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INVERSE POLYNOMIAL MODULES INDUCED BY AN *R*-LINEAR MAP

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Abstract. In this paper we show that the flat property of a left R-module does not imply (carry over) to the corresponding inverse polynomial module. Then we define an induced inverse polynomial module as an $R[x]$-module, i.e., given an R-linear map $f : M \to N$ of left R-modules, we define $N + x^{-1}M[x^{-1}]$ as a left $R[x]$-module. Given an exact sequence of left R-modules

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow 0,$$

where E^0, E^1 injective, we show $E^1 + x^{-1}E^0[[x^{-1}]]$ is not an injective left $R[x]$-module, while $E^0[[x^{-1}]]$ is an injective left $R[x]$-module. Make a left R-module N as a left $R[x]$-module by $xN = 0$. We show

$$\operatorname{inj}\dim_R N = n \qquad \text{implies} \qquad \operatorname{inj}\dim_{R[x]} N = n + 1$$

by using the induced inverse polynomial modules and their properties.

1. Introduction

If R is a left Noetherian ring, then for an injective left R-module E, $E[x^{-1}]$ is an injective left $R[x]$-module ([2], [3]). But for a projective left R-module P, $P[x^{-1}]$ is not a projective left $R[x]$-module, in general ([5]). We extend this question to the flat module and we show that for a flat left R-module F, $F[x^{-1}]$ is not a flat left $R[x]$-module, in general. Then we construct an induced inverse polynomial as an $R[x]$-module. Let M and N be left R-modules and $f : M \to N$ be an R-linear map. Then we can define $N + x^{-1}M[x^{-1}]$ as a left $R[x]$-module defined by

$$x(b_0 + a_1x^{-1} + \cdots + a_nx^{-n}) = b_1 + a_2x^{-1} + \cdots + a_nx^{-n+1},$$

where $f(a_1) = b_1,\ b_0 \in N$, and $a_i \in M$. Given an exact sequence of R-modules

$$0 \longrightarrow N \longrightarrow E^0 \longrightarrow E^1 \longrightarrow 0,$$

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where E^0, E^1 are injective, we show $E^1 + x^{-1}E^0[[x^{-1}]]$ is not an injective left $R[x]$-module, while $E^0[[x^{-1}]]$ is an injective left $R[x]$-module. Make a left R-module N as a left $R[x]$-module by $xN = 0$. We show

$$\operatorname{inj}\dim_R N = n \quad \text{implies} \quad \operatorname{inj}\dim_{R[x]} N = n+1$$

by using the inverse polynomial modules. Inverse polynomial modules were developed in ([1], [6], [7], [8]) recently.

Definition 1.1 ([4])**.** Let R be a ring and M be a left R-module. Then $M[x^{-1}]$ is a left $R[x]$-module defined by

$$x(m_0 + m_1x^{-1} + \cdots + m_ix^{-i}) = m_1 + m_2x^{-1} + \cdots + m_ix^{-i+1}$$

and such that

$$r(m_0 + m_1x^{-1} + \cdots + m_nx^{-n}) = rm_0 + rm_1x^{-1} + \cdots + rm_nx^{-n},$$

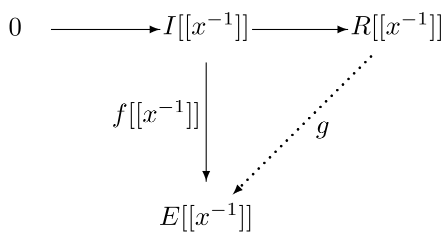
where $r \in R$. We call $M[x^{-1}]$ as an inverse polynomial module.

Similarly, we can define $M[[x^{-1}]]$, $M[x, x^{-1}]$, $M[[x, x^{-1}]]$, $M[x, x^{-1}]]$ and $M[[x, x^{-1}]$ as left $R[x]$-modules where, for example, $M[[x, x^{-1}]$ is the set of Laurent series in x with coefficients in M, i.e., the set of all formal sums $\sum_{k \geq n_0} m_kx^k$ with n_0 any element of $\mathbb{Z}$ ($\mathbb{Z}$ is the set of all integers).

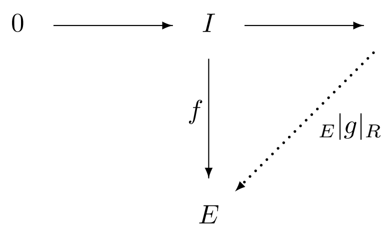
Lemma 1.2 ([8])**.** *Let E be a left R-module. Then $E[[x^{-1}]]$ is an injective left $R[x]$-module.*

Lemma 1.3. *If $E[[x^{-1}]]$ is an injective left $R[x]$-module, then E is an injective left R-module.*

Proof. Let I be a left ideal of R and $f : I \to E$ be an R-linear map. Then since $E[[x^{-1}]]$ is an injective left $R[x]$-module, we can complete the following diagram by g



as a commutative diagram, where $f[[x^{-1}]](\sum_{i=0}^{\infty} r_ix^{-i}) = \sum_{i=0}^{\infty} f(r_i)x^{-i}$. Since $xR = 0$, $xg(R) = 0$ in $E[[x^{-1}]]$. But this implies $g(R) \subset E$. Thus ${}_E|g|_R : R \to E$ can complete the following diagram



as a commutative diagram. Hence, E is an injective left R-module. $\square$

Lemma 1.4. *Let M be a left R-module. Then*

$$\operatorname{inj}\dim_{R[x]} M[[x^{-1}]] = \operatorname{inj}\dim_R M.$$

Proof. Suppose $\operatorname{inj}\dim_R M = n$ and

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to 0$$

is an injective resolution of M. Then by Lemma 1.2, for each i, $E^i[[x^{-1}]]$ is an injective left $R[x]$-module and

$$0 \to M[[x^{-1}]] \to E^0[[x^{-1}]] \to E^1[[x^{-1}]] \to \cdots \to E^n[[x^{-1}]] \to 0$$

is an injective resolution of $M[[x^{-1}]]$. Let $K^i = \operatorname{Ker}(E^i \to E^{i+1})$ for $0 \leq i < n$. Then K^i is not an injective left R-module for $0 \leq i < n$. So by Lemma 1.3, $K^i[[x^{-1}]]$ is not an injective left $R[x]$-module. So then we get $\operatorname{inj}\dim_{R[x]} M[[x^{-1}]] = n$. Suppose $\operatorname{inj}\dim_R M = \infty$ and

$$0 \to M \to E^0 \to E^1 \to \cdots \to E^n \to \cdots$$

is an injective resolution of M. Then

$$0 \to M[[x^{-1}]] \to E^0[[x^{-1}]] \to E^1[[x^{-1}]] \to \cdots \to E^n[[x^{-1}]] \to \cdots$$

is an injective resolution of $M[[x^{-1}]]$. But K^i is not an injective left R-module for all i. Thus $K^i[[x^{-1}]]$ is not an injective left $R[x]$-module for all i. Therefore, $\operatorname{inj}\dim_{R[x]} M[[x^{-1}]] = \infty$. Similarly, if $\operatorname{inj}\dim_{R[x]} M[[x^{-1}]] = n$, then $\operatorname{inj}\dim_R M = n$, and if $\operatorname{inj}\dim_{R[x]} M[[x^{-1}]] = \infty$, then $\operatorname{inj}\dim_R M = \infty$. Hence, $\operatorname{inj}\dim_{R[x]} M[[x^{-1}]] = \operatorname{inj}\dim_R M$. $\square$

2. Flat module

Lemma 2.1. *Let M be a left R-module. Then $R[x] \otimes_{R[x]} M[x^{-1}] \cong M[x^{-1}]$.*

Proof. Define $\phi : M[x^{-1}] \to R[x] \otimes M[x^{-1}]$ by $\phi(f) = 1 \otimes f$ and $\psi : R[x] \otimes M[x^{-1}] \to M[x^{-1}]$ by $\psi(x \otimes f) = xf$. Then ϕ and ψ are $R[x]$-linear maps. And

$$(\phi \circ \psi)(x \otimes f) = \phi(\psi(x \otimes f)) = \phi(xf) = 1 \otimes xf = x \otimes f,$$
$$(\psi \circ \phi)(f) = \psi(\phi(f)) = \psi(1 \otimes f) = f.$$

Hence, $R[x] \otimes_{R[x]} M[x^{-1}] \cong M[x^{-1}]$. $\square$

Similarly, we can get $R[x] \otimes_{R[x]} M[[x^{-1}]] \cong M[[x^{-1}]]$.

Theorem 2.2. *If F is a flat left R-module, then $F[x^{-1}]$ is not a flat left $R[x]$-module, in general.*

Proof. Let $R = \mathbb{R}$ (the ring of real numbers). Let $\phi : R[x] \to R[x]$ by $\phi(f) = xf$. Then ϕ is an injective $R[x]$-linear map. Then $\phi \otimes_{\mathbb{R}[x]} \mathrm{id}_{F[x^{-1}]} : \mathbb{R}[x] \otimes_{\mathbb{R}[x]} F[x^{-1}] \to \mathbb{R}[x] \otimes_{\mathbb{R}[x]} F[x^{-1}]$ is defined by $\phi \otimes_{\mathbb{R}[x]} \mathrm{id}_{F[x^{-1}]}(ax \otimes bx^{-1}) = ax^2 \otimes bx^{-1}$, where $a,\ b \in \mathbb{R}$. Since $\mathbb{R}[x] \otimes_{\mathbb{R}[x]} F[x^{-1}] \cong F[x^{-1}]$, we have the following

$$\begin{array}{ccc} \mathbb{R}[x] \otimes F[x^{-1}] & \longrightarrow & \mathbb{R}[x] \otimes F[x^{-1}] \\ \Big\downarrow f & & \Big\downarrow g \\ F[x^{-1}] & \xrightarrow{\ h\ } & F[x^{-1}] \end{array}$$

commutative diagram. But $(h \circ f)(ax \otimes bx^{-1}) = (g \circ \phi \otimes_{\mathbb{R}[x]} \mathrm{id}_{F[x^{-1}]})(ax \otimes bx^{-1})$ implies $h(ab) = 0$. Thus $h : F[x^{-1}] \to F[x^{-1}]$ is not injective, so that $\phi \otimes_{\mathbb{R}[x]} \mathrm{id}_{F[x^{-1}]}$ is not injective. Hence, $F[x^{-1}]$ is not a flat left $R[x]$-module. □

Remark 1. Since $R[x] \otimes_{R[x]} M[[x^{-1}]] \cong M[[x^{-1}]]$, we also see that $F[[x^{-1}]]$ is not a flat left $R[x]$-module.

3. Induced inverse polynomial modules

Definition 3.1. Let $f : M \to N$ be an R-linear map. Then $N + x^{-1}M[x^{-1}]$ is a left $R[x]$-module defined by

$$x(b_0 + a_1 x^{-1} + \cdots + a_n x^{-n}) = b_1 + a_2 x^{-1} + \cdots + a_n x^{-n+1},$$

where $f(a_1) = b_1,\ b_0 \in N,\ a_i \in M$.

Similarly, we can define $N + x^{-1}M[[x^{-1}]]$ as a left $R[x]$-module.

Note. Given a left R-module M, we can make M as a left $R[x]$-module by defining $xM = 0$.

Lemma 3.2. *If $0 \to L \xrightarrow{f} M \xrightarrow{g} N \to 0$ is a short exact sequence of R-modules, then*

$$0 \to L \to M[x^{-1}] \to N + x^{-1}M[x^{-1}] \to 0$$

is a short exact sequence of $R[x]$-modules.

Proof. Let $f[x^{-1}] : L \to M[x^{-1}]$ be defined by $f[x^{-1}](n) = f(n)$ for $n \in L$. Then since f is an injective R-linear map, $f[x^{-1}]$ is an injective $R[x]$-linear map. Let $g[x^{-1}] : M[x^{-1}] \to N + x^{-1}M[x^{-1}]$ be defined by

$$g[x^{-1}](e_0 + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i}) = g(e_0) + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i}.$$

Then easily $g[x^{-1}]$ is an $R[x]$-linear map. Let $b_0 + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i} \in N + x^{-1}M[x^{-1}]$. Then since g is a surjective R-linear map, there exists $e_0 \in M$ such that $g(e_0) = b_0$. So, $g[x^{-1}]$ is a surjective $R[x]$-linear map. Now

$$\begin{aligned}(g[x^{-1}] \circ f[x^{-1}])(n) &= g[x^{-1}](f(n)) \\ &= g(f(n)) \\ &= 0.\end{aligned}$$

And if $e_0 + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i} \in \operatorname{Ker} g[x^{-1}]$, where $e_i \in M$, then

$$\begin{aligned}&g[x^{-1}](e_0 + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i}) \\ &= g(e_0) + e_1x^{-1} + e_2x^{-2} + \cdots + e_ix^{-i} \\ &= 0.\end{aligned}$$

So $g(e_0) = 0,\ e_1 = e_2 = \cdots = e_i = 0$, which implies $e_0 \in \operatorname{Ker} g = \operatorname{Im} f = f(L)$.

Hence,

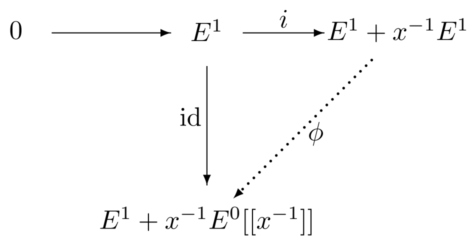
$$0 \to L \to M[x^{-1}] \to N + x^{-1}M[x^{-1}] \to 0$$

is a short exact sequence of $R[x]$-modules. □

Similarly, given a short exact sequence $0 \to L \to M \to N \to 0$ of R-modules, we get a short exact sequence $0 \to L \to M[[x^{-1}]] \to N + x^{-1}M[[x^{-1}]] \to 0$ of $R[x]$-modules.

Lemma 3.3. *Let* $0 \to N \xrightarrow{f} E^0 \xrightarrow{g} E^1 \to 0$ *be a short exact sequence of R-modules, where E^0, E^1 are injective with* $\operatorname{inj\,dim}_R N = 1$*. Then* $E^1 + x^{-1}E^0[[x^{-1}]]$ *is not an injective left $R[x]$-module.*

Proof. Suppose $E^1 + x^{-1}E^0[[x^{-1}]]$ is an injective left $R[x]$-module. Then there exists a $R[x]$-linear map ϕ which completes the following diagram



as a commutative diagram.

Then there exists an R-linear map $h : E^1 \to E^0$ such that $g \circ h = \mathrm{id}_{E^1}$. But since $\operatorname{inj\,dim}_R N = 1$, $0 \to N \xrightarrow{f} E^0 \xrightarrow{g} E^1 \to 0$ is not split, which implies a contradiction. Hence, $E^1 + x^{-1}E^0[[x^{-1}]]$ is not an injective left $R[x]$-module. □

Similarly, given a short exact sequence $0 \to N \to E^0 \to E^1 \to 0$ of R-modules with E^0, E^1 injective and $\operatorname{inj\,dim}_R N = 1$, we see that $E^1 + x^{-1}E^0[x^{-1}]$ is not an injective left $R[x]$-module.

Theorem 3.4. *Let* $\operatorname{inj\,dim}_R N = n$ *(with* $N \neq 0$*). Make* N *into an left* $R[x]$*-module so that* $xN = 0$*. Then*

$$\operatorname{inj\,dim}_{R[x]} N = n + 1.$$

Proof. Let N be a left R-module. Then

$$\operatorname{inj\,dim}_R N = \operatorname{inj\,dim}_{R[x]} N[[x^{-1}]] = n.$$

And we have the short exact sequence of $R[x]$-modules

$$0 \to N \to N[[x^{-1}]] \to N[[x^{-1}]] \to 0.$$

Then $\operatorname{inj\,dim}_{R[x]} N \leq (\operatorname{inj\,dim}_R N) + 1 = n + 1$. Since if N is an injective $R[x]$-module, then N is an injective R-module so that

$$\operatorname{inj\,dim}_R N \leq \operatorname{inj\,dim}_{R[x]} N \leq (\operatorname{inj\,dim}_R N) + 1.$$

Now by induction on n, if $n = 0$, then we want to show $\operatorname{inj\,dim}_{R[x]} N = 1$. But $\operatorname{inj\,dim}_R N = 0$ means that N is an injective R-module. If N is an injective $R[x]$-module, then N is divisible by x. But $xN = 0$. Thus N is not divisible by x. Thus N is not an injective $R[x]$-module. Therefore, $\operatorname{inj\,dim}_{R[x]} N \neq 0$, i.e., $\operatorname{inj\,dim}_{R[x]} N = 1$.

If $n = 1$, then we have a short exact sequence $0 \to N \to E^0 \to E^1 \to 0$ of R-modules with E^0, E^1 injective. Then by Lemma 3.3, $E^1 + x^{-1}E^0[[x^{-1}]]$ is not an injective left $R[x]$-module and by Lemma 3.2, $0 \to N \to E^0[[x^{-1}]] \to E^1 + x^{-1}E^0[[x^{-1}]] \to 0$ is a short exact sequence. Therefore, $\operatorname{inj\,dim}_{R[x]} N = 2$.

Now we suppose $\operatorname{inj\,dim}_R N = n > 1$ and make the obvious induction hypothesis. Let $0 \to N \to E \to L \to 0$ be an exact sequence of left R-modules with E injective. Then $\operatorname{inj\,dim}_R L = n - 1$. Now make N, E, L into $R[x]$-modules with $xN = 0$, $xE = 0$, $xL = 0$. Then $\operatorname{inj\,dim}_{R[x]} E = 1$ and by the induction hypothesis we know $\operatorname{inj\,dim}_{R[x]} L = n$. Using the long exact sequence of $\operatorname{Ext}_{R[x]}(A, -)$ where A is any left R-module, we get that $\operatorname{inj\,dim}_{R[x]} N = n + 1$. □

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