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# Finding Optimal Controls for Helicopter Maneuvers Using the Direct Multiple-Shooting Method

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## Abstract

The purpose of this paper deals with direct multiple-shooting method (DMS) to resolve helicopter maneuver problems of helicopters. The maneuver problem is transformed into nonlinear problems and solved DMS technique. The DMS method is easy in handling constraints and it has large convergence radius compared to other strategies. When parameterized with piecewise constant controls, the problems become most effectively tractable because the search direction is easily estimated by solving the structured Karush-Kuhn-Tucker (KKT) system. However, generally the computation of function, gradients and Hessian matrices has considerably time-consuming for complex system such as helicopter. This study focused on the approximation of the KKT system using the matrix exponential and its integrals. The propose method is validated by solving optimal control problems for the linear system where the KKT system is exactly expressed with the matrix exponential and its integrals. The trajectory tracking problem of various maneuvers like bob up, sidestep near hovering flight speed and hurdle hop, slalom, transient turn, acceleration and deceleration are analyzed to investigate the effects of algorithmic details. The results show the matrix exponential approach to compute gradients and the Hessian matrix is most efficient among the implemented methods when combined with the mixed time integration method for the system dynamics. The analyses with the proposed method show good convergence and capability of tracking the prescribed trajectory. Therefore, it can be used to solve critical areas of helicopter flight dynamic problems.

Key Words : Helicopter Flight Controller, Optimal Control, Direct Multiple-Shooting method

## Introduction

There exist various strategies[1,2] for reaching numerical solutions to the nonlinear optimal control problem. The direct multiple-shooting (DMS) method is usually preferred for analyzing general nonlinear optimal control problems due to its convenience in handling system constraints and a large convergence radius compared to other methods. However, these advantages can be decreased in case the estimation of the related Karush-Kuhn-Tucker (KKT) system is not accurate enough to guarantee robust analyses. The errors in estimating the KKT system are originated from time integration and finite difference method in the standard DMS method.

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## Design of Direct Multiple-Shooting Controller

### 2.1 Formulation of nonlinear optimal control problems

Nonlinear optimal control problems can be represented by the standard Bolza form[3, 4].

$$\min_{x,u,t_f} J(x,u,t_f) = \phi(x(t_f)) + \int_{t_0}^{t_f} \Phi(x,u)dt \tag{1}$$

$$\text{s.t.} \quad \begin{aligned} &\dot{x}(t) = f(x(t),u(t),t), \quad t \in [0,t_f] \\ &x(t_0) = x_0 \end{aligned} \tag{2}$$

$$h(x(t_f),t_f) = 0 \tag{3}$$

$$g(x(t),u(t),t) \le 0 \tag{4}$$

In the above equations $x, u, t_o, t_f$ represent states, controls, initial time and final time, respectively and $\phi, \Phi$ denote final cost and const function. The system dynamics, initial condition, final condition, and system constraints are defined in Eq (2)-Eq(4).

## 2.2 Direct Multiple Shooting Method

The DMS method transforms the above equations into a solvable nonlinear programming problem in finite dimension using suitable state and control parameterization methods. The resulting nonlinear programming (NLP) problem can be effectively resolved using the SQP method. The control parameterization, with piecewise constant controls, formulates one of the most efficient DMS methods because the related QP solution benefits from the sparseness in the KKT system. In the interest of completeness, the major procedures of DMS methods are introduced, in which the mathematical notations are similar to those used in Ref. 7. In a case where controls are parameterized using piecewise constant inputs, the related NLP can be derived by applying the following steps:

For the application of the DMS method, *N*-shooting nodes and initial states at each shooting node are defined in Eq (5) and the control at each shooting-interval is assumed to be piecewise constant as defined in Eq (6).

$$t_0 < t_1 < \ldots < t_{N-1} < t_N := t_f \tag{5}$$

$$x(t_j) = s_j \quad j = 0,\ldots,N$$

$$u(t) = q_j \quad t \in [t_j, t_{j+1}] \quad j = 0,\ldots,N-1 \tag{6}$$

By solving the initial value problems the states and cost function contribution over each shooting-interval can be written as

$$x_j(t) = X_j(t; s_j, q_j) = s_j + \int_{t_j}^{t} f(x_j, q_j, t)dt \tag{7}$$

$$J_j(s_j, q_j) = \int_{t_j}^{t_{j+1}} \Phi\left(x_j(t; s_j, q_j), q_j\right)dt \tag{8}$$

Where $X_j(t; s_j, q_j)$ defines the system states over the *j*-th shooting interval with initial state $s_j$ and control input $q_j$. Then the system dynamics are transformed into the equality constraints as shown in Eq (9) and the optimal control problem can be converted into a NLP problem as in Eq (10).

$$s_{j+1} = X_j(t_{j+1}; s_j, q_j) = s_j + \int_{t_j}^{t_{j+1}} f(x_j, q_j, t)dt \tag{9}$$

$$\min J(\mathbf{s}, \mathbf{q}) = \phi(s_N) + \sum_{i=0}^{N-1} J_j(s_j, q_j) \tag{10}$$

s.t.

$$s_0 = x_0$$
$$h_j(s_j, q_j) = X_j(t_{j+1}; s_j, q_j) - s_{j+1} = 0, \quad j = 0,\ldots,N-1$$
$$h_N(s_N, t_N) = 0$$
$$g_j(s_j, q_j, t_j) \le 0, \quad j = 1,\ldots,N$$

## 2.3 Sequential Quadratic Problem (SQP)

The SQP framework is one of the most popular iterative method to solve the NLP problem. In this framework the cost function is approximated using the quadratic function and the constraints are linearly approximated. If we define $\left(p_j^s, p_j^q\right)$ as iterative corrections in initial states $s_j$ and controls $q_j$ over the *j*-th shooting interval, these correction can be obtained by solving the following QP sub-problem.

$$\min J(\mathbf{p}) = \sum_{j=0}^{N} \bar{J}_j\left(p_j^s, p_j^q\right) \tag{11}$$

s.t.

$$F_j p_j^s + D_j p_j^q + h_j - p_{j+1}^s = 0, \quad j = 0,\ldots,N \tag{12}$$

$$G_j p_j^s + E_j p_j^q + g_j \le 0, \quad j = 1,\ldots,N-1 \tag{13}$$

where $\bar{J}$ is the quadratic approximation of local cost functions and $F_j, D_j, G_j, E_j$ are Jacobean matrices for the constraint functions defined as following

$$\mathbf{p} = \left[p_0^q\ , p_1^s\ , p_1^q\ ,, \cdots, p_{N-1}^s\ , p_{N-1}^q, p_N^s\ \right]^T \tag{14}$$

$$\bar{J}_0 = J_0^q p_0^q + \frac{1}{2}(p_0^q)^T H_0^{qq} p_0^q$$
$$\bar{J}_j = \left(J_j^s \mid J_j^q\right)\begin{pmatrix} p_j^s \\ p_j^q \end{pmatrix} + \frac{1}{2}\left((p_j^s)^T \mid (p_j^q)^T\right)\begin{bmatrix} H_j^{ss} & (H_j^{qs})^T \\ H_j^{qs} & H_j^{qq} \end{bmatrix}\begin{pmatrix} p_j^s \\ p_j^q \end{pmatrix} \tag{15}$$
$$j = 1,\ldots,N-1$$
$$\bar{J}_N = \left(\phi_N^s\right)p_N^s + \frac{1}{2}(p_N^s)^T\left(\phi_N^{ss}\right)p_N^s$$

$$J_j^s = \frac{\partial J_j}{\partial s_j}, J_j^q = \frac{\partial J_j}{\partial q_j}$$

$$H_j^{ss} = \frac{\partial^2 J_j}{\partial s_j^2}, H_j^{qs} = \frac{\partial^2 J_j}{\partial q_j \partial s_j}, H_j^{qq} = \frac{\partial^2 J_j}{\partial q_j^2}$$

$$\phi_N^s = \frac{\partial \phi}{\partial s_N}, \phi_N^{ss} = \frac{\partial^2 \phi}{\partial s_N^2}$$

$$F_j = \frac{\partial h}{\partial s_j}, D_j = \frac{\partial h}{\partial q_j}, h_j = h(s_j, q_j, t_j), \quad j = 0, \ldots, N$$

$$F_0 = D_N = 0$$

$$G_j = \frac{\partial g}{\partial s_j}, E_j = \frac{\partial g}{\partial q_j}, g_j = g(s_j, q_j, t_j), \quad j = 1, \ldots, N-1$$

### 2.4 Karush-Kuhn-Tucker (KKT)System

The adjoined cost function can be defined using Lagrange multipliers ($\lambda_j$ and $\mu_j$) to derive the KKT-condition as follows:

$$L(p,\lambda,\mu) = \sum_{j=0}^{N} J_j\left(p_j^s, p_j^q\right) + \sum_{j=0}^{j=N} \lambda_j^T (h_j + F_j p_j^s + D_j p_j^q - p_{j+1}^s) + \sum_{j=1}^{j=N-1} \mu_j^T (g_j + G_j p_j^s + E_j p_j^q) \quad (16)$$

The detailed derivation of the KKT-condition can be found in Ref. 7, and the related KKT system can be summarized as for $j=2,\cdots,N-1$:

$$\begin{bmatrix} H_j^{ss} & (H_j^{qs})^T & (F_j)^T & (G_j)^T \\ H_j^{qs} & H_j^{qq} & (D_j)^T & (E_j)^T \\ F_j & D_j & & \\ G_j & E_j & & \end{bmatrix} \begin{bmatrix} p_j^s \\ p_j^q \\ \lambda_j \\ \mu_j \end{bmatrix} + \begin{bmatrix} (J_j^s)^T - \lambda_{j-1} \\ (J_j^q)^T \\ h_j - p_{j+1}^s \\ g_j \end{bmatrix} = 0 \quad (17)$$

In Eq. (17), only active inequality constraints should be included with the positivity condition of $\mu_j \geq 0$. Therefore, the resultant KKT system is written as a linear system with a banded structure. Ref. 7 introduces the Schur complement method and the reduced Hessian method as a means to obtain an efficient solution for the above KKT system. Since the local cost function and continuity conditions include time integration terms, the gradient vectors of those functions requires repeated time integrations of motion equations, which are generally the most time-consuming elements in the SQP-based DMS method for highly complex nonlinear systems.

## Dynamic Models and Numerical Method

### 3.1 Rotorcraft Model

This paper presents the usage of a rotor dynamic model, proposed by R. T. N. Chen[20], where hub fixed flap states are used to derive the closed form expression for aerodynamic forces and moments based on quasi-linear aerodynamic theory. If the tip-path-plane dynamics are assumed to approximate first-harmonic components, then the higher-harmonic components can be removed in the rotor dynamics and aerodynamic forces and moments. In this study, the aerodynamic forces and moments, generated by rotors, are calculated using the main and tail rotor trim solution. This approach allows the use of relatively large time intervals for the time integration of the motion equations. However, any compromise in the model selection is still a huge drawback and high-fidelity models will become more acceptable with the advancement of the computer technology and with the improvements in the solution methodology. If high-fidelity models are required, the level 2 modeling formulated in Ref. 18 can be applied. The resultant computational burden can then be estimated based on the results from Ref. 10. For LQR problems, a linear time-invariant model is derived using a finite difference formula around trim flight conditions.

### 3.2 Numerical Methods for the SQP-based DMS

Gradients and Hessian matrices related to the continuity condition and to cost function can be estimated by using the SDME[25](State-Dependent matrix Exponential) technique Where numerical estimation of these matrices is required, a central differencing method is used with the following formula[23] for the cost function gradient vector:

$$\frac{\partial J}{\partial y_j} \approx \frac{J(y_j + \mathbf{e}_j \Delta y_j) - J(y_j - \mathbf{e}_j \Delta y_j)}{2\Delta y_j} \quad (18)$$

Here, $y_j$, $\mathbf{e}_j$, and $\Delta y_j$ denote the $j^{th}$ design variable, a unit vector whose components are zero except j-th row equal to 1 and a small increment, respectively. The Hessian matrix can also be updated using the BFGS method. The BFGS method iteratively update the Hessian matrix, $H$, for the following general NLP problem with the design variable $x$:

$$\begin{aligned} &\min J(x) \\ &\text{s.t. } h(x) = 0 \end{aligned} \quad (19)$$

The corresponding BFGS formula[23] has the following expression at the k[th] iteration stage:

$$H^{k+1} = H^k - \frac{H^k pp^T H^k}{p^T H^k p} + \frac{\eta\eta^T}{p^T \eta} \quad (20)$$

$$\begin{aligned} &p = x^{k+1} - x^k \\ &\widetilde{J} = J(x^k) + \sum \lambda^T h(x^k) \\ &y = \nabla_x \widetilde{J}^{k+1} - \nabla_x \widetilde{J}^k \\ &\eta = \theta y + (1-\theta) H^k p \\ &\theta = \begin{cases} 1.0 & \text{if } p^T y \ge 0.2 p^T Bp \\ \dfrac{0.8 p^T Bp}{p^T Bp - p^T y} & \text{if } p^T y < 0.2 p^T Bp \end{cases} \end{aligned}$$

The BFGS method is the most popular technique for the iterative Hessian update in the SQP method because it is fast and widely applicable despite its simplicity.

The line search procedure based on the Powell' s method[24] is applied in this study, using the L1-penality function (or the L1-merit function) $P_P$, which can be used to represent a system having equality constraints $h_i(y)$, $i = 1, \ldots, m$ and active inequality constraints $g_j(y), j = 1, \ldots, m$, as follows:

$$P_P(x, \sigma, \tau) = J(x) + \sum_{i=1}^{m} \sigma_i |h_i(x)| + \sum_{j=1}^{l} \tau_j |\max(0, g_j(x))| \quad (21)$$

Penalty parameters are iteratively updated using initial guesses $\sigma_i^{(0)} = |\lambda_i^{(0)}|, \tau_j^{(0)} = \mu_j^{(0)}$, where $\lambda_i^{(0)}$ and $\mu_j^{(0)}$ are Lagrange multipliers corresponding to the first SQP iteration. These parameters are recursively updated using Lagrange multipliers $\lambda_i^{(k)}$ and $\mu_j^{(k)}, k = 1, 2, \cdots, k_{\max}$, for the k[th] SQP iteration as follows.

$$\begin{aligned} \sigma_i^{(k)} &= \max\left( |\lambda_i^{(k)}|, \frac{1}{2}(\sigma_i^{(k)} + |\lambda_i^{(k)}|) \right) \\ \tau_j^{(k)} &= \max\left( |\mu_j^{(k)}|, \frac{1}{2}(\tau_{k-1,j} + |\mu_j^{(k)}|) \right) \end{aligned} \quad (22)$$

Next, the one-dimensional line search and the update of design variables are performed using the following formulae:

$$x^{k+1} = x^k + \alpha_k d_k \quad (23)$$

$$\alpha_k = \arg\{\min T(\alpha)\} \quad (24)$$

$$T(\alpha) = P_P(x^k + \alpha d_k, \sigma_k, \tau_k) \quad (25)$$

One-dimensional optimization, shown in Eq. (23), is performed using the following steps of the Powell' s algorithm:

[step 1] set $\alpha_k^{(0)} := 1$ with $x_k$ :=current states and controls and $d_k$: search direction.

[step 2] check the Wolf condition for each iteration index j, with initial setting j:=0

$$T(x_k + \alpha_k^{(j)} d_k) \le T(x_k) + \rho \alpha_k^{(j)} \frac{\partial T(0)}{\partial x} d_k, \quad \rho \in \left(0, \frac{1}{2}\right) \quad (26)$$

If true, then set $x_{k+1} = x_k + \alpha_k d_k^{(j)}$ and go to [step 3]

Otherwise, set $\alpha_k^{(j+1)} = \kappa_\alpha \alpha_k^{(j)}$ and return to the beginning of [step 2]

(In this paper we use $\kappa_\alpha = 0.7$ for a linear system, $\kappa_\alpha = 0.5$ for a nonlinear system)

[step 4] terminate the one-dimensional search with the step length of $\|\alpha_k^{(j)} d_k\|$.

The above steps generally work well in the DMS method with a linear system but sometimes the initial step size can becomes too large to reach a converged solution when applied to a DMS with nonlinear dynamics. In this case, the initial step size is limited in order to guarantee numerical convergence by using this formula:

$$\alpha_k^{(0)} = \min(1.0, \kappa_1 \|x_{k-1}\| / \|d_{k-1}\|), \text{ and } \kappa_1 = 0.8 \quad (27)$$

In some cases, the step size controlled by step 2 and by above equation can be too small to continue efficient SQP iteration. In this paper two different strategies are used to cope with such cases. If the condition $\|\Delta x_{k-1}\| \le \kappa_2 \|x_{k-1}\|$ with $\Delta x_{k-1} = \alpha_{k-1} d_{k-1}$ is satisfied for a given constant $\kappa_1 = 0.2$, an the initial step size is increased by using $\alpha_k^{(0)} = \min(2\alpha_{k-1}, 1)$, in order to speed up subsequent iterations of the line search. Also, where $\alpha_k \le \alpha_{\min}$ for an allowed minimum value of $\alpha_{\min}$, the line search routine is completed and the next SQP iteration begins. In this paper, the parameter $\alpha_{\min}$ is specified as $\alpha_{\min} = 1.0 \times 10^{-3}$.

There exist various other strategies to improve the numerical efficiency of the one-dimensional search, such as higher-order correction methods to prevent the Maratos effect, the watchdog technique to cope with the cycling effect, etc. However, the appropriate selection from such methods depends on the problem at hand and, in most cases, is heuristically determined resulting in increased computational burden. For this reason, the convergence characteristics of the one-dimensional search algorithm in question were investigated during the code development stage.

## Applications

### 4.1 Maneuver Trajectory

The numerical methods outlined in the previous sections are applied to an optimal control formulation for slalom maneuver of the BO-105 helicopter and the modeling details are covered in Ref. 8. The maneuver trajectory for this study is prescribed using trigonometric functions, as in Ref. 21 and 22. A trajectory can be expressed as the sum of states at maneuver entry and its variation during the maneuver.

$$x(t) = x(t_{entry}) + \Delta x(t) \tag{28}$$

Or $$x(t) = x(t_{entry}) + \int_{t_{entry}}^{t} \Delta\dot{x}(\tau)d\tau \tag{29}$$

The lateral-position change during a slalom maneuver is initially described with the following formula:

$$\Delta x(\bar{t}) = \frac{(\Delta x)_{\max}}{46.8}\begin{bmatrix} 32 + \sin(2\pi\bar{t}) - 20\sin(4\pi\bar{t}) \\ +2\sin(8\pi\bar{t}) \end{bmatrix} \tag{30}$$

Where $\bar{t} = (t - t_{entry})/(t_{finish} - t_{entry}) \quad 0 \le \bar{t} \le 1$

The times $t_{entry}$ and $t_{finish}$ designate maneuver entry and finish times, respectively, and $(\Delta x)_{\max}$ is the maximum amplitude of the Y-position, which determines the maneuver aggressiveness for a given duration. Since a vehicle' s maneuver is carried out in three-dimensional space, its trajectory must be defined in all three axes. The maneuver accuracy in other axes is commonly specified in terms of bounded deviations from stabilized trim reference parameters such as flight speed, altitude, sideslip, heading, and positions, etc. These trajectory deviations are considered when we define a cost function for optimal control problems. The following form of the quadratic cost function with no terminal cost is implemented in this study[14]:

$$f_{CO}(\bar{x}_R(t), u(t), t) = 0.5(\bar{x}_R - \bar{x}_{target})^T Q_C(\bar{x}_R - \bar{x}_{target}) + 0.5(u - u_{trim})^T R(u - u_{trim}) \tag{31}$$

Where

$\bar{x}_R$ : reduced rigid body states

$\bar{x}_{target}$ : target states

$\bar{x}_R(t) = [u, v, w, p, q, \dot{\psi}, \phi, \theta, \psi, x_E, y_N, h]^T$

$R = diag(r_{\delta_0},\ r_{\delta_{1C}},\ r_{\delta_{1S}},\ r_{\delta_{TR}})$

$Q_C = diag(\ q_u,\ q_v,\ q_w,\ q_p,\ q_q,\ q_{\dot{\psi}},\ q_\phi,\ q_\theta,\ q_\psi,\ q_{xE},\ q_{yN},\ q_H)$

For simplicity of analysis, no system constraints are imposed. The target states $\bar{x}_{target}(t)$ are set to be the trim states $(x_R)_{trim}$, except in cases where they require a description of their time variation for a specific maneuver.

The control weighting matrix $R$ and the semi-positive definite weighting matrix $Q$ are selected with those diagonal components as:

$$\begin{aligned} R &= diag(50, 50, 50, 50) \\ Q &= diag(1,1,1, 0.4, 0.2, 2, 0.1, 0.1, 50, 0, 5, 100) \end{aligned} \tag{32}$$

The initial conditions for the state variables are specified by the results of the trim analysis because the maneuver considered for this study is initiated from a steady trim condition. Terminal conditions are defined in terms of the target states at the end of a maneuver:

$$\begin{aligned} x_i(t_0) &= x_{i,\,trim}, \quad i = 1, \dots, n \\ \bar{x}(t_f) &= \bar{x}_{target}(t_f) \end{aligned} \tag{33}$$

### 4.2 Rotorcraft Trajectory Tracking Problem

The SDME technique[25] was applied to the slalom maneuver problems of the Bo-105 helicopter, where the slalom trajectory was defined by maneuver parameters of $(\Delta y)_{\max} = 5.0$ m, $t_{entry} = 1.0$ seconds, and $t_{finish} = 9.0$ seconds. In this setup, the slalom maneuver begins in a steady level flight at a forward speed of 60 knots. The optimal control problem was formulated with $t_0 = 0.0$ and $t_f = 10.0$ seconds. The 4-stage Runge-Kutta time integrator was used for the forward simulations. If not otherwise specified, 500 shooting nodes for linear systems and 800 shooting nodes for nonlinear systems were evenly distributed over $t \in [0,10]$ and the time interval between two adjacent shooting nodes was divided into 32 integration stages. In case of the LQR problem, KKT system matrices corresponding to the cost function and continuity conditions could be expressed exactly, with the SDME under the DMS framework.

For the comparative study among the time integration method and the way of building the KKT system, the following classifications are defined for (a) HE: Function value of equality constraint function, (b) GHE: Gradient of equality constraint function, and (c) GOB: Gradient of the objective function.

(1) Full Time Integration method
- All of HE, GHE, and GOB are computed using time integration and finite difference formula

(2) Only GOB Time Integration
- HE and GHE are computed using SDME
- GOB is computed using time integration and finite difference formula

(3) Only GOB Matrix exponential
- GOB is computed using SDME
- HE and GHE are computed using time integration and finite difference formula

(4) Only HE Matrix exponential
- HE is computed using SDME
- GHE and GOB are computed using time integration and finite difference formula

(5) Only HE Time Integration
- GHE and GOB are computed using SDME
- HE is computed using time integration and finite difference formula

Fig. 1 shows the computed slalom trajectory with different methods of building the KKT system. Fig. 2 ~ Fig. 6 present the variations of control inputs and state variables. Regardless of methods, the present DMS methods track well the prescribed trajectory. However, the details in controls and state variables show the scattering depending on the method used. Fig. 7 ~ Fig. 10 show the comparison of convergence history. All methods presents nearly the same convergence character. About 25 iterations are enough to get the fully converged solution. Fig. 11 shows the comparison in computing time. The SDME method shows the most efficient computation and the results using Runge−Kutta time integration with different numbers of internal stages shows the time integration for system dynamics consumes most of required computing time.

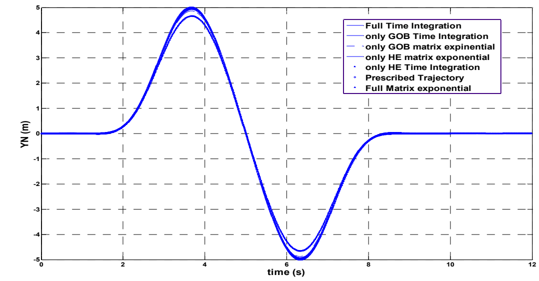
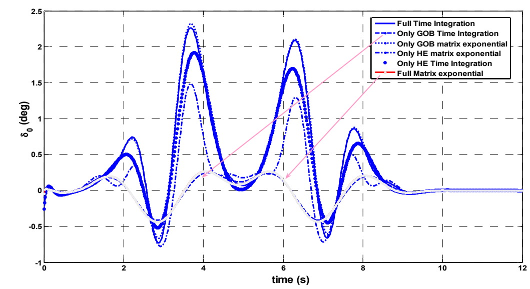
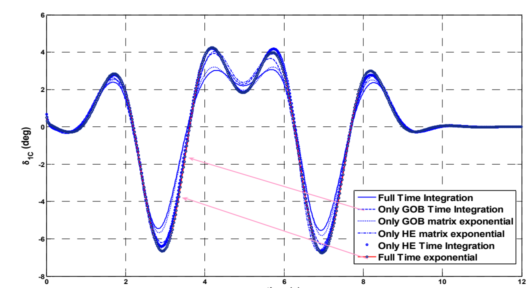


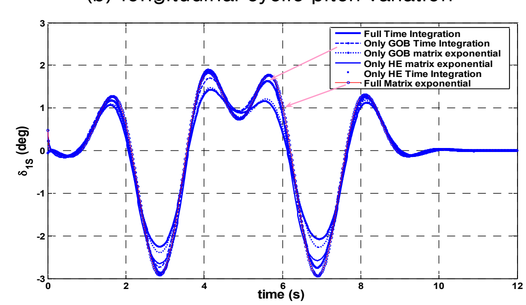

Fig. 1. The lateral position variation during slalom maneuver



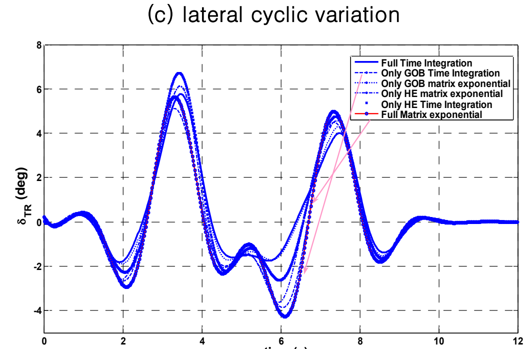

(a) collective pitch variation




(b) longitudinal cyclic pitch variation




(c) lateral cyclic variation




(d) tail rotor collective pitch variation

Fig. 2. Computed controls for slalom maneuver

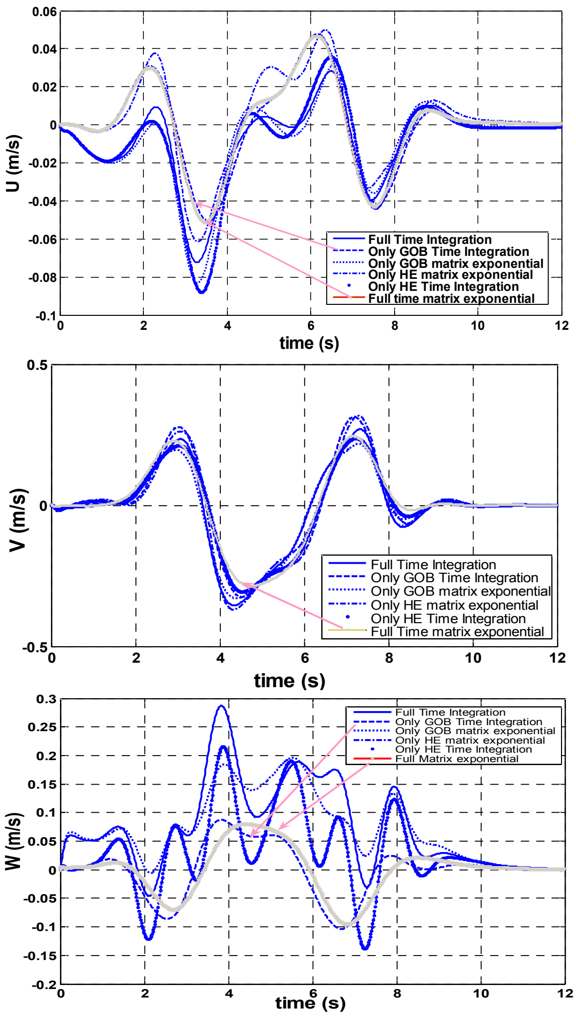



Fig. 3. velocity variation during slalom maneuver

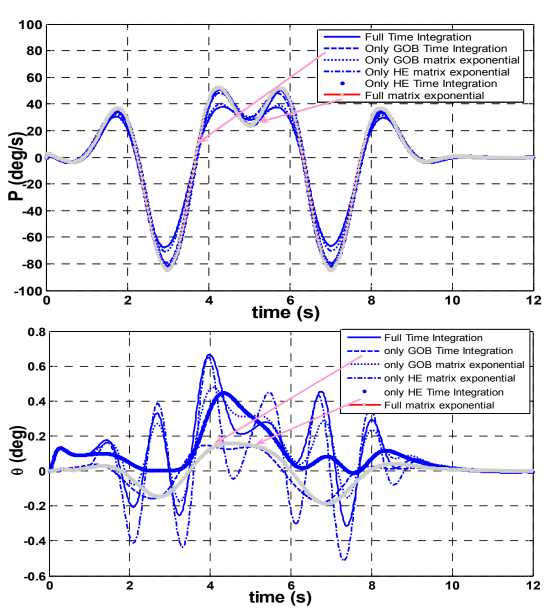



Fig. 4. pitch rate and pitch attitude variation

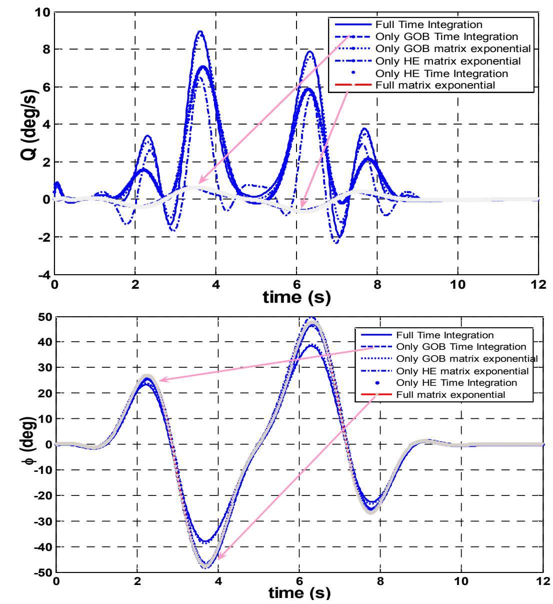



Fig. 5. roll rate and bank angle variation

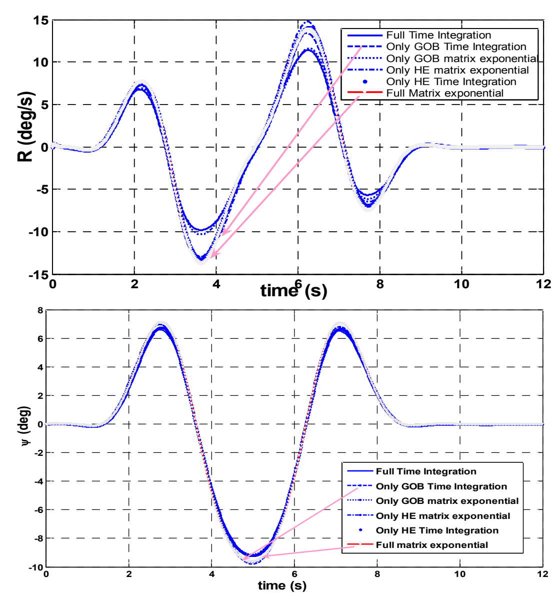



Fig. 6. yaw rate and heading angle variation

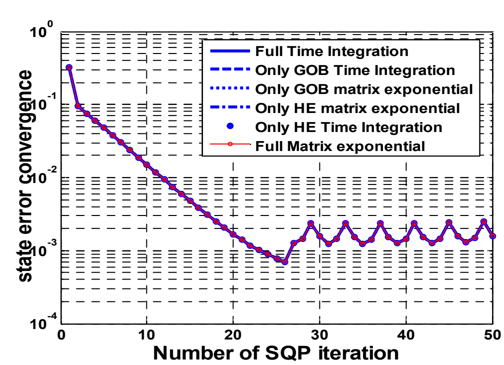



Fig. 7. The error convergence of state variable

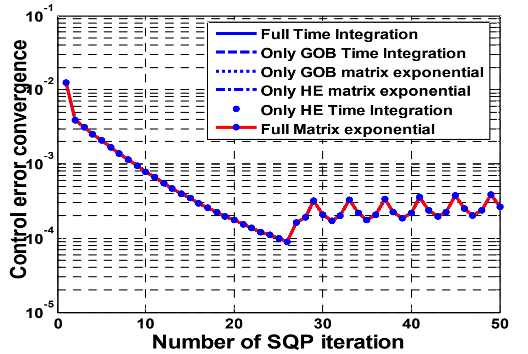



Fig. 8. correction in control input with SQP iteration

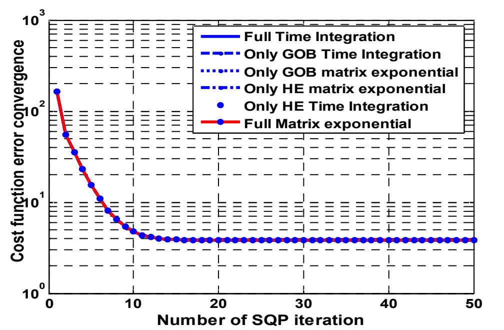



Fig. 9. correction in cost function with SQP iteration

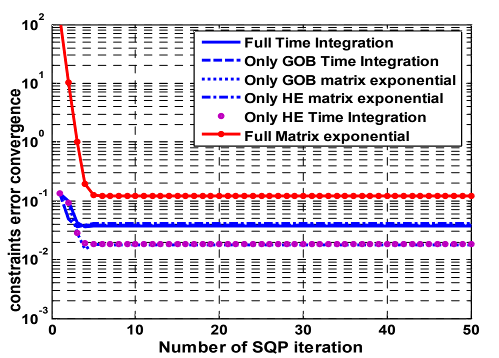



Fig. 10. correction in constraints with SQP iteration

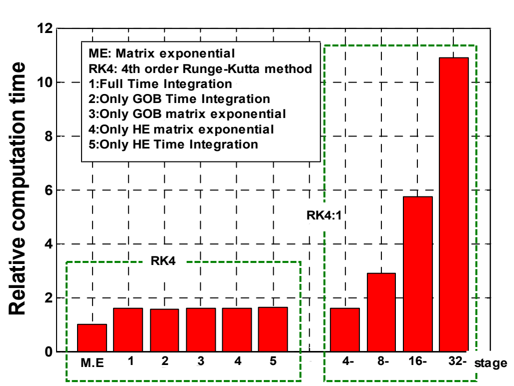



Fig. 11. comparison of relative computation time between matrix exponential and mixed time integration.

## Conclusion

Applications of the SDME technique to linear quadratic regulator problems showed that the matrix exponential approach yields better numerical efficiency with the direct multiple-shooting method than the conventional estimations of the KKT system matrices. More importantly, this approach simultaneously calculates system states at the end of each shooting node as well as gradients and Hessian matrices for the cost function and continuity constraints. Here, repeated computations for the state-dependent factorization and integrals weighted by matrix exponential are the major contributors to long computing times for nonlinear optimal control analyses.

Here, repeated computations for the state-dependent However, the related computational burden is generally much less than that associated with the finite difference methods to estimate gradients of the cost function and continuity constraints. The proposed method calculates the converged solutions for the nonlinear trajectory tracking problem, even though the solutions using conventional approaches are mostly divergent. In addition, the state-dependent matrix exponential approach can be used to integrate nonlinear motion equations. The update frequency of state-dependent coefficient matrices had a minor effect on the accuracy of trajectory tracking over the present time horizon. Therefore, the compared results could be utilized to design an efficient MPC framework using the present method.

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