definition 1. Let $0 < \omega < 2$ and $A = D - L_k - U_k$ for $k = 1, 2, \ldots, \ell$, where $D = \text{diag}(A)$, L_k's are strictly lower triangular matrices, and U_k's are general matrices. $(M_k(\omega), N_k(\omega), E_k)$, $k = 1, 2, \ldots, \ell$, is called the *SOR multisplitting* of A if $(M_k(\omega), N_k(\omega), E_k)$, $k = 1, 2, \ldots, \ell$, is a multisplitting of A, $M_k(\omega) = \frac{1}{\omega}(D - \omega L_k)$, and $N_k(\omega) = \frac{1}{\omega}\left((1-\omega)D + \omega U_k\right)$.

If $\omega = 1$ in Definition 1, then the SOR multisplitting of A is called the *Gauss-Seidel multisplitting* of A. In this case, $M_k(\omega) = D - L_k$ and $N_k(\omega) = U_k$.

3. Convergence results of relaxed two-stage multisplitting method

In this section, we consider convergence of relaxed two-stage multisplitting method (Algorithm 1) with a relaxation parameter $\beta > 0$ using an outer splitting $A = M - N$ and inner splittings $M = B_k - C_k$. Then, Algorithm 1 can be written as $x_i = H_\beta x_{i-1} + P_\beta b$, $i = 1, 2, \ldots$, where

$$H_\beta = \sum_{k=1}^{\ell} E_k (R_{\beta,k})^s + \beta \sum_{k=1}^{\ell} E_k \left(\sum_{j=0}^{s-1} (R_{\beta,k})^j \right) B_k^{-1} N,$$

$$P_\beta = \beta \sum_{k=1}^{\ell} E_k \left(\sum_{j=0}^{s-1} (R_{\beta,k})^j \right) B_k^{-1},$$

where $R_{\beta,k} = \beta B_k^{-1} C_k + (1-\beta)I$. The H_β is called an *iteration matrix* for the relaxed two-stage multisplitting method with a relaxation parameter $\beta > 0$ and s inner iterations. Then, it can be shown that $P_\beta A = I - H_\beta$ and the relaxed two-stage multisplitting method with a relaxation parameter $\beta > 0$ converges to the exact solution of $Ax = b$ for any initial vector x_0 if and only if $\rho(H_\beta) < 1$. First, we provide convergence result of the relaxed two-stage multisplitting method using H-compatible splittings.

Theorem 3.1. *Let* $A \in \mathbb{R}^{n\times n}$ *be an* H*-matrix and* $A = M - N$ *be an* H*-compatible splitting of* A*. Let* $M = B_k - C_k$ *be an* H*-compatible splitting such that* $\operatorname{diag}(B_k) = \operatorname{diag}(M) = D$ *for each* $1 \le k \le \ell$*, and let* $B = D - M$*. Then, the relaxed two-stage multisplitting method using an outer splitting* $A = M - N$ *and inner splittings* $M = B_k - C_k$*,* $k = 1, 2, \ldots, \ell$*, converges to the exact solution of* $Ax = b$ *for any initial vector* x_0 *if* $0 < \beta < \frac{2}{1+\alpha}$*, where* $\alpha = \rho(|D|^{-1}(|B| + |N|))$*.*

Proof. For $0 < \beta \le 1$, this theorem follows directly from Theorem 2.2. Now we consider the case of $1 < \beta < \frac{2}{1+\alpha}$. Let

$$H_\beta = \sum_{k=1}^{\ell} E_k H_{\beta,k}\,, \quad H_{\beta,k} = (R_{\beta,k})^s + \beta \sum_{j=0}^{s-1} (R_{\beta,k})^j B_k^{-1} N,$$

$$\tilde{H}_\beta = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,k}\,, \quad \tilde{H}_{\beta,k} = (\tilde{R}_{\beta,k})^s + \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j \langle B_k\rangle^{-1} |N|,$$

where $R_{\beta,k} = \beta B_k^{-1} C_k + (1-\beta) I$ and $\tilde{R}_{\beta,k} = \beta \langle B_k\rangle^{-1}|C_k| + (\beta - 1)I$. Let

$$\tilde{A}_{\beta,k} = \frac{2-\beta}{\beta}\langle B_k\rangle - |C_k| - |N| \text{ and } \tilde{A}_\beta = \frac{2-\beta}{\beta}|D| - |B| - |N|.$$

Notice that M and B_k are H-matrices since $A = M - N$ and $M = B_k - C_k$ are H-compatible splittings. Since it can be easily shown that $|R_{\beta,k}| \le \tilde{R}_{\beta,k}$, $|H_{\beta,k}| \le \tilde{H}_{\beta,k}$ and thus

$$|H_\beta| \le \tilde{H}_\beta. \tag{2}$$

Since $\operatorname{diag}(B_k) = \operatorname{diag}(M) = D$, $\langle M\rangle = |D| - |B|$ and $\langle B_k\rangle \le |D|$. Since $\langle A\rangle = |D| - (|B| + |N|) = \langle B_k\rangle - (|C_k| + |N|)$ are regular splittings of $\langle A\rangle$ and $\langle B_k\rangle^{-1} \ge |D|^{-1}$, $\rho\left(\langle B_k\rangle^{-1}(|C_k| + |N|)\right) \le \alpha < 1$. Since $\beta < \frac{2}{1+\alpha}$, $\frac{\beta\alpha}{2-\beta} < 1$ and thus $I - \tilde{R}_{\beta,k} = (2-\beta)I - \beta\langle B_k\rangle^{-1}|C_k|$ is nonsingular. Hence, one obtains

$$\begin{aligned}
\tilde{H}_{\beta,k} &= (\tilde{R}_{\beta,k})^s + \beta(I - (\tilde{R}_{\beta,k})^s)(I - \tilde{R}_{\beta,k})^{-1}\langle B_k\rangle^{-1}|N| \\
&= I - (I - (\tilde{R}_{\beta,k})^s)(I - \beta(I - \tilde{R}_{\beta,k})^{-1}\langle B_k\rangle^{-1}|N|) \\
&= I - (I - (\tilde{R}_{\beta,k})^s)(I - \tilde{R}_{\beta,k})^{-1}(I - \tilde{R}_{\beta,k} - \beta\langle B_k\rangle^{-1}|N|) \\
&= I - \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j((2-\beta)I - \beta\langle B_k\rangle^{-1}|C_k| - \beta\langle B_k\rangle^{-1}|N|) \\
&= I - \beta\sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j \langle B_k\rangle^{-1}\left(\frac{2-\beta}{\beta}\langle B_k\rangle - |C_k| - |N|\right) \\
&= I - \beta\sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j \langle B_k\rangle^{-1}\tilde{A}_{\beta,k}.
\end{aligned} \tag{3}$$

Let $E_k = D - B_k$ for each k. Then $\langle B_k \rangle = |D| - |E_k|$ and $|B| = |E_k| + |C_k|$. Since $1 < \beta < \frac{2}{1+\alpha}$, one obtains

$$\tag{4} \begin{aligned} \tilde{A}_{\beta,k} &= \frac{2-\beta}{\beta}\langle B_k \rangle - |C_k| - |N| \\ &= \frac{2-\beta}{\beta}|D| - \frac{2-\beta}{\beta}|E_k| - |C_k| - |N| \\ &\geq \frac{2-\beta}{\beta}|D| - |E_k| - |C_k| - |N| \\ &= \frac{2-\beta}{\beta}|D| - |B| - |N| \\ &= \tilde{A}_\beta \end{aligned}$$

for each k. Since $\tilde{A}_\beta = \frac{2-\beta}{\beta}|D| - (|B| + |N|)$ is a regular splitting of $\tilde{A}_\beta$ and $\frac{\beta\alpha}{2-\beta} < 1$, $\tilde{A}_\beta^{-1} \geq 0$. Since $\tilde{R}_{\beta,k}$ and $\langle B_k \rangle^{-1}$ are nonnegative, from (3) and (4) one obtains

$$\tag{5} \tilde{H}_{\beta,k} \leq I - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j \langle B_k \rangle^{-1} \tilde{A}_\beta.$$

Let $e = (1, 1, \ldots, 1)^T$ and $v = \tilde{A}_\beta^{-1} e$. Then $v > 0$ and $\langle B_k \rangle^{-1} e > 0$. Using these relations and (5),

$$\tag{6} \tilde{H}_{\beta,k} v \leq v - \beta \sum_{j=0}^{s-1} (\tilde{R}_{\beta,k})^j \langle B_k \rangle^{-1} e \leq v - \beta \langle B_k \rangle^{-1} e < v.$$

From (6), there exists $\theta_{\beta,k} \in [\,0, 1)$ such that

$$\tag{7} \tilde{H}_{\beta,k} v \leq \theta_{\beta,k} v$$

for each k. Let $\theta_\beta = \max\{\theta_{\beta,k} \mid 1 \leq k \leq \ell\}$. It is clear that $\theta_\beta < 1$. From (2) and (7),

$$\tag{8} |H_\beta| v \leq \tilde{H}_\beta v = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,k} v \leq \sum_{k=1}^{\ell} \theta_{\beta,k} E_k v \leq \theta_\beta v.$$

From (8), $\rho(\,|H_\beta|\,) \leq \theta_\beta < 1$ and hence $\rho(H_\beta) \leq \theta_\beta < 1$ for $1 < \beta < \frac{2}{1+\alpha}$. Therefore, the proof is complete. □

In Theorem 3.1, notice that $\frac{2}{1+\alpha} > 1$ since $\alpha < 1$. It means that Theorem 3.1 can be viewed as an extension of Theorem 2.2. The following corollary for an M-matrix A is directly obtained from Theorem 3.1

Corollary 3.2. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix and $A = M - N$ be an M-splitting of A. Let $M = B_k - C_k$ be an M-splitting such that* $\operatorname{diag}(B_k) = \operatorname{diag}(M) = D$ *for each $1 \leq k \leq \ell$, and let $B = D - M$. Then, the relaxed two-stage multisplitting method using an outer splitting $A = M - N$ and inner* *splittings* $M = B_k - C_k$, $k = 1, 2, \ldots, \ell$, *converges to the exact solution of* $Ax = b$ *for any initial vector* x_0 *if* $0 < \beta < \frac{2}{1+\alpha}$, *where* $\alpha = \rho(D^{-1}(B+N))$.

We next provide convergence results of the relaxed two-stage multisplitting method using SOR multisplitting.

Theorem 3.3. *Let* $A \in \mathbb{R}^{n\times n}$ *be an* H*-matrix and* $A = M - N$ *be an* H*-compatible splitting of* A. *Let* $M = D - B = D - L_k - U_k$ $(1 \le k \le \ell)$ *such that* $\langle M\rangle = |D| - |L_k| - |U_k|$, *where* $D = \mathrm{diag}(M)$, L_k *is a strictly lower triangular matrix, and* U_k *is a general matrix, and let* $(B_k(\omega), C_k(\omega), E_k)$, $k = 1, 2, \ldots, \ell$, *be the SOR multisplitting of* M. *Let* $\alpha = \max\{\max\{\rho\left(\langle B_k(\omega)\rangle^{-1}|C_k(\omega)|\right) \mid 1 \le k \le \ell\}, \rho(|D|^{-1}(|B|+|N|))\}$. *Then, the relaxed two-stage multisplitting method using an outer splitting* $A = M - N$ *and inner splittings* $M = B_k(\omega) - C_k(\omega)$, $k = 1, 2, \ldots, \ell$, *converges to the exact solution of* $Ax = b$ *for any initial vector* x_0 *if* $0 < \omega \le 1$ *and* $0 < \beta < \frac{2}{2-\omega(1-\alpha)}$.

Proof. Notice that $\langle B_k(\omega)\rangle = \frac{1}{\omega}(|D| - \omega|L_k|)$ and $|C_k(\omega)| = \frac{1}{\omega}((1-\omega)|D| + \omega|U_k|)$ for $0 < \omega \le 1$. It follows that $M = B_k(\omega) - C_k(\omega)$ is an H-compatible splitting of M, that is, $\langle M\rangle = \langle B_k(\omega)\rangle - |C_k(\omega)|$. Since $\langle A\rangle = |D| - (|B| + |N|)$ and $\langle M\rangle = \langle B_k(\omega)\rangle - |C_k(\omega)|$ are regular splittings, it is clear that $\alpha < 1$. For $0 < \omega \le 1$ and $0 < \beta \le 1$, this theorem follows directly from Theorem 2.2 since both $A = M - N$ and $M = B_k(\omega) - C_k(\omega)$ are H-compatible splittings. Now we consider the case of $0 < \omega \le 1$ and $1 < \beta < \frac{2}{2-\omega(1-\alpha)}$. Let

$$R_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}C_k(\omega) + (1-\beta)I,$$

$$\tilde{R}_{\beta,\omega,k} = \beta\langle B_k(\omega)\rangle^{-1}|C_k(\omega)| + (\beta-1)I,$$

$$H_{\beta,\omega} = \sum_{k=1}^{\ell} E_k H_{\beta,\omega,k}, \quad H_{\beta,\omega,k} = (R_{\beta,\omega,k})^s + \beta\sum_{j=0}^{s-1}(R_{\beta,\omega,k})^j(B_k(\omega))^{-1}N,$$

$$\tilde{H}_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,\omega,k}, \quad \tilde{H}_{\beta,\omega,k} = (\tilde{R}_{\beta,\omega,k})^s + \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j\langle B_k(\omega)\rangle^{-1}|N|.$$

Let

$$\tilde{A}_{\beta,\omega,k} = \frac{2-\beta}{\beta}\langle B_k(\omega)\rangle - |C_k(\omega)| - |N| \text{ and } \tilde{A}_{\beta,\omega} = \frac{2-2\beta+\beta\omega}{\beta\omega}|D| - |B| - |N|.$$

Since $B_k(\omega)$ is an H-matrix, $|(B_k(\omega))^{-1}| \le \langle B_k(\omega)\rangle^{-1}$ and thus $|R_{\beta,\omega,k}| \le \tilde{R}_{\beta,\omega,k}$. It follows that

$$|H_{\beta,\omega}| \le \tilde{H}_{\beta,\omega}. \tag{9}$$

Since $\beta < \frac{2}{2-\omega(1-\alpha)} \leq \frac{2}{1+\alpha}$, $\frac{\beta\alpha}{2-\beta} < 1$ and thus $I - \tilde{R}_{\beta,\omega,k} = (2-\beta)I - \beta\langle B_k(\omega)\rangle^{-1}|C_k(\omega)|$ is nonsingular. Hence, one obtains

$$\begin{aligned}
\tilde{H}_{\beta,\omega,k} &= (\tilde{R}_{\beta,\omega,k})^s + \beta(I - (\tilde{R}_{\beta,\omega,k})^s)(I - \tilde{R}_{\beta,\omega,k})^{-1}\langle B_k(\omega)\rangle^{-1}|N| \\
&= I - (I - (\tilde{R}_{\beta,\omega,k})^s)(I - \beta(I - \tilde{R}_{\beta,\omega,k})^{-1}\langle B_k(\omega)\rangle^{-1}|N|) \\
&= I - (I - (\tilde{R}_{\beta,\omega,k})^s)(I - \tilde{R}_{\beta,\omega,k})^{-1}(I - \tilde{R}_{\beta,\omega,k} - \beta\langle B_k(\omega)\rangle^{-1}|N|) \\
&= I - \sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j((2-\beta)I - \beta\langle B_k(\omega)\rangle^{-1}|C_k(\omega)| - \beta\langle B_k(\omega)\rangle^{-1}|N|) \\
&= I - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j\langle B_k(\omega)\rangle^{-1}\left(\frac{2-\beta}{\beta}\langle B_k(\omega)\rangle - |C_k(\omega)| - |N|\right) \\
&= I - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j\langle B_k(\omega)\rangle^{-1}\tilde{A}_{\beta,\omega,k}.
\end{aligned} \tag{10}$$

Since $\beta > 1$ and $|B| = |L_k| + |U_k|$ for every k, one obtains

$$\begin{aligned}
\tilde{A}_{\beta,\omega,k} &= \frac{2-2\beta+\beta\omega}{\beta\omega}|D| - \frac{2-\beta}{\beta}|L_k| - |U_k| - |N| \\
&\geq \frac{2-2\beta+\beta\omega}{\beta\omega}|D| - |L_k| - |U_k| - |N| \\
&= \frac{2-2\beta+\beta\omega}{\beta\omega}|D| - |B| - |N| \\
&= \tilde{A}_{\beta,\omega}
\end{aligned} \tag{11}$$

for each k. Since $\beta < \frac{2}{2-\omega(1-\alpha)}$, $2 - 2\beta + \beta\omega > 0$ and $\frac{\beta\omega\alpha}{2-2\beta+\beta\omega} < 1$. It follows that $\tilde{A}_{\beta,\omega} = \frac{2-2\beta+\beta\omega}{\beta\omega}|D| - (|B| + |N|)$ is a regular splitting of $\tilde{A}_{\beta,\omega}$ and thus $\tilde{A}_{\beta,\omega}^{-1} \geq 0$. Since $\tilde{R}_{\beta,\omega,k}$ and $\langle B_k(\omega)\rangle^{-1}$ are nonnegative, from (10) and (11) one obtains

$$\tilde{H}_{\beta,\omega,k} \leq I - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j\langle B_k(\omega)\rangle^{-1}\tilde{A}_{\beta,\omega}. \tag{12}$$

Let $e = (1, 1, \ldots, 1)^T$ and $v = \tilde{A}_{\beta,\omega}^{-1}e$. Then $v > 0$ and $\langle B_k(\omega)\rangle^{-1}e > 0$. Using these relations and (12),

$$\tilde{H}_{\beta,\omega,k}v \leq v - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j\langle B_k(\omega)\rangle^{-1}e \leq v - \beta\langle B_k(\omega)\rangle^{-1}e < v. \tag{13}$$

From (13), there exists a $\theta_{\beta,\omega,k} \in [0, 1)$ such that

$$\tilde{H}_{\beta,\omega,k}v \leq \theta_{\beta,\omega,k}v \tag{14}$$

for each k. Let $\theta_{\beta,\omega} = \max\{\theta_{\beta,\omega,k} \mid 1 \le k \le \ell\}$. It is clear that $\theta_{\beta,\omega} < 1$. From (9) and (14),

$$|H_{\beta,\omega}|v \le \tilde{H}_{\beta,\omega}v = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,\omega,k}v \le \sum_{k=1}^{\ell} \theta_{\beta,\omega,k} E_k v \le \theta_{\beta,\omega} v. \tag{15}$$

From (15), $\rho(\,|H_{\beta,\omega}|\,) \le \theta_{\beta,\omega} < 1$ and hence $\rho(H_{\beta,\omega}) \le \theta_{\beta,\omega} < 1$ for $0 < \omega \le 1$ and $1 < \beta < \frac{2}{2-\omega(1-\alpha)}$. Therefore, the proof is complete. □

Theorem 3.4. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix and $A = M - N$ be an H-compatible splitting of A. Let $M = D - B = D - L_k - U_k$ $(1 \le k \le \ell)$ such that $\langle M\rangle = |D| - |L_k| - |U_k|$, where $D = \operatorname{diag}(M)$, L_k is a strictly lower triangular matrix, and U_k is a general matrix, and let $(B_k(\omega), C_k(\omega), E_k)$, $k = 1, 2, \ldots, \ell$, be the SOR multisplitting of M. Let $\delta = \rho(|D|^{-1}(|B| + |N|))$ and $\alpha = \max\{\delta, \max\{\rho\left(\langle B_k(\omega)\rangle^{-1}|C_k(\omega)|\right) \mid 1 \le k \le \ell\}\}$. Let*

$$H_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \left((R_{\beta,\omega,k})^s + \beta \sum_{j=0}^{s-1} (R_{\beta,\omega,k})^j (B_k(\omega))^{-1} N \right)$$

be an iteration matrix of the relaxed two-stage multisplitting method using an outer splitting $A = M - N$ and inner splittings $M = B_k(\omega) - C_k(\omega)$, where $R_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}C_k(\omega) + (1-\beta)I$. Then the following hold.

(a) *If $1 < \omega < \frac{2}{1+\delta}$ and $0 < \beta \le 1$, then $\rho(H_{\beta,\omega}) < 1$.*

(b) *If $\omega > 1$ is chosen so that $\omega(1+\alpha) < 2$ and if $0 < \beta < \frac{2}{\omega(1+\alpha)}$, then $\rho(H_{\beta,\omega}) < 1$.*

Proof. Let $\hat{M} = \langle B_k(\omega)\rangle - |C_k(\omega)|$ for $1 < \omega < \frac{2}{1+\delta}$. Since $\langle B_k(\omega)\rangle = \frac{1}{\omega}(|D| - \omega|L_k|)$ and $|C_k(\omega)| = \frac{1}{\omega}\left((\omega-1)|D| + \omega|U_k|\right)$, $\hat{M} = \langle B_k(\omega)\rangle - |C_k(\omega)| = \frac{2-\omega}{\omega}|D| - |B|$ are regular splittings of $\hat{M}$. Since $\frac{\omega\delta}{2-\omega} < 1$, $\hat{M}$ is an M-matrix and thus $\alpha < 1$. Let

$$\hat{R}_{\beta,\omega,k} = \beta\langle B_k(\omega)\rangle^{-1}|C_k(\omega)| + (1-\beta)I,$$
$$\tilde{R}_{\beta,\omega,k} = \beta\langle B_k(\omega)\rangle^{-1}|C_k(\omega)| + (\beta-1)I,$$
$$\hat{H}_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \hat{H}_{\beta,\omega,k}, \quad \hat{H}_{\beta,\omega,k} = (\hat{R}_{\beta,\omega,k})^s + \beta\sum_{j=0}^{s-1}(\hat{R}_{\beta,\omega,k})^j \langle B_k(\omega)\rangle^{-1}|N|,$$
$$\tilde{H}_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \tilde{H}_{\beta,\omega,k}, \quad \tilde{H}_{\beta,\omega,k} = (\tilde{R}_{\beta,\omega,k})^s + \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j \langle B_k(\omega)\rangle^{-1}|N|.$$

We first prove part (a). Let $\hat{A} = \hat{M} - |N|$. Then $\hat{A} = \hat{M} - |N| = \frac{2-\omega}{\omega}|D| - (|B| + |N|)$ are regular splittings of $\hat{A}$. Since $\frac{\omega\delta}{2-\omega} < 1$, $\hat{A}$ is also an M-matrix. Since $\hat{H}_{\beta,\omega}$ can be viewed as an iteration matrix of the relaxed two-stage multisplitting method using an outer splitting $\hat{A} = \hat{M} - |N|$ and inner splittings $\hat{M} = \langle B_k(\omega)\rangle - |C_k(\omega)|$, $\rho(\hat{H}_{\beta,\omega}) < 1$ is obtained from Theorem 2.1. Since it can be easily shown that $|H_{\beta,\omega}| \leq \hat{H}_{\beta,\omega}$, $\rho(H_{\beta,\omega}) < 1$. Next we prove part (b). Assume that $\omega > 1$ is chosen so that $\omega(1+\alpha) < 2$. Then $\omega(1+\delta) < 2$. For $0 < \beta \leq 1$, $\rho(H_{\beta,\omega}) < 1$ is directly obtained from part (a). Now we consider the case of $1 < \beta < \frac{2}{\omega(1+\alpha)}$. Let

$$\tilde{A}_{\beta,\omega,k} = \frac{2-\beta}{\beta}\langle B_k(\omega)\rangle - |C_k(\omega)| - |N| \text{ and } \tilde{A}_{\beta,\omega} = \frac{2-\beta\omega}{\beta\omega}|D| - |B| - |N|.$$

Since $B_k(\omega)$ is an H-matrix, $|R_{\beta,\omega,k}| \leq \tilde{R}_{\beta,\omega,k}$ and thus $|H_{\beta,\omega}| \leq \tilde{H}_{\beta,\omega}$. Since $\omega > 1$, $\beta < \frac{2}{\omega(1+\alpha)} < \frac{2}{(1+\alpha)}$ and so $\frac{\beta\alpha}{2-\beta} < 1$. It follows that $I - \tilde{R}_{\beta,\omega,k} = (2-\beta)I - \beta\langle B_k(\omega)\rangle^{-1}|C_k(\omega)|$ is nonsingular. Hence, one obtains

$$\tilde{H}_{\beta,\omega,k} = I - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j\langle B_k(\omega)\rangle^{-1}\tilde{A}_{\beta,\omega,k}. \tag{16}$$

Since $\beta > 1$ and $|B| = |L_k| + |U_k|$ for every k, one obtains

$$\begin{aligned} \tilde{A}_{\beta,\omega,k} &= \frac{2-\beta\omega}{\beta\omega}|D| - \frac{2-\beta}{\beta}|L_k| - |U_k| - |N| \\ &\geq \frac{2-\beta\omega}{\beta\omega}|D| - |L_k| - |U_k| - |N| \\ &= \frac{2-\beta\omega}{\beta\omega}|D| - |B| - |N| \\ &= \tilde{A}_{\beta,\omega} \end{aligned} \tag{17}$$

for each k. Since $\beta < \frac{2}{\omega(1+\alpha)}$, $2-\beta\omega > 0$ and $\frac{\beta\omega\alpha}{2-\beta\omega} < 1$. It follows that $\tilde{A}_{\beta,\omega} = \frac{2-\beta\omega}{\beta\omega}|D| - (|B| + |N|)$ is a regular splitting of $\tilde{A}_{\beta,\omega}$ and thus $\tilde{A}_{\beta,\omega}^{-1} \geq 0$. Since $\tilde{R}_{\beta,\omega,k}$ and $\langle B_k(\omega)\rangle^{-1}$ are nonnegative, from (16) and (17) one obtains

$$\tilde{H}_{\beta,\omega,k} \leq I - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j\langle B_k(\omega)\rangle^{-1}\tilde{A}_{\beta,\omega}. \tag{18}$$

Let $e = (1,1,\ldots,1)^T$ and $v = \tilde{A}_{\beta,\omega}^{-1}e$. Then $v > 0$ and $\langle B_k(\omega)\rangle^{-1}e > 0$. Using these relations and (18),

$$\tilde{H}_{\beta,\omega,k}v \leq v - \beta\sum_{j=0}^{s-1}(\tilde{R}_{\beta,\omega,k})^j\langle B_k(\omega)\rangle^{-1}e \leq v - \beta\langle B_k(\omega)\rangle^{-1}e < v. \tag{19}$$

The remaining part of the proof can be done in a similar way as was done in that of Theorem 3.3. Hence, $\rho(H_{\beta,\omega}) < 1$ is obtained for $1 < \beta < \frac{2}{\omega(1+\alpha)}$. Therefore, the proof is complete. □

Notice that $1 < \frac{2}{1+\alpha} \leq \frac{2}{1+\delta}$ in Theorem 3.4. The following theorem is directly obtained by combining Theorems 3.3 and 3.4.

Theorem 3.5. *Let $A \in \mathbb{R}^{n\times n}$ be an H-matrix and $A = M - N$ be an H-compatible splitting of A. Let $M = D - B = D - L_k - U_k$ $(1 \le k \le \ell)$ such that $\langle M\rangle = |D| - |L_k| - |U_k|$, where $D = \operatorname{diag}(M)$, L_k is a strictly lower triangular matrix, and U_k is a general matrix, and let $(B_k(\omega), C_k(\omega), E_k)$, $k = 1, 2, \ldots, \ell$, be the SOR multisplitting of M. Let $\delta = \rho(|D|^{-1}(|B| + |N|))$ and $\alpha = \max\{\delta, \max\{\rho\left(\langle B_k(\omega)\rangle^{-1}|C_k(\omega)|\right) \mid 1 \le k \le \ell\}\}$. Let*

$$H_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \left((R_{\beta,\omega,k})^s + \beta \sum_{j=0}^{s-1} (R_{\beta,\omega,k})^j (B_k(\omega))^{-1} N \right)$$

be an iteration matrix of the relaxed two-stage multisplitting method using an outer splitting $A = M - N$ and inner splittings $M = B_k(\omega) - C_k(\omega)$, where $R_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}C_k(\omega) + (1-\beta)I$. Then the following hold.

(a) *If $1 < \omega < \frac{2}{1+\delta}$ and $0 < \beta \le 1$, then $\rho(H_{\beta,\omega}) < 1$.*

(b) *If $\omega > 0$ is chosen so that $\omega(1+\alpha) < 2$ and if $0 < \beta < \frac{2}{1+\omega\alpha+|1-\omega|}$, then $\rho(H_{\beta,\omega}) < 1$.*

In Theorem 3.5, notice that if $0 < \omega \le 1$, then the condition $\omega(1+\alpha) < 2$ in part (b) is automatically satisfied from the fact that $\alpha < 1$. Also notice that the upper bound of β, which is $\frac{2}{1+\omega\alpha+|1-\omega|}$, is greater than 1 when $\omega(1+\alpha) < 2$. The following corollary for an M-matrix A is directly obtained from Theorem 3.5.

Corollary 3.6. *Let $A \in \mathbb{R}^{n\times n}$ be an M-matrix and $A = M - N$ be an M-splitting of A. Let $M = D - B = D - L_k - U_k$ $(1 \le k \le \ell)$, where $D = \operatorname{diag}(M)$, $L_k \ge 0$ is a strictly lower triangular matrix, and $U_k \ge 0$ is a general matrix, and let $(B_k(\omega), C_k(\omega), E_k)$, $k = 1, 2, \ldots, \ell$, be the SOR multisplitting of M. Let $\delta = \rho(D^{-1}(B + N))$ and $\alpha = \max\{\delta, \max\{\rho\left((B_k(\omega))^{-1}|C_k(\omega)|\right) \mid 1 \le k \le \ell\}\}$. Let*

$$H_{\beta,\omega} = \sum_{k=1}^{\ell} E_k \left((R_{\beta,\omega,k})^s + \beta \sum_{j=0}^{s-1} (R_{\beta,\omega,k})^j (B_k(\omega))^{-1} N \right)$$

be an iteration matrix of the relaxed two-stage multisplitting method using an outer splitting $A = M - N$ and inner splittings $M = B_k(\omega) - C_k(\omega)$, where $R_{\beta,\omega,k} = \beta(B_k(\omega))^{-1}C_k(\omega) + (1-\beta)I$. Then the following hold.

(a) *If $1 < \omega < \frac{2}{1+\delta}$ and $0 < \beta \le 1$, then $\rho(H_{\beta,\omega}) < 1$.*

(b) *If $\omega > 0$ is chosen so that $\omega(1+\alpha) < 2$ and if $0 < \beta < \frac{2}{1+\omega\alpha+|1-\omega|}$, then $\rho(H_{\beta,\omega}) < 1$.*

4. Numerical experiments

In this section, we provide numerical experiments for the convergence of the relaxed two-stage multisplitting method using SOR multisplitting as inner splittings. All numerical values are computed using MATLAB.

Example 4.1. Suppose that $\ell = 3$. Consider an H-matrix A of the form

$$A = \begin{pmatrix} F & I & 0 \\ -I & F & I \\ 0 & -I & F \end{pmatrix}, \quad F = \begin{pmatrix} 4 & -1 & 0 \\ 1 & 4 & -1 \\ 0 & 1 & 4 \end{pmatrix}, \quad I = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

Let $A = M - N$, where

$$M = \begin{pmatrix} F & 0 & 0 \\ 0 & F & 0 \\ 0 & 0 & F \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -I & 0 \\ I & 0 & -I \\ 0 & I & 0 \end{pmatrix}.$$

Let $D = \operatorname{diag}(M)$, $B = D - M$,

$$L_{11} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; U_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix},$$

$$L_{12} = \begin{pmatrix} 0 & 0 & 0 \\ -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}, \; U_{12} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$

$$L_1 = \begin{pmatrix} L_{11} & 0 & 0 \\ 0 & L_{12} & 0 \\ 0 & 0 & L_{12} \end{pmatrix}, \; L_2 = \begin{pmatrix} L_{12} & 0 & 0 \\ 0 & L_{11} & 0 \\ 0 & 0 & L_{12} \end{pmatrix}, \; L_3 = \begin{pmatrix} L_{12} & 0 & 0 \\ 0 & L_{12} & 0 \\ 0 & 0 & L_{11} \end{pmatrix},$$

$$U_1 = \begin{pmatrix} U_{11} & 0 & 0 \\ 0 & U_{12} & 0 \\ 0 & 0 & U_{12} \end{pmatrix}, \; U_2 = \begin{pmatrix} U_{12} & 0 & 0 \\ 0 & U_{11} & 0 \\ 0 & 0 & U_{12} \end{pmatrix}, \; U_3 = \begin{pmatrix} U_{12} & 0 & 0 \\ 0 & U_{12} & 0 \\ 0 & 0 & U_{11} \end{pmatrix},$$

$$E_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Then, $A = M - N$ is an H-compatible splitting of A and $M = D - L_k - U_k$ is such that $\langle M \rangle = |D| - |L_k| - |U_k|$ for $k = 1, 2, 3$. Let $(B_k(\omega), C_k(\omega), E_k)$, $k = 1, 2, 3$, be the SOR multisplitting of M. That is, $B_k(\omega) = \frac{1}{\omega}(D - \omega L_k)$ and $C_k(\omega) = \frac{1}{\omega}\left((1-\omega)D + \omega U_k\right)$ for $k = 1, 2, 3$. Then δ, α and $H_{\beta,\omega}$ are defined as in Theorem 3.5. Note that $\delta = \rho(|D|^{-1}(|B| + |N|)) \approx 0.7071$ and $\frac{2}{1+\delta} \approx 1.1716$. For various values of ω, the numerical values of α, $\omega(1+\alpha)$ and $\frac{2}{1+\omega\alpha+|1-\omega|}$ are listed in Table 1. Numerical values of $\rho(H_{\beta,\omega})$ for various values of ω, β and s are listed in Table 2.

We next consider the more general case where A is a large sparse block-tridiagonal H-matrix which is usually constructed from five-point finite difference discretization of the elliptic second order partial differential equations. For simplicity of exposition, suppose that $\ell = 3$. Then A can be partitioned into an $\ell \times \ell$ block-tridiagonal matrix of the form

$$A = \begin{pmatrix} A_{11} & A_{12} & 0 \\ A_{21} & A_{22} & A_{23} \\ 0 & A_{32} & A_{33} \end{pmatrix},$$

where A_{ii} is a square matrix for $i = 1, 2, 3$. Let $A = M - N$, where

$$M = \begin{pmatrix} A_{11} & 0 & 0 \\ 0 & A_{22} & 0 \\ 0 & 0 & A_{33} \end{pmatrix}, \quad N = \begin{pmatrix} 0 & -A_{12} & 0 \\ -A_{21} & 0 & -A_{23} \\ 0 & -A_{32} & 0 \end{pmatrix}.$$

Let $A_{ii} = D_{ii} - L_{ii} - U_{ii}$ for $i = 1, 2, 3$, where $D_{ii} = \operatorname{diag}(A_{ii})$, L_{ii} is a strictly lower triangular matrix and U_{ii} is a strictly upper triangular matrix. Let

$$L_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & L_{22} & 0 \\ 0 & 0 & L_{33} \end{pmatrix}, \; L_2 = \begin{pmatrix} L_{11} & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & L_{33} \end{pmatrix}, \; L_3 = \begin{pmatrix} L_{11} & 0 & 0 \\ 0 & L_{22} & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

$$U_1 = \begin{pmatrix} L_{11} + U_{11} & 0 & 0 \\ 0 & U_{22} & 0 \\ 0 & 0 & U_{33} \end{pmatrix}, \; U_2 = \begin{pmatrix} U_{11} & 0 & 0 \\ 0 & L_{22} + U_{22} & 0 \\ 0 & 0 & U_{33} \end{pmatrix},$$

$$U_3 = \begin{pmatrix} U_{11} & 0 & 0 \\ 0 & U_{22} & 0 \\ 0 & 0 & L_{33} + U_{33} \end{pmatrix}, \; D = \begin{pmatrix} D_{11} & 0 & 0 \\ 0 & D_{22} & 0 \\ 0 & 0 & D_{33} \end{pmatrix},$$

$$E_1 = \begin{pmatrix} I & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & I & 0 \\ 0 & 0 & 0 \end{pmatrix}, \; E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & I \end{pmatrix}.$$

Then, $A = M - N$ is an H-compatible splitting of A and $M = D - L_k - U_k$ is such that $\langle M \rangle = |D| - |L_k| - |U_k|$ for $k = 1, 2, 3$. Let $(B_k(\omega), C_k(\omega), E_k)$, $k = 1, 2, 3$, be the SOR multisplitting of M. That is, $B_k(\omega) = \frac{1}{\omega}(D - \omega L_k)$ and $C_k(\omega) = \frac{1}{\omega}((1-\omega)D + \omega U_k)$ for $k = 1, 2, 3$. Using the ideas and techniques mentioned above, we provide numerical results for the following example.

Example 4.2. Consider the following Poisson PDE

$$\text{(20)} \qquad \begin{cases} -u_{xx} - u_{yy} & = g \quad \text{in } \Omega \\ u(x, y) & = 0 \quad \text{on } \partial\Omega, \end{cases}$$

where $\Omega = (0, 1) \times (0, 1)$ and $\partial\Omega$ denotes the boundary of Ω. The five-point finite difference discretization for the PDE (20) is used. We have used a uniform mesh of $\Delta x = \Delta y = 1/46$, which lead to a block-tridiagonal H-matrix A of order $n = 45^2 = 2025$, where Δx and Δy refer to the mesh sizes in the x-direction and y-direction, respectively. δ, α and $H_{\beta,\omega}$ are defined as in Theorem 3.5. Note that $\delta = \rho(|D|^{-1}(|B| + |N|)) \approx 0.9977$ and $\frac{2}{1+\delta} \approx 1.0012$ for this example. For various values of ω, the numerical values of α, $\omega(1 + \alpha)$ and $\frac{2}{1+\omega\alpha+|1-\omega|}$ are listed in Table 3. Numerical values of $\rho(H_{\beta,\omega})$ for various values of ω, β and s are listed in Table 4.

For test problems used in this paper, the upper bound of β, which is

$$\frac{2}{1 + \omega\alpha + |1 - \omega|},$$

becomes maximum when $\omega = 1$ (see Tables 1 and 3). All numerical results are consistent with the theoretical results provided in this paper (see Tables 1 to 4). From Tables 2 and 4, it may be concluded that the optimal pairs of β and ω for which $\rho(H_{\beta,\omega})$ is minimized vary depending upon s and the problem to be considered.

TABLE 1. Numerical values of α, $\omega(1+\alpha)$ and $\frac{2}{1+\omega\alpha+|1-\omega|}$ for Example 4.1.

ω	α	$\omega(1+\alpha)$	$\frac{2}{1+\omega\alpha+\|1-\omega\|}$
0.2	0.8683	0.3737	1.0133
0.3	0.8006	0.5402	1.0308
0.4	0.7316	0.6926	1.0567
0.5	0.7071	0.8536	1.0790
0.8	0.7071	1.3657	1.1327
0.9	0.7071	1.5364	1.1518
1.0	0.7071	1.7071	1.1716
1.1	0.7071	1.8778	1.0651
1.15	0.7071	1.9632	1.0188
1.17	0.7071	1.9973	1.0013
1.18	0.7071	2.0144	0.9929

TABLE 2. Numerical values of $\rho(H_{\beta,\omega})$ for Example 4.1.

s	ω	β	$\rho(H_{\beta,\omega})$	s	ω	β	$\rho(H_{\beta,\omega})$	s	ω	β	$\rho(H_{\beta,\omega})$
1	0.2	0.8	0.8452	2	0.2	0.8	0.7226	3	0.2	0.8	0.6275
		1.0	0.8093			1.0	0.6688			1.0	0.5688
		1.01	0.8075			1.01	0.6662			1.01	0.5662
	0.5	0.8	0.6432		0.5	0.8	0.4821		0.5	0.8	0.4204
		1.0	0.5851			1.0	0.4504			1.0	0.4084
		1.07	0.5696			1.07	0.4455			1.07	0.4060
	0.8	0.6	0.5775		0.8	0.6	0.4353		0.8	0.6	0.3958
		0.8	0.5160			0.8	0.4236			0.8	0.3858
		1.0	0.5257			1.0	0.4224			1.0	0.3574
		1.13	0.5696			1.13	0.4069			1.13	0.3532
	1.0	0.6	0.5079		1.0	0.6	0.4110		1.0	0.6	0.3826
		0.8	0.5042			0.8	0.4118			0.8	0.3542
		1.0	0.6101			1.0	0.3646			1.0	0.3536
		1.17	0.7538			1.17	0.4511			1.17	0.3553
	1.1	0.6	0.4838		1.1	0.6	0.4067		1.1	0.6	0.3745
		0.8	0.5243			0.8	0.3953			0.8	0.3529
		1.0	0.6810			1.0	0.4065			1.0	0.3539
		1.06	0.7412			1.06	0.4448			1.06	0.3552
	1.15	0.6	0.4751		1.15	0.6	0.4051		1.15	0.6	0.3696
		0.8	0.5406			0.8	0.3827			0.8	0.3534
		1.0	0.7218			1.0	0.4335			1.0	0.3548
		1.01	0.7326			1.01	0.4402			1.01	0.3551

TABLE 3. Numerical values of α, $\omega(1+\alpha)$ and $\frac{2}{1+\omega\alpha+|1-\omega|}$ for Example 4.2.

ω	α	$\omega(1+\alpha)$	$\frac{2}{1+\omega\alpha+\|1-\omega\|}$
0.2	0.9978	0.3996	1.0002
0.5	0.9977	0.9988	1.0006
0.8	0.9977	1.5981	1.0009
1.0	0.9977	1.9977	1.0012
1.001	0.9977	1.9997	1.0002
1.002	0.9977	2.0017	0.9992

TABLE 4. Numerical values of $\rho(H_{\beta,\omega})$ for Example 4.2.

s	ω	β	$\rho(H_{\beta,\omega})$	s	ω	β	$\rho(H_{\beta,\omega})$	s	ω	β	$\rho(H_{\beta,\omega})$
1	0.2	0.8	0.9996	2	0.2	0.8	0.9993	3	0.2	0.8	0.9989
		1.0	0.9995			1.0	0.9991			1.0	0.9986
		1.0001	0.9995			1.0001	0.9991			1.0001	0.9986
	0.5	0.8	0.9991		0.5	0.8	0.9981		0.5	0.8	0.9972
		1.0	0.9988			1.0	0.9977			1.0	0.9966
		1.0005	0.9988			1.0005	0.9977			1.0005	0.9966
	0.8	0.6	0.9989		0.8	0.6	0.9978		0.8	0.6	0.9967
		0.8	0.9985			0.8	0.9971			0.8	0.9956
		1.0	0.9981			1.0	0.9963			1.0	0.9946
		1.0008	0.9981			1.0008	0.9963			1.0008	0.9946
	1.0	0.6	0.9986		1.0	0.6	0.9972		1.0	0.6	0.9959
		0.8	0.9981			0.8	0.9963			0.8	0.9946
		1.0	0.9977			1.0	0.9954			1.0	0.9933
		1.0011	0.9999			1.0011	0.9954			1.0011	0.9933
	1.001	0.6	0.9986		1.001	0.6	0.9972		1.001	0.6	0.9959
		0.8	0.9981			0.8	0.9963			0.8	0.9946
		1.0	0.9997			1.0	0.9954			1.0	0.9933
		1.0001	0.9999			1.0001	0.9954			1.0001	0.9933

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