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Asymptotics of the Variance Ratio Test for MA Unit Root Processes

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Abstract

We consider the asymptotic results of the variance ratio statistic when the underlying processes have moving average(MA) unit roots. This degenerate situation of zero spectral density near the origin cause the limit of the variance ratio to become zero. Its asymptotic behaviors are different from non-degenerating case, where the convergence rate of the variance ratio statistic is formally derived.

Keywords: Variance ratio test, moving average unit root, spectral density, degeneracy.

1. Introduction

The variance ratio(VR) statistic has been widely used to test for random walk hypothesis or for efficient market hypothesis in the context of economics and finance. Campbell *et al.* (1997) provide excellent summary and review of the VR and its financial applications. Among many useful empirical works, Lo and MacKinlay (1988) use the VR to examine the random walk hypothesis using the US stock price data. In economics context, Cochrane (1988) also make use of the VR testing to investigate the magnitude of random walk components in the GDP series. To figure out deterministic trend components and random walk components out of certain economic time series have drawn a lot of attention among economists, where the VR statistic is popularly used.

Define the second-order stationary process r_t and let M compounded sums be $r_t(M) = r_t + r_{t-1} + \cdots + r_{t-M+1}$. The population VR is defined as the ratio of the variance of M compounded sums and M times the variance of r_t,

$$\mathrm{VR}(M) = \frac{\mathrm{Var}(r_t(M))}{M \cdot \mathrm{Var}(r_t)}. \tag{1.1}$$

For example, when two-periods are considered, we have $\mathrm{VR}(2) = 1 + \rho(1)$, where $\rho(1) = R(1)/R(0)$, and covariance function is given as

$$R(j) = E(r_t - \mu)(r_{t-j} - \mu), \quad \text{for } \mu = E(r_t) \tag{1.2}$$

and $R(0) = \mathrm{Var}(r_t)$ is the variance of r_t. One is interested to test whether $\mathrm{VR}(2) = 1$.

The VR can be also seen as

$$\mathrm{VR}(M) = R(0)^{-1}\left[R(0) + 2\sum_{j=1}^{M-1}\left(1 - \frac{j}{M}\right)R(j)\right]. \tag{1.3}$$

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In order to see the limit of the VR as the number of periods M increases, we introduce the spectral density function evaluated at the zero frequency, denoted as $f(0)$,

$$f(0) = \frac{1}{2\pi} \sum_{j=-\infty}^{\infty} R(j) \tag{1.4}$$

(*cf*: Priestley, 1981). Then, the limit of the VR equals to

$$\lim_{M\to\infty} \text{VR}(M) = \frac{2\pi f(0)}{R(0)}. \tag{1.5}$$

It is often assumed that the stationary processes r_t have strictly positive spectrum near the origin, say $f(0) > 0$. The resulting theoretical results have been well established (*cf*: Campbell *et al.*, 1997, Ch.2; Andrews, 1991; Newey and West, 1994). On the other hand, we concentrate on the case of $f(0) = 0$, which has been little known in econometrics context. One popular example is that if a certain process is trend stationary,

$$y_t = a + bt + e_t, \tag{1.6}$$

for $t = 1, 2, \ldots, T$, where T is the sample size. Then, the first differenced series r_t can be put as

$$r_t = y_t - y_{t-1} = b + \Delta e_t, \quad \text{for } \Delta e_t = e_t - e_{t-1}. \tag{1.7}$$

Thus, r_t reduces to have the zero spectral density evaluated at zero frequency, *i.e.*, $f(0) = 0$. It then follows that the limit of the VR simply becomes zero. This is typically known as moving average(MA) unit roots. See Leybourne *et al.* (1996) and Saikkonen and Lukkonen (1993). The arguments can be easily extended to MA(q) unit root processes with $q > 1$. Another example includes first-differencing fractionally integrated (long memory) processes with the memory parameter less than the unity.

In this work, we study theoretical properties of the VR when the series yields zero spectrum near the origin, particularly focusing on the convergence rate of the VR statistic in comparison with existing results of non-degenerate case.

2. Main Results

We verify the asymptotic order of magnitude of the sample VR statistic under degenerate case. The sample VR statistic is written as

$$\widehat{\text{VR}}(M) = \hat{R}(0)^{-1}\left[\hat{R}(0) + 2\sum_{j=1}^{M-1}\left(1 - \frac{j}{M}\right)\hat{R}(j)\right], \tag{2.1}$$

where sample covariances are defined as

$$\hat{R}(j) = \frac{1}{T}\sum_{t=|j|+1}^{T}(r_t - \bar{r})(r_{t-j} - \bar{r}),$$

where $\bar{r} = T^{-1}\sum_{t=1}^{T} r_t$. The limiting form, denoted as $\widehat{\text{VR}}(\infty)$ can be written as

$$\widehat{\text{VR}}(\infty) = \lim_{M\to\infty} \widehat{\text{VR}}(M) = \frac{2\pi\hat{f}(0)}{\hat{R}(0)}, \tag{2.2}$$

where $\hat{f}(0)$ is a consistent estimator of $f(0)$. Consider a kernel-based spectral density estimator for $f(0)$,

$$\hat{f}(0) = (2\pi)^{-1} \sum_{j=1-T}^{T-1} k\left(\frac{j}{M}\right)\hat{R}(j), \tag{2.3}$$

where, $k(x)$ is a kernel function and M is a bandwidth parameter.

To see the asymptotic properties of the $\widehat{\mathrm{VR}}(\infty)$ particularly when the true $f(0) = 0$, we assume that r_t is stationary Gaussian process throughout this work. As it is necessary to treat higher-order expansions for kernel-based spectral density estimators, thus Gaussian assumptions significantly ease derivation of asymptotic variance of $\hat{f}(0)$. Note that without Gaussianity, however, general dependence of fourth order cumulants will arise, which are very difficult to handle. Velasco and Robinson (2001) also assume Gaussianity in Edgeworth expansion of the studentized sample mean.

To show the main result, we impose some conditions for the VR statistic.

Assumption 1.

(a) $k(x) : R \to [-1, 1]$ *is symmetric and continuous at zero with* $k(0) = 1$, *and the Fourier transform of* $k(x)$ *is defined as*

$$K(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} k(x)e^{-i\lambda x}dx$$

for all $\lambda \in [-\pi, \pi]$.

(b)
$$\int_{-\infty}^{\infty} \lambda^r K(\lambda)d\lambda = 0, \quad \text{for } r = 1, 2, \ldots, q-1, \quad \text{and} \quad \neq 0, \quad \text{for } r = q.$$

Assumption 1 lists a regularity condition for kernel functions. which is standard in the nonparametric literature. The function $K(\lambda)$, a spectral window generator, has the property that $\int_{-\pi}^{\pi} K(\lambda)d\lambda = 1$, which is equivalent to $k(0) = 1$. The moment condition in (b) can be equivalently understood as derivative of its inverse Fourier transforms, $k(\bullet)$ evaluated at zero, that is to say, $d^r k(x)/dx^r|_{x=0} = 0$, for $r = 1, 2, \ldots, q-1$ and $\neq 0$, for $r = q$, where $k(x) = \int_{-\pi}^{\pi} K(\lambda)e^{ix\lambda}d\lambda$. This is related with higher-order expansions of $\hat{f}(0)$.

Quadratic kernels such as Parzen and quadratic spectral kernel satisfy the above conditions with $q = 2$. In our analysis, the value of q needs to be at least 4, which is directly related with higher-order Taylor expansions of the estimators. For example, the following is the fourth-order kernel,

$$K(\lambda) = \frac{15}{32}\left[7\left(\frac{\lambda}{\pi}\right)^4 - 10\left(\frac{\lambda}{\pi}\right)^2 + 3\right]. \tag{2.4}$$

Other higher-order kernels with $q > 4$ can be considered (*e.g.* Velasco and Robinson, 2001).

Also, we need some smoothness conditions for the spectral density function $f(\lambda)$ at $\lambda = 0$.

Assumption 2.

$$\sum_{j=-\infty}^{\infty} \|R(j)\| < \infty \quad \text{and} \quad \sum_{j=-\infty}^{\infty} |j|^q |R(j)| < \infty, \qquad \text{for } q \in [0, \infty).$$

The smoothness for $f(\bullet)$ near zero frequency is given by the q^{th} order generalized spectral derivative,

$$f^{(q)}(0) = (2\pi)^{-1} \sum_{j=-\infty}^{\infty} |j|^q R(j), \quad \text{for } q \in [0, \infty), \tag{2.5}$$

where q may not be integer-valued. The generalized spectral derivative $f^{(q)}(0)$ is not necessarily equal to q^{th} derivative since $f_{(q)}(0) = d^q f(\lambda)/d\lambda^q|_{\lambda=0}$. If q is even-numbered, then

$$f^{(q)}(0) = (-1)^{\frac{q}{2}} f_{(q)}(0). \tag{2.6}$$

For example, when $q = 2$, then $f^{(2)}(0) = -f_{(2)}(0)$. If $q = 4$, then $f^{(4)}(0) = f_{(4)}(0)$. The larger values of q, the more smooth the spectral density function near origin. When $f(0) = 0$, it is necessary to derive higher-order expansions of the estimator near the zero frequency, which requires a large values of q. Our main results need $q \geq 4$.

Now, we present the main result.

Theorem 1. *Suppose Assumptions 1~2 hold, and* $M = C \times T^{\alpha}$, *where* $0 < C < \infty$, $0 < \alpha < 1$, $q \geq 4$. *Under* $f(0) = 0$,

$$\lim_{M\to\infty} \left(T \times M^3\right) Var\left(\widehat{VR}(M)\right) = \Omega < \infty,$$

where $\Omega = R(0)^{-2}(16\pi^5) f_{(2)}^2(0) \int_{-\pi}^{\pi} u^4 K^2(u) du$.

The Theorem 1 implies that the convergence rate of the VR test with sufficiently large values of M is $\sqrt{T \times M^3}$, *i.e.*,

$$\lim_{M\to\infty} \left(T \times M^3\right)^{\frac{1}{2}} \widehat{\text{VR}}(M) = O_p(1). \tag{2.7}$$

The above result is fundamentally different from existing asymptotic results, which show that convergence rate of the VR equals to $\sqrt{T \times M}$. In the non-degenerate case, it is well known that the sample VR has convergence rate equal to $\sqrt{T \times M}$ (*cf.* Campbell *et al.*, 1997). The convergence rate derived in Theorem 1 is mainly due to degeneracy of $f(0)$, requiring certain higher-order expansions of the spectrum near zero frequency. It is immediate that if we use the sample VR with the rate $\sqrt{T \times M}$ for over-differenced series, then the test behaves as $O_p(M^{-1})$ and decays to zero as M goes to the infinity. As a digression, based on this result, it is possible to obtain asymptotic normality of the VR with additional assumptions on data generating processes (*cf.* Phillips and Solo, 1992; Lee, 2009). As our main focus is the asymptotic order of magnitude in the VR test statistic, we skip the part of normality.

3. Conclusion

We study the variance ratio test statistic when the underlying processes have moving average unit roots such as over-differenced series, generating zero spectrum at the zero frequency. Under degeneracy, the convergence rate of the variance ratio statistic is derived, which is different from the known rate in non-degenerate case.

Appendix:

Proof of Theorem 1: The sample variance ratio test is given as

$$\widehat{\mathrm{VR}}(\infty) = \hat{R}(0)^{-1}(2\pi)\hat{f}(0). \tag{A.1}$$

First, we show that the limit of $\mathrm{Var}(\hat{f}(0)) = 0$ when $f(0) = 0$. Given Gaussian assumption, we get

$$\mathrm{Var}\left(\hat{f}(0)\right) = \sum_{j=1-M}^{M-1} \sum_{j'=1-M}^{M-1} k\left(\frac{j}{M}\right) k\left(\frac{j'}{M}\right) \mathrm{Cov}\left[\hat{R}(j), \hat{R}(j')\right], \tag{A.2}$$

where

$$\mathrm{Cov}\left[\hat{R}(j), \hat{R}(j')\right] = T^{-1} \sum_{h=1-T}^{T-1} \left[R(h)R(h+j'-j) + R(h+j')R(h-j)(1+o(1))\right].$$

For the exact form of $\mathrm{Cov}[\hat{R}(j), \hat{R}(j')]$, see, for example, Priestley (1981, p.326). Then, we have

$$\mathrm{Var}\left(\hat{f}(0)\right) = (V_{1T} + V_{2T})(1 + o(1)), \tag{A.3}$$

where

$$V_{1T} = T^{-1} \sum_{j=1-M}^{M-1} \sum_{j'=1-M}^{M-1} k\left(\frac{j}{M}\right) k\left(\frac{j'}{M}\right) \sum_{h=1-T}^{T-1} R(h)R(h+j'-j),$$

$$V_{2T} = T^{-1} \sum_{j=1-M}^{M-1} \sum_{j'=1-M}^{M-1} k\left(\frac{j}{M}\right) k\left(\frac{j'}{M}\right) \sum_{h=1-T}^{T-1} R(h+j')R(h-j).$$

To ease the technical proof, we treat $k(j/M) = 0$ for $j \geq M$. This does not affect the main results. As for the first term V_{1T}, we use Fourier and inverse Fourier transforms to obtain

$$\begin{aligned}
T \times V_{1T} &= \sum_{h=1-T}^{T-1} R(h) \sum_{j=1-M}^{M-1} k\left(\frac{j}{M}\right) \sum_{j'=1-M}^{M-1} \int_{-\pi}^{\pi} MK(M\lambda) e^{-ij'\lambda} R(h+j'-j) d\lambda \\
&= \left(2\pi^3\right) \left[(2\pi)^{-1} \sum_{h=1-T}^{T-1} R(h)\right] \left[(2\pi)^{-1} \sum_{j=1-M}^{M-1} k\left(\frac{j}{M}\right)\right] \\
&\quad \times \left[\int_{-\pi}^{\pi} MK(M\lambda)(2\pi)^{-1} \sum_{j'=1-M}^{M-1} R(h+j'-j) e^{-i(h+j'-j)\lambda} e^{i(h-j)\lambda} d\lambda\right]
\end{aligned}$$

which can be further decomposed as follows,

$$\begin{aligned}
T \times V_{1T} &= \left(2\pi^3\right) \left[(2\pi)^{-1} \sum_{h=1-T}^{T-1} R(h)\right] \left[(2\pi)^{-1} \sum_{j=1-M}^{M-1} k\left(\frac{j}{M}\right)\right] \left[\int_{-\pi}^{\pi} MK(M\lambda) f(\lambda) e^{i(h-j)} d\lambda\right] \\
&= \left(2\pi^3\right) \int_{-\pi}^{\pi} MK(M\lambda) f(\lambda) \left[f(\lambda) - (2\pi)^{-1} \sum_{h \geq T} R(h) e^{ih\lambda}\right] [MK(M\lambda)]\, d\lambda \\
&= A_{1T} - B_{1T},
\end{aligned}$$

where

$$A_{1T} = \left(2\pi^3\right) \int_{-\pi}^{\pi} f^2(\lambda) M^2 K^2(M\lambda) d\lambda \tag{A.4}$$

$$B_{1T} = \left(2\pi^3\right) \int_{-\pi}^{\pi} f(\lambda) M^2 K^2(M\lambda) \sum_{|h|\geq T} R(h) e^{ih\lambda},$$

and $K(\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} k(x) e^{-ix\lambda} dx$ and $MK(M\lambda) = (2\pi)^{-1} \int_{-\infty}^{\infty} k(x/M) e^{-ix\lambda} dx$, (*e.g.*, Priestley, 1981, p.447).

Below we show that A_{1T} is a dominant term as big as $O(M^{-3})$. Using change of variable techniques, we obtain

$$\begin{aligned} A_{1T} &= \left(2\pi^3\right) \int_{-\pi}^{\pi} f^2(\lambda) M^2 K^2(M\lambda) d\lambda \\ &= \left(2\pi^3\right) M \int_{-\pi}^{\pi} f^2\left(\frac{u}{M}\right) K^2(u) du \\ &= \left(2\pi^3\right) M \left[\left(\frac{1}{4}\right) f_{(2)}^2(0) \int_{-\pi}^{\pi} \left(\frac{u}{M}\right)^4 K^2(u) du + O\left(M^{-6}\right)\right] \\ &= 2\pi^3 M^{-3} f_{(2)}^2(0) \int_{-\pi}^{\pi} u^4 K^2(u) du + O\left(M^{-5}\right), \end{aligned} \tag{A.5}$$

where the third line follows from Taylor expansions of the squared $f(0)$, given that $f(0) = f_{(1)}(0) = 0$.

On the other hand, for B_{1T}, we have

$$\begin{aligned} B_{1T} &\leq (2\pi)^2 \sum_{|h|\geq T} R(h) \int_{-\pi}^{\pi} f(\lambda) M^2 K^2(M\lambda) d\lambda \\ &\leq (2\pi)^2 \sum_{|h|\geq T} |h|^q R(h) \int_{-\pi}^{\pi} f(\lambda) M^2 K^2(M\lambda) d\lambda. \end{aligned} \tag{A.6}$$

By Taylor expansions, it is written as

$$\begin{aligned} \int_{-\pi}^{\pi} f(\lambda) M^2 K^2(M\lambda) d\lambda &= M \int_{-\pi}^{\pi} f\left(\frac{u}{M}\right) K^2(u) du \\ &= M \left[\left(\frac{1}{2}\right) f_{(2)}(0) \int_{-\pi}^{\pi} \left(\frac{u}{M}\right)^2 K^2(u) du\right] + O\left(M^{-4}\right). \end{aligned}$$

Thus, we have $\int_{-\pi}^{\pi} f(\lambda) M^2 K^2(M\lambda) d\lambda = O(M^{-1})$ and as a result,

$$B_{1T} = O\left(T^{-q} M^{-1}\right) = o\left(M^{-3}\right), \tag{A.7}$$

given that $\alpha < q/2$.

Similar arguments can be applied to V_{2T} term above by repeating the same techniques. The dominant term in V_{2T} is identical to that in V_{1T}. Therefore, we finally obtain

$$\left(T \times M^3\right) [V_{1T} + V_{2T}] = \left(4\pi^3\right) f_{(2)}^2(0) \int_{-\pi}^{\pi} u^4 K^2(u) du + o(1), \tag{A.8}$$

which leads to

$$\lim_{M\to\infty}\left(T\times M^3\right)\mathrm{Var}\left(\widehat{\mathrm{VR}}(\infty)\right)=R(0)^{-2}\left(16\pi^5\right)f_{(2)}^2(0)\int_{-\pi}^{\pi}u^4K^2(u)du. \tag{A.9}$$

This completes the proof of the Theorem 1.

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