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MODULE EXTENSION OF DUAL BANACH ALGEBRAS

Madjid Eshaghi Gordji, Fereydoun Habibian, and Ali Rejali

Abstract. This work was intended as an attempt to introduce and investigate the Connes-amenability of module extension of dual Banach algebras. It is natural to try to study the *weak**-continuous derivations on the module extension of dual Banach algebras and also the weak Connes-amenability of such Banach algebras.

Introduction

In [6], B. E. Johnson introduced the notion of an amenable Banach algebra, and proved that a locally compact group G is amenable if and only if its group algebra $L^1(G)$ is amenable. The theory of amenable Banach algebras has been a very active field of research ever since. Once of the deepest result in this theory is due to Connes [2] and A. Haagerup [5]: a C^*-algebra is amenable if and only if it is nuclear. In [11], S. Wassermann showed that a von Neumann algebra is nuclear/amenable if and only if it is subhomogenuous [8]. This suggests that the definition of amenability from [6] has to be modified to yield a sufficiently rich theory for von Neumann algebras. A variant of that definition, one that takes the dual space structure of von Neumann algebra into account, was introduced in [7], but is most commonly associated with A. Connes paper [1]. For this reason, we refer to this notion of amenability as Connes-amenability. The definition of Connes-amenability makes sense for large class of Banach algebras (called dual Banach algebras in [8]). Examples of dual Banach algebras are: $B(\mathrm{E})$, where E is a reflexive Banach space; $M(G)$, where G is a locally compact group; $\mathfrak{A}^{**}$, where $\mathfrak{A}$ is an Arens regular Banach algebra.

This paper is organized as follows. Section 1 is devoted to the notations and definitions which are needed throughout of the paper. The Connes-amenability of module extension of dual Banach algebras is studied in Section 2. Finally in Section 3, we investigate the weak Connes-amenability of module extension of dual Banach algebras.

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1. Preliminaries

This section is preliminary in character. For a Banach algebra $\mathfrak{A}$, an $\mathfrak{A}$-bimodule will always refer to a *Banach* $\mathfrak{A}$*-bimodule* X, that is a Banach space which is algebraically an $\mathfrak{A}$-bimodule, and for which there is a constant $C_{\mathfrak{A},X} > 0$ such that

$$\|a.x\|, \|x.a\| \le C_{\mathfrak{A},X}\|a\|\|x\| \quad (a \in \mathfrak{A}, x \in X).$$

Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. Then X^* is a Banach $\mathfrak{A}$-bimodule by the operations,

$$\langle ax^*, x\rangle = \langle x^*, xa\rangle, \qquad \langle x^*a, x\rangle = \langle x^*, ax\rangle,$$

where $a \in \mathfrak{A}$, $x \in X$ and $x^* \in X^*$.

Definition 1.1. Let $\mathfrak{A}$ be a Banach algebra. A Banach $\mathfrak{A}$-bimodule E is called dual if there is a closed submodule E_* of E^* such that $E = (E_*)^*$. E_* is called the predual of E. A Banach algebra $\mathfrak{A}$ is called dual if it is dual as a Banach $\mathfrak{A}$-bimodule.

Let $\mathfrak{A}$ be a dual Banach algebra. A dual Banach $\mathfrak{A}$-bimodule E is called normal if, for every $x \in E$, the maps

$$\mathfrak{A} \longrightarrow E, \qquad a \mapsto a.x$$

and

$$\mathfrak{A} \longrightarrow E, \qquad a \mapsto x.a$$

are *weak**-continuous.

Let $\mathfrak{A}$ and $\mathfrak{B}$ be dual Banach algebras and let $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$ be a *weak**-continuous Banach algebra homomorphism. Then $\mathfrak{B}$ is a normal $\mathfrak{A}$-bimodule by the following module actions

$$a.b = \varphi(a)b, \quad b.a = b\varphi(a) \quad (a \in \mathfrak{A}, b \in \mathfrak{B}).$$

We denote the above $\mathfrak{A}$-bimodule by $\mathfrak{B}_\varphi$. Let X be a Banach $\mathfrak{A}$-bimodule. A derivation from $\mathfrak{A}$ into an $\mathfrak{A}$-bimodule X is a bounded linear map D such that $D(ab) = D(a).b + a.D(b)$ for all $(a, b \in \mathfrak{A})$. If $x \in X$, then $\delta_x : \mathfrak{A} \longrightarrow X$ defined by

$$\delta_x(a) = a.x - x.a \qquad (a \in \mathfrak{A}),$$

is a derivation. Such derivations are called inner. A Banach algebra $\mathfrak{A}$ is amenable if, for every $\mathfrak{A}$-bimodule X, every derivation $D : \mathfrak{A} \longrightarrow X^*$ is inner, equivalently if $H^1(\mathfrak{A}, X^*) = \{0\}$ for every Banach $\mathfrak{A}$-bimodule X, where the quotient space $H^1(\mathfrak{A}, X^*)$ of all continuous derivations from $\mathfrak{A}$ into X^* modulo the subspace of all inner derivations from $\mathfrak{A}$ into X^* is called *the first cohomology group* with coefficients in X^*[3] (see [1] and [5] for more details).

Let $\mathfrak{A}$ be a dual Banach algebra. $\mathfrak{A}$ is called Connes-amenable if, for every dual Banach $\mathfrak{A}$-bimodule X, every *weak**-continuous derivation $D : \mathfrak{A} \longrightarrow X$ is inner; or equivalently, $H^1_{w^*}(\mathfrak{A}, X) = \{0\}$. This definition was introduced by

V. Runde (see Section 4 of [9]). A dual Banach algebra $\mathfrak{A}$ is weakly Connes-amenable if every *weak**-continuous derivation from $\mathfrak{A}$ into $\mathfrak{A}$ is inner; or equivalently, $H^1_{w^*}(\mathfrak{A},\mathfrak{A}) = \{0\}$ [4]. The weak amenability of module extension Banach algebras was studied by Y. Zhang in [12]. We define the module extensions of dual Banach algebras and then we study the Connes-amenability and the weak Connes-amenability of such Banach algebras.

2. Connes-amenability

In this section we give necessary and sufficient conditions for module extension of dual Banach algebras to be Connes-amenable.

Lemma 2.1. *Let $\mathfrak{A}$ be a Banach algebra and X be a Banach $\mathfrak{A}$-bimodule. $\mathfrak{A} \oplus_\infty X$ is a Banach algebra with the algebra product,*

$$(a,x)(b,y) = (ab, ay + xb)$$

and with the norm,

$$\|(a,x)\| = \max\{\|x\|, \|a\|\} \qquad (a \in \mathfrak{A}\ ,\ x \in X).$$

Proof. It is easily seen that $\mathfrak{A} \oplus_\infty X$ is a Banach space. But $\mathfrak{A}$ is a Banach algebra, then there exists $C_{\mathfrak{A}} > 0$ such that $\|ab\| \le C_{\mathfrak{A}}\|a\|\|b\|$ for all $a, b \in \mathfrak{A}$ (see page 152 of [3] for more details). Also there is a constant $C_{\mathfrak{A},X} > 0$ such that

$$\|a.x\|, \|x.a\| \le C_{\mathfrak{A},X}\|a\|\|x\| \quad (a \in \mathfrak{A}, x \in X).$$

Fix $(a,x),(b,y) \in \mathfrak{A} \oplus_\infty X$. The proof falls naturally into four cases, but we give the proof for one case, the other cases are similar. If $\|(a,x)\| = \mathrm{Max}\{\|x\|,\|a\|\} = \|a\|$ and $\|(b,y)\| = \mathrm{Max}\{\|y\|,\|b\|\} = \|b\|$, then

$$\|ay + xb\| \le \|ay\| + \|xb\| \le C_{\mathfrak{A},X}\|a\|\|y\| + C_{\mathfrak{A},X}\|x\|\|b\| \le 2C_{\mathfrak{A},X}\|a\|\|b\|.$$

Set $C = \mathrm{Max}\{C_{\mathfrak{A}}, 2C_{\mathfrak{A},X}\}$. Now, if $\mathrm{Max}\{\|ab\|,\|ay+xb\|\} = \|ab\|$, then

$$\begin{aligned}\|(a,x)(b,y)\| = \|(ab, ay+xb)\| &= \|ab\| \le C_{\mathfrak{A}}\|a\|\|b\| \\ &= C_{\mathfrak{A}}\|(a,x)\|\|(b,y)\| \\ &\le C\|(a,x)\|\|(b,y)\|.\end{aligned}$$

If $\mathrm{Max}\{\|ab\|,\|ay+xb\|\} = \|ay+xb\|$, then

$$\begin{aligned}\|(a,x)(b,y)\| &= \|(ab, ay+xb)\| = \|ay+xb\| \\ &\le 2C_{\mathfrak{A},X}\|a\|\|b\| \le C\|a\|\|b\| \\ &= C\|(a,x)\|\|(b,y)\|.\end{aligned}$$

Then $(\mathfrak{A} \oplus_\infty X, \|\cdot\|)$ is a Banach algebra. Also if we define a new norm on $\mathfrak{A} \oplus_\infty X$, say, $|||(a,x)||| = C\|(a,x)\|$, then $(\mathfrak{A} \oplus_\infty X, |||\cdot|||)$ is a Banach algebra with

$$|||(a,x)(b,y)||| \le |||(a,x)|||\,|||(b,y)|||. \qquad \square$$

We define a new class of dual Banach algebras. Let $\mathfrak{A}$ be a dual Banach algebra with predual $\mathfrak{A}_*$, and let X be a normal dual Banach $\mathfrak{A}$-bimodule with predual X_*.

It is a simple matter to check that $\mathfrak{A}_* \oplus_1 X_*$ is a Banach space with the norm

$$\|(a', x')\| = \|a'\| + \|x'\| \qquad (a' \in \mathfrak{A}_* \ , \ x' \in X_*).$$

and $\mathfrak{A} \oplus_\infty X = (\mathfrak{A}_* \oplus_1 X_*)^*$. Since $\mathfrak{A}$ is a dual Banach algebra and X is a normal dual Banach $\mathfrak{A}$-bimodule, then clearly the multiplication in $\mathfrak{A} \oplus_\infty X$ is separately *weak**-continuous. According to Exercise 4.4.1 of [9], we get the following lemma.

Lemma 2.2. *Let $\mathfrak{A}$ and X be as above. Then $\mathfrak{A} \oplus_\infty X$ is a dual Banach algebra with predual $\mathfrak{A}_* \oplus_1 X_*$.*

The Banach algebra $\mathfrak{A} \oplus_\infty X$ in Lemma 2.2, is said to be *the module extension of dual Banach algebras.*

Theorem 2.3. *Let $\mathfrak{A}$ be a dual Banach algebra. Then the following assertions are equivalent*:

(i) $\mathfrak{A}$ *is Connes-amenable.*

(ii) *For every dual Banach algebra $\mathfrak{B}$ and every weak*-continuous homomorphism $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$, $H^1_{w^*}(\mathfrak{A}, \mathfrak{B}_\varphi) = \{0\}$.*

(iii) *For every dual Banach algebra $\mathfrak{B}$ and every injective weak*-continuous homomorphism $\varphi : \mathfrak{A} \longrightarrow \mathfrak{B}$, $H^1_{w^*}(\mathfrak{A}, \mathfrak{B}_\varphi) = \{0\}$.*

Proof. It is straightforward to verify (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii). We prove (iii) $\Rightarrow$ (i). Let X be a normal dual Banach $\mathfrak{A}$-bimodule, and let $D : \mathfrak{A} \longrightarrow X$ be a *weak**-continuous derivation. Lemma 2.2 shows that the map

$$\varphi : a \mapsto (a, 0), \qquad \mathfrak{A} \longrightarrow \mathfrak{A} \oplus_\infty X$$

is an injective *weak**-continuous homomorphism. Hence $H^1_{w^*}(\mathfrak{A}, ((\mathfrak{A} \oplus_\infty X)_\varphi)) = \{0\}$. We define $D_1 : \mathfrak{A} \longrightarrow \mathfrak{A} \oplus_\infty X$ by $D_1(a) = (0, D(a))$. For each $a, b \in \mathfrak{A}$,

$$\begin{aligned} D_1(ab) &= (0, D(ab)) = (0, D(a)b + aD(b)) \\ &= (0, D(a))(b, 0) + (a, 0)(0, D(b)) \\ &= D_1(a)\varphi(b) + \varphi(a)D_1(b). \end{aligned}$$

Therefore D_1 is a *weak**-continuous derivation from $\mathfrak{A}$ into $(\mathfrak{A} \oplus_\infty X)_\varphi$. From this we conclude that D_1 is an inner derivation. On the other word, $D_1 = \delta_{(b,x)}$ for some $b \in \mathfrak{A}, x \in X$. For every $a \in \mathfrak{A}$, we have

$$\begin{aligned} (0, D(a)) = D_1(a) &= \delta_{(b,x)}(a) \\ &= \varphi(a)(b, x) - (b, x)\varphi(a) \\ &= (a, 0)(b, x) - (b, x)(a, 0) \\ &= (ab - ba, ax - xa). \end{aligned}$$

Hence $D = \delta_x$ and $\mathfrak{A}$ is Connes-amenable. ∎

We are thus led to give the main result.

Theorem 2.4. *Let $\mathfrak{A}$ be a dual Banach algebra and let X be a reflexive Banach $\mathfrak{A}$-bimodule. If for every $x' \in X^*$ and $a \in \mathfrak{A}$, the mappings*

$$(1) \qquad (x'\widehat{\otimes}a). : b \longmapsto (x'\widehat{\otimes}ab) \ , \qquad .(x'\widehat{\otimes}a) : b \longmapsto (bx'\widehat{\otimes}a); \qquad \mathfrak{A} \longrightarrow X^*\widehat{\otimes}\mathfrak{A},$$

are weak-weak continuous, then $\mathfrak{A} \oplus_\infty X$ is Connes-amenable if and only if $X = 0$ and $\mathfrak{A}$ is Connes-amenable.*

Proof. Let $\mathfrak{A} \oplus_\infty X$ be Connes-amenable and the mappings defined in (1), are *weak**-weak continuous. We have to show that $X = 0$. It is easy to check that $X^*\widehat{\otimes}\mathfrak{A}$ is a Banach $\mathfrak{A} \oplus_\infty X$-bimodule with the following module actions:

$$(x'\widehat{\otimes}a).(b,x) = x'\widehat{\otimes}ab,\ (b,x).(x'\widehat{\otimes}a) = bx'\widehat{\otimes}a,\ (x'\widehat{\otimes}a \in X^*\widehat{\otimes}\mathfrak{A}, (b,x) \in \mathfrak{A}\oplus_\infty X).$$

Let

$$(b_\alpha, x_\alpha) \xrightarrow{weak^*} (b,x) \qquad \text{in} \quad \mathfrak{A} \oplus_\infty X,$$

hence $b_\alpha \xrightarrow{weak^*} b$ in $\mathfrak{A}$. Then for each $x' \in X^*$ and each $a \in \mathfrak{A}$,,

$$b_\alpha x'\widehat{\otimes}a \xrightarrow{\text{weakly}} bx'\widehat{\otimes}a \qquad \text{in} \quad X^*\widehat{\otimes}\mathfrak{A}.$$

From this, for each $F \in (X^*\widehat{\otimes}\mathfrak{A})^*$,

$$\langle F.(b_\alpha, x_\alpha), x'\widehat{\otimes}a\rangle = \langle F, b_\alpha x'\widehat{\otimes}a\rangle \longrightarrow \langle F, bx'\widehat{\otimes}a\rangle = \langle F.(b,x), x'\widehat{\otimes}a\rangle.$$

Consequently

$$F.(b_\alpha, x_\alpha) \xrightarrow{weak^*} F.(b,x) \qquad \text{in} \quad (X^*\widehat{\otimes}\mathfrak{A})^*.$$

Similarly

$$(b_\alpha, x_\alpha).F \xrightarrow{weak^*} (b,x).F \qquad \text{in} \quad (X^*\widehat{\otimes}\mathfrak{A})^*.$$

Thus $(X^*\widehat{\otimes}\mathfrak{A})^*$ is a normal dual $\mathfrak{A} \oplus_\infty X$-bimodule. Define $D : \mathfrak{A} \oplus_\infty X \longrightarrow (X^*\widehat{\otimes}\mathfrak{A})^*$ by;

$$\langle D(b,x), x'\widehat{\otimes}a\rangle = \langle x', ax\rangle \qquad (x'\widehat{\otimes}a \in X^*\widehat{\otimes}\mathfrak{A}, (b,x) \in \mathfrak{A} \oplus_\infty X).$$

For each $(b_1, x_1), (b_2, x_2) \in \mathfrak{A} \oplus_\infty X$, and $x'\widehat{\otimes}a \in X^*\widehat{\otimes}\mathfrak{A}$, we have

$$\begin{aligned}
&\langle D((b_1,x_1)(b_2,x_2)), x'\widehat{\otimes}a\rangle \\
&= \langle x', a(b_1x_2 + x_1b_2)\rangle = \langle x', ab_1x_2\rangle + \langle x', ax_1b_2\rangle \\
&= \langle D(b_2,x_2), x'\widehat{\otimes}ab_1\rangle + \langle D(b_1,x_1), b_2x'\widehat{\otimes}a\rangle \\
&= \langle D(b_2,x_2), (x'\widehat{\otimes}a)(b_1,x_1)\rangle + \langle D(b_1,x_1), (b_2,x_2)(x'\widehat{\otimes}a)\rangle \\
&= \langle (b_1,x_1).(D(b_2,x_2)) + (D(b_1,x_1)).(b_2,x_2), x'\widehat{\otimes}a\rangle.
\end{aligned}$$

Hence D is a derivation. Also if

$$(b_\alpha, x_\alpha) \xrightarrow{weak^*} (b,x) \qquad \text{in} \quad \mathfrak{A} \oplus_\infty X,$$

then $x_\alpha \xrightarrow{weak^*} x$ in X. Since X is a normal dual $\mathfrak{A}$-bimodule, we have $ax_\alpha \xrightarrow{weak^*} ax$ in X. On the other hand, X is reflexive, then $ax_\alpha \xrightarrow{\text{weakly}} ax$ in X. Thus

$$\langle D(b_\alpha, x_\alpha), x'\widehat{\otimes}a\rangle = \langle x', ax_\alpha\rangle \longrightarrow \langle x', ax\rangle = \langle D(b,x), x'\widehat{\otimes}a\rangle$$

for every $x'\widehat{\otimes}a \in X^*\widehat{\otimes}\mathfrak{A}$. Therefore D is *weak**-continuous. Connes-amenability of $\mathfrak{A}\oplus_\infty X$ implies that $D = \delta_F$ for some $F \in (X^*\widehat{\otimes}\mathfrak{A})^*$. For each $x'\widehat{\otimes}a \in X^*\widehat{\otimes}\mathfrak{A}$ and $(b,x) \in \mathfrak{A}\oplus_\infty X$, we have

$$\begin{aligned}\langle x', ax\rangle &= \langle D((b,x)), x'\widehat{\otimes}a\rangle \\ &= \langle (b,x).F - F.(b,x), x'\widehat{\otimes}a\rangle \\ &= \langle F, (x'\widehat{\otimes}a).(b,x) - (b,x)(x'\widehat{\otimes}a)\rangle \\ &= \langle F, x'\widehat{\otimes}ab - bx'\widehat{\otimes}a\rangle.\end{aligned}$$

Then $\langle x', ax\rangle = 0$ for each $a \in \mathfrak{A}, x \in X$ and $x' \in X^*$. We have to show that $\mathfrak{A}X = X$. To this end, we know that if $\mathfrak{A}\oplus_\infty X$ is Connes-amenable, then it is unital [9]. Let (e,x) be the unite element of $\mathfrak{A}\oplus_\infty X$. It is easy to show that $x = 0$ and $ey = y$ for every $y \in X$, and this finishes the proof. □

Corollary 2.5. *Let $\mathfrak{A}$ be a dual Banach algebra and let X be a non-trivial Banach $\mathfrak{A}$-bimodule. If $\mathfrak{A}$ and X are reflexive, then $\mathfrak{A}\oplus_\infty X$ is not Connes-amenable.*

Corollary 2.6. *Let $\mathfrak{A}$ be a non-trivial reflexive dual Banach algebra. Then the Banach algebras $\mathfrak{A}\oplus_\infty \mathfrak{A}$ and $\mathfrak{A}\oplus_\infty \mathfrak{A}^*$ are not Connes-amenable.*

3. Weak Connes-amenability

Let $\mathfrak{A}$ be a dual Banach algebra with predual $\mathfrak{A}_*$, and let X be a normal dual Banach $\mathfrak{A}$-bimodule with predual X_*. In this section we investigate the weak Connes-amenability of $\mathfrak{A}\oplus_\infty X$.

Lemma 3.1. *Let X be a normal, dual Banach $\mathfrak{A}$-bimodule and $T : X \longrightarrow X$ be a weak***-continuous $\mathfrak{A}$-bimodule morphism. Then $\bar{T} : \mathfrak{A}\oplus_\infty X \longrightarrow \mathfrak{A}\oplus_\infty X$, defined by $\bar{T}((a,x)) = (0, T(x))$ is a weak***-continuous derivation. $\bar{T}$ is inner if and only if there exists $b \in \mathfrak{A}$ such that $ba = ab$ for each $a \in \mathfrak{A}$ and $T(x) = xb - bx$ for all $x \in X$.*

Proof. Let $(a,x),(b,y) \in \mathfrak{A}\oplus_\infty X$, we have

$$\bar{T}((a,x).(b,y)) = \bar{T}((ab, ay + xb)) = (0, T(ay + xb)) = (0, aT(y)) + (0, T(x)b).$$

On the other hand, $\bar{T}((a,x)).(b,y) = (0,T(x)).(b,y) = (0, T(x)b)$, similarly

$$(a,x).\bar{T}((b,y)) = (a,x).(0,T(x)) = (0, aT(y)),$$

and hence $\bar{T}$ is a derivation. From *weak**-continuity of T, it is clear that $\bar{T}$ is *weak**-continuous. If $\bar{T}$ is inner, then there exists $\xi = (b,y) \in \mathfrak{A}\oplus_\infty X$ such that $\bar{T}((a,x)) = (a,x).\xi - \xi.(a,x)$. In particular $(0,0) = (a,0).\xi - \xi.(a,0)$ and $(0,T(x)) = (0,x).\xi - \xi.(0,x)$. Then $(0,0) = (ab - ba, ay - ya)$ and $(0,T(x)) =$

$(0, xb - bx)$ and so there exists $b \in \mathfrak{A}$ such that $ba = ab$ for $a \in \mathfrak{A}$ and $T(x) = xb - bx$ for all $x \in X$. Conversely, if there exists $b \in \mathfrak{A}$ such that $ba = ab$ for $a \in \mathfrak{A}$ and $T(x) = xb - bx$ for all $x \in X$, then

$$\bar{T}((a, x)) = (0, T(x)) = (ab - ba, xb - bx) = (a, x).(b, 0) - (b, 0).(a, x).$$

Therefore $\bar{T}$ is inner. □

Lemma 3.2. *Let $\mathfrak{A}$ be a dual Banach algebra and let X be a normal, dual Banach $\mathfrak{A}$-bimodule. If $D : \mathfrak{A} \longrightarrow X$ is a weak*-continuous derivation, then $\bar{D} : (\mathfrak{A} \oplus_\infty X) \longrightarrow (\mathfrak{A} \oplus_\infty X)$ defined by $\bar{D}((a, x)) = (0, D(a))$, is a weak*-continuous derivation. Furthermore, $\bar{D}$ is inner if and only if D is inner.*

Proof. It is straightforward to check that $\bar{D}$ is a *weak**-continuous derivation. Let $\bar{D}$ be inner. Then there exists $\xi = (b, y) \in \mathfrak{A} \oplus_\infty X$ such that $\bar{D}((a, x)) = (a, x).\xi - \xi.(a, x)$. In particular

$$(0, D(a)) = \bar{D}((a, 0)) = (a, 0).(b, y) - (b, y).(a, 0) = (ab - ba, ay - ya),$$

then $D(a) = ay - ya$ for some $y \in X$ and hence D is inner. The same proof works for the converse. □

Theorem 3.3. *Let $\mathfrak{A}$ be a dual Banach algebra and let X be a normal, dual Banach $\mathfrak{A}$-bimodule. Then $\mathfrak{A} \oplus_\infty X$ is weakly Connes-amenable if and only if the following conditions hold*:

1. *The only weak*-continuous derivations $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ for which there is a weak*-continuous operator $T : X \longrightarrow X$ such that $T(ax) = D(a)x + aT(x)$ and $T(xa) = xD(a) + T(x)a$ $(a \in \mathfrak{A}, x \in X)$, are the inner derivations.*
2. $H^1_{w^*}(\mathfrak{A}, X) = \{0\}$.
3. *The only weak*-continuous $\mathfrak{A}$-bimodule morphism $\Gamma : X \longrightarrow \mathfrak{A}$ for which $x\Gamma(y) + \Gamma(x)y = 0$ $(x, y \in X)$, is zero.*
4. *For every weak*-continuous $\mathfrak{A}$-bimodule morphism $T : X \longrightarrow X$, there exists $b \in \mathfrak{A}$ for which $ab = ba$ for $a \in \mathfrak{A}$ and $T(x) = xb - bx$ for $x \in X$.*

Proof. Denote by τ_1 and τ_2 the inclusion mappings from, respectively, $\mathfrak{A}$ and X into $\mathfrak{A} \oplus_\infty X$, and denote by Δ_1 and Δ_2 the natural projections from $\mathfrak{A} \oplus_\infty X$ onto $\mathfrak{A}$ and X, respectively. Then Δ_1 and Δ_2 are $\mathfrak{A}$-bimodule morphisms, so τ_1 and τ_2 are algebra homomorphisms. To prove the sufficiency we assume that Conditions 1-4 hold. Let $D : \mathfrak{A} \oplus_\infty X \longrightarrow \mathfrak{A} \oplus_\infty X$ be a *weak**-continuous derivation. Then $\Delta_1 \circ D \circ \tau_1 : \mathfrak{A} \longrightarrow \mathfrak{A}$ and $\Delta_2 \circ D \circ \tau_1 : \mathfrak{A} \longrightarrow X$ are *weak**-continuous derivations. Now we show that $\Gamma = \Delta_1 \circ D \circ \tau_2 : X \longrightarrow \mathfrak{A}$ is trivial. By Condition 3 it suffices to show that Γ is an $\mathfrak{A}$-bimodule morphism satisfying $x\Gamma(y) + \Gamma(x)y = 0$ $(x, y \in X)$. We have

$$\begin{aligned} 0 = D((0,0)) &= D((0,x).(0,y)) \\ &= D((0,x)).(0,y) + (0,x).D((0,y)) \\ &= (0, \Gamma(x)y) + (0, x\Gamma(y)). \end{aligned}$$

On the other hand,

$$\begin{aligned}\Gamma(ax) &= \Delta_1 \circ D((0,ax)) = \Delta_1 \circ D((a,0).(0,x)) \\ &= \Delta_1(D((a,0)).(0,x) + (a,0).D((0,x))) \\ &= \Delta_1((a,0).D((0,x))) = \Delta_1(aD \circ \tau_2(x)) \\ &= a\Gamma(x).\end{aligned}$$

Similarly, $\Gamma(xa) = \Gamma(x)a$. Then Γ is an $\mathfrak{A}$-bimodule morphism such that $x\Gamma(y) + \Gamma(x)y = 0$. Therefore Γ is trivial. Now let $T = \Delta_2 \circ D \circ \tau_2 : X \longrightarrow X$ and $D_1 = \Delta_1 \circ D \circ \tau_1 : \mathfrak{A} \longrightarrow \mathfrak{A}$. For every $a \in \mathfrak{A}$ and $x \in X$,

$$\begin{aligned}(0,T(ax)) &= (0,\Delta_2 \circ D((0,ax)) = D((0,ax)) \\ &= D((a,0).(0,x)) = D((a,0)).(0,x) + (a,0).D((0,x)) \qquad (1) \\ &= (0,D_1(a)x) + a(0,T(x)) = (0,D_1(a)x + aT(x)).\end{aligned}$$

This gives $T(ax) = D_1(a)x + aT(x)$. Similarly, for every $a \in \mathfrak{A}$ and $x \in X$, we have

$$(0,T(xa)) = (0,xD_1(a) + T(x)a). \qquad (2)$$

Therefore by Condition 1, $D_1 = \Delta_1 \circ D \circ \tau_1$ is inner.

Now suppose that $b \in \mathfrak{A}$ satisfies $D_1(a) = ab - ba$ for $a \in \mathfrak{A}$. Let $T_1 : X \longrightarrow X$ be defined by $T_1(x) = xb - bx$ for $x \in X$. Then $T - T_1 : X \longrightarrow X$ is a *weak**-continuous $\mathfrak{A}$-bimodule morphism. In fact, from (1), for every $a \in \mathfrak{A}$ and $x \in X$, we have

$$\begin{aligned}(T - T_1)(ax) &= T(ax) - T_1(ax) \\ &= (D_1(a)x + aT(x)) - (axb - bax) \\ &= (ab - ba)x + aT(x) - (axb - bax) \\ &= a(bx - xb) + aT(x) = a(T - T_1)(x).\end{aligned}$$

Similarly, $T - T_1$ is a right $\mathfrak{A}$-bimodule morphism. From Condition 4 there is $c \in \mathfrak{A}$ such that $ac = ca$ for $a \in \mathfrak{A}$ and $(T - T_1)(x) = xc - cx$ for $x \in X$. By Lemma 3.1, we know that

$$\overline{T - T_1} : (a,x) \longrightarrow (0,(T - T_1)(x)), \mathfrak{A} \oplus_\infty X \longrightarrow \mathfrak{A} \oplus_\infty X$$

is an inner derivation. Since $\Delta_2 \circ D \circ \tau_1 : \mathfrak{A} \longrightarrow X$ is a *weak**-continuous derivation, it is inner by Condition 2. By Lemma 3.2, the mapping

$$\overline{\Delta_2 \circ D \circ \tau_1} : (a,x) \longrightarrow (0,\Delta_2 \circ D \circ \tau_1(a)), \mathfrak{A} \oplus_\infty X \longrightarrow \mathfrak{A} \oplus_\infty X$$

is also an inner derivation. Since Γ is trivial, we now have

$$\begin{aligned}D((a,x)) &= (D_1(a), \Delta_2 \circ D \circ \tau_1(a) + T(x)) \\ &= \overline{\Delta_2 \circ D \circ \tau_1}((a,x)) + \overline{(T - T_1)}((a,x)) + (D_1(a), T(x)).\end{aligned}$$

Since

$$(D_1(a), T_1(x)) = (ab - ba, xb - bx) = (a,x).(u,0) - (u,0).(a,x)$$

for $a \in \mathfrak{A}$ and $x \in X$, it gives an inner derivation from $\mathfrak{A} \oplus_\infty X$ into $\mathfrak{A} \oplus_\infty X$. Hence as a sum of three inner derivations, D is inner. According to Conditions 1-4, $\mathfrak{A} \oplus_\infty X$ is weakly Connes-amenable.

Now we prove the necessity. Suppose that $\mathfrak{A} \oplus_\infty X$ is weakly Connes-amenable. Let $D : \mathfrak{A} \longrightarrow \mathfrak{A}$ be a *weak**-continuous derivation with the property given in Condition 1. We define $\bar{D} : \mathfrak{A} \oplus_\infty X \longrightarrow \mathfrak{A} \oplus_\infty X$ by

$$\bar{D}((a,x)) = (D(a), T(x)) \quad (a,x) \in (\mathfrak{A} \oplus_\infty X).$$

Then $\bar{D}$ is a *weak**-continuous derivation. But $\bar{D}$ is inner, so there exists $(b,y) \in \mathfrak{A} \oplus_\infty X$ such that

$$\bar{D}((a,x)) = (a,x).(b,y) - (b,y).(a,x),$$

and then for some $b \in \mathfrak{A}$, we have $(D(a), T(x)) = (ab-ba, xb-bx)$. Thus $D(a) = ab-ba$ for all $a \in \mathfrak{A}$, i.e., D is inner, and Condition 1 holds. Condition 2 follows from Lemma 3.2. Let now $\Gamma : X \longrightarrow \mathfrak{A}$ be an arbitrary *weak**-continuous $\mathfrak{A}$-bimodule morphism for which $x\Gamma(y) + \Gamma(x)y = 0$ $(x, y \in X)$. Define $\bar{\Gamma} : \mathfrak{A} \oplus_\infty X \longrightarrow \mathfrak{A} \oplus_\infty X$ by $\bar{\Gamma}((a,x)) = (\Gamma(x), 0)$ then $\bar{\Gamma}$ is a *weak**-continuous derivation, but $\bar{\Gamma}$ is inner, hence there exists $\xi = (b,y) \in \mathfrak{A} \oplus_\infty X$ such that $\bar{\Gamma}((a,x)) = (a,x).(b,y) - (b,y).(a,x)$. In particular

$$(\Gamma(x), 0) = \bar{\Gamma}((0,x)) = (0,x).(b,y) - (b,y).(0,x) = (0, xb - bx)$$

and then $\Gamma = 0$, and Condition 3 holds. Let $T : X \longrightarrow X$ be a *weak**-continuous $\mathfrak{A}$-bimodule morphism. $\bar{D} : \mathfrak{A} \oplus_\infty X \longrightarrow \mathfrak{A} \oplus_\infty X$ defined by $\bar{D}((a,x)) = (0, T(x))$ is a *weak**-continuous derivation, and Condition 4 holds by Lemma 3.1. □

Let $X = \mathfrak{A}$. If in Condition 4 of above theorem, we suppose that $T = id : \mathfrak{A} \longrightarrow \mathfrak{A}$, thus we get:

Corollary 3.4. *Let $\mathfrak{A}$ be a non-trivial dual Banach algebra $\mathfrak{A}$. Then,*

$$H^1_{w^*}(\mathfrak{A} \oplus_\infty \mathfrak{A}, \mathfrak{A} \oplus_\infty \mathfrak{A}) \neq \{0\}.$$

Lemma 3.5. *Let X and Y be dual Banach spaces. Then every weak*-continuous linear map from X into Y is bounded.*

Proof. Let $T : X \longrightarrow Y$ be an unbounded linear map. Then there exists a sequence $\{x_n\}$ in X such that $\lim_n \|x_n\| = 0$ and $\lim_n \|T(x_n)\| = \infty$. By uniform boundedness theorem ([10]), $T(x_n) \overset{weak^*}{\not\longrightarrow} 0$. On the other hand *weak**-$\lim_n x_n = 0$, therefore T is not *weak**-continuous. □

Let us mention an important consequence of Corollary 3.4 and Lemma 3.5.

Corollary 3.6. *Let $\mathfrak{A}$ be a non-trivial dual Banach algebra. Then*

$$H^1(\mathfrak{A} \oplus_\infty \mathfrak{A}, \mathfrak{A} \oplus_\infty \mathfrak{A}) \neq \{0\}.$$

Let $\mathfrak{A}$ be a dual Banach algebra, and let $X = \mathfrak{A}$ by module actions

$$a.x = ax, \qquad x.a = 0, \qquad (a \in \mathfrak{A}, x \in X),$$

We follow the notation of [12] to show that X by $\mathfrak{A}_0$.

Corollary 3.7. *$\mathfrak{A}$ is unital and weakly Connes-amenable if and only if $\mathfrak{A} \oplus_\infty \mathfrak{A}_0$ is weakly Connes-amenable.*

Proof. Let $\mathfrak{A}$ be a unital weakly Connes-amenable Banach algebra. Since $\mathfrak{A}$ is weakly Connes-amenable, then the Conditions 1 and 2 in Theorem 3.3, hold. But $\mathfrak{A}$ is unital then Conditions 3 and 4 hold when $X = \mathfrak{A}_0$. For the converse let $\mathfrak{A} \oplus_\infty \mathfrak{A}_0$ be weakly Connes-amenable, then by Condition 2, $\mathfrak{A}$ is weakly Connes-amenable. The mapping $id : \mathfrak{A}_0 \longrightarrow \mathfrak{A}_0$ is a *weak**-continuous $\mathfrak{A}$-bimodule morphism, then by Condition 4 of Theorem 3.3, there exists $b \in \mathfrak{A}$ such that $ab = ba$ for $a \in \mathfrak{A}$, and $x = id(x) = x.b - b.x = bx$ for $x \in \mathfrak{A}_0$. Thus b is the unit element of $\mathfrak{A}$. $\square$

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Madjid Eshaghi Gordji
Department of Mathematics
Semnan University
P. O. Box 35195-363, Semnan, Iran
E-mail address: madjid.eshaghi@gmail.com; maj_ess@yahoo.com

Fereydoun Habibian
Department of Mathematics
Science Building
Isfahan University
Isfahan, Iran
and
Department of Mathematics
Semnan University
P. O. Box 35195-363, Semnan, Iran
E-mail address: fhabibian@math.ui.ac.ir

Ali Rejali
Department of Mathematics
Science Building
Isfahan University
Isfahan, Iran
E-mail address: rejali@sci.ui.ac.ir