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EXPANSIONS OF REAL NUMBERS IN NON-INTEGER BASES

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Abstract. The works of Erdös et al. about expansions of 1 with respect to a non-integer base q, referred to as q-expansions, are investigated to determine how far they continue to hold when the number 1 is replaced by a positive number x. It is found that most results about q-expansions for real numbers greater than or equal to 1 are in somewhat opposite direction to those for real numbers less than or equal to 1. The situation when a real number has a unique q-expansion, and when it has exactly two q-expansions are studied. The smallest base number q yielding a unique q-expansion is determined and a particular sequence is shown, in certain sense, to be the smallest sequence whose corresponding base number q yields exactly two q-expansions.

1. Introduction

Let $q \in (1, 2]$. By a *q-expansion* of 1, we mean a sequence $(e_i)_{i\geq 1}$ of integers in $\{0, 1\}$ satisfying the equality $1 = \sum_{i=1}^{\infty} e_i/q^i$. Such an expansion is not unique in general. There exist two particular expansions, constructed via the so-called greedy and lazy algorithms. In the greedy algorithm, we choose the biggest possible value for e_i, while in the lazy algorithm, we choose the smallest possible value for e_i.

In 1990, Erdös, Joo, and Komornik [4] began the work about characterizing the unique q-expansion of 1 for non-integer base q. In 1991, Erdös, Horváth, and Joo [3] showed that for almost all $q \in (1, 2]$, there are uncountably many different q-expansions, and surprisingly, there exist as well uncountably many exceptional $q \in (0, 1)$ for which there is only one q-expansion. In 1998, Komornik and Loreti [5] determined the smallest base $q \in (1, 2]$ for which the q-expansion of 1 is unique. In 1999, Komornik and Loreti [6] gave a sufficient condition for which the number 1 has exactly two different q-expansions as well as using this information to construct the smallest base q for which the number 1 has exactly two different q-expansions. In 2002, Dajani and Kraaikamp [2] studied the ergodic properties of non-greedy series expansions to non-integer

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bases $\beta > 1$. It was shown that the so-called lazy expansion is isomorphic to the greedy expansion. Furthermore, a class of expansions to bases $\beta > 1$, $\beta \notin \mathbb{Z}$, in between the lazy and the greedy expansions are introduced and studied. These expansions are of the form $Tx = \beta x + \alpha \mod 1$. A more recent article with contents related to this work is [7].

In this paper, our overall objective is to investigate how far the results mentioned above, excluding the cardinality and the ergodicity ones, continue to hold for the positive number x replacing the number 1. In the next section, general results about greedy and lazy q-expansions are derived. It is found that most results about q-expansions for real numbers greater than or equal to 1 are in somewhat opposite direction to those for real numbers less than or equal to 1, which illustrate the remarkable standing of the number 1 in this regard. Through the concept of U-sequences, we then investigate the situation when a real number has unique q-expansion and determine the smallest such base. Finally, the situation with exactly two q-expansions is studied and a particular sequence, first treated in [6], which becomes in certain sense the smallest sequence for certain positive number with corresponding base q yielding exactly two q-expansions is considered.

Let $q \in (1, 2]$. By an expansion with respect to q, or *q-expansion*, of a positive real number x we mean a sequence $(e_i)_{i\geq 1} \subseteq \{0, 1\}$ satisfying

$$\sum_{i=1}^{\infty} \frac{e_i}{q^i} = x.$$

It is easily checked that x has an expansion if and only if $0 \leq x \leq 1/(q-1)$.

The *lexicographical order* $\prec$ is defined as follows: given two real sequences (a_i) and (b_i), we write $(a_i) \prec (b_i)$ or $a_1a_2 \cdots \prec b_1b_2 \cdots$ if there exists a positive integer n such that $a_i = b_i$ for all $i < n$, but $a_n < b_n$. It is easily checked that this is a complete ordering.

Using this lexicographical order, we now define three special sequences, termed D-, U- and T-sequences. The notions of these three sequences were first considered by Komornik and Loreti [6].

A sequence $(a_i)_{i\geq 1} \subseteq \{0, 1\}$ is called a D-*sequence* if

$$(a_{n+i}) \prec (a_i) \text{ whenever } a_n = 0. \tag{1.1}$$

A sequence $(a_i)_{i\geq 1} \subseteq \{0, 1\}$ is called a U-*sequence* if

$$(a_{n+i}) \prec (a_i) \text{ whenever } a_n = 0 \text{ (i.e., being a D-sequence)}$$

and

$$(\overline{a_{n+i}}) := (1 - a_{n+i}) \prec (a_i) \text{ whenever } a_n = 1,$$

where for brevity we write $\overline{\varepsilon_i}$ for $1 - \varepsilon_i$ and $\overline{s}$ for $\overline{\varepsilon_1\varepsilon_2\cdots}$ if $s = (\varepsilon_i) \subseteq \{0, 1\}$. If (a_i) begins with $N\,(\geq 2)$ consecutive 1 digits and if there are neither N consecutive 1 digits, nor N consecutive 0 digits later, then it is easily checked that (a_i) is a U-sequence.

A sequence $(e_i)_{i\geq 1} \subseteq \{0,1\}$ is called a T-*sequence* if the following three conditions hold:

(1) $(e_{n+i}) \prec (e_i)$ whenever $e_n = 0$ (i.e., (e_i) is D-sequence);
(2) there exists a positive integer m such that $e_m = 1$, and
(3) there exists a sequence $(\varepsilon_i)_{i\geq 1} \subseteq \{0,1\}$ defined by $e_{i+m}+\varepsilon_i \in \{0,1\}$ ($i \geq 1$), such that if the sequence $(\delta_i)_{i\geq 1} \subseteq \{0,1\}$ is defined by

$$\delta_i = \begin{cases} e_i & \text{if } i < m \\ 0 & \text{if } i = m \\ e_i + \varepsilon_{i-m} & \text{if } i > m, \end{cases} \tag{1.2}$$

then the following three requirements hold:

$$\overline{\delta_{n+1}\delta_{n+2}\cdots} \prec e_1e_2\cdots \quad \text{whenever } \delta_n = 1, \tag{1.3}$$

$$\overline{e_{n+1}e_{n+2}\cdots} \prec e_1e_2\cdots \quad \text{whenever } e_n = 1 \text{ and } n > m, \tag{1.4}$$

$$\delta_{n+1}\delta_{n+2}\cdots \prec e_1e_2\cdots \quad \text{whenever } \delta_n = 0 \text{ and } n > m. \tag{1.5}$$

Komornik and Loreti [6] showed that if (e_i) is a T-sequence with $e_i = \varepsilon_i$, then there exists a $q \in (1,2]$ such that 1 has exactly two expansions.

A real number $q \in (1,2]$ is called a T-*base number* if there exists a positive real number x with exactly two different q-expansions.

As a general, preliminary result, we have:

Theorem 1.1. *Let* $(e_i) \subseteq \{0,1\}$*. Then the map*

$$q \mapsto \sum_{i\geq 1} e_i/q^i$$

is continuous and strictly decreasing from the interval $(1,2]$ *onto the interval* $\left[\sum_{i\geq 1} e_i/2^i,\ \sum_{i\geq 1} e_i\right)$.

Proof. Let $q \in (1,2]$ and $F(q) = \sum_{i\geq 1} e_i/q^i$. That this map is strictly decreasing is clear. If $q_1 < q_2$, then

$$|F(q_1) - F(q_2)| = \sum_{i\geq 1} \left| \frac{e_i(q_1^i - q_2^i)}{q_1^i q_2^i} \right| \leq \frac{|q_1 - q_2|}{q_1 q_2} \sum_{i\geq 1} \frac{i}{q_1^{i-1}},$$

showing that F is continuous. That this map is onto follows from the intermediate value theorem for continuous functions. □

2. Greedy expansions

Let $q \in (1,2]$ and $x \in [0, 1/(q-1)]$. We define the *greedy* q*-expansion* $(a_i) \subseteq \{0,1\}$ of x as follows: if for some positive integer n, the numbers a_i are defined for all $i < n$, then set $a_n = 1$ whenever

$$\sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n} \leq x,$$

and $a_n = 0$ otherwise, where the summation is taken as 0 if $n = 1$.

Our next result reveals some intrinsic relations between the greedy q-expansion of a number in $[1, 1/(q-1)]$ and that of any non-negative real number in $[0, 1/(q-1)]$.

Theorem 2.1. *Let $q \in (1, 2]$, $y \geq 1$ and let (e_i) be the greedy q-expansion of y.*

(a) *The greedy q-expansion, (a_i), of any $x \in [0, 1/(q-1)]$ satisfies*

$$a_{n+1}a_{n+2}\cdots \prec e_1e_2\cdots \quad \text{whenever } a_n = 0. \tag{2.1}$$

(b) *If the sequence (e_i) is finite with a last nonzero digit e_k, then no greedy q-expansion is eventually periodic with the period $e_1e_2\cdots e_{k-1}(e_k-1)$.*

Proof. (a) Assume that $a_n = 0$. If $(a_{n+i}) \succ (e_i)$, then there exists an integer k such that $a_{n+i} = e_i$ for $i = 1, 2, \ldots, k-1$, but $a_{n+k} > e_k$. Thus $e_k = 0$ and $a_{n+k} = 1$ and so, by the definition of greedy q-expansion of y,

$$\sum_{i=1}^{k-1} \frac{e_i}{q^i} + \frac{1}{q^k} > y.$$

Thus

$$\sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} \geq \sum_{i=1}^{k-1} \frac{a_{n+i}}{q^i} + \frac{1}{q^k} = \sum_{i=1}^{k-1} \frac{e_i}{q^i} + \frac{1}{q^k} > y,$$

and so

$$x = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{1}{q^n}\left(\frac{a_{n+1}}{q} + \frac{a_{n+2}}{q^2} + \cdots\right) > \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{y}{q^n} \geq \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n},$$

contradicting the definition of the greedy q-expansion of x (because $a_n = 0$).

If $(a_{n+i}) = (e_i)$, then $a_{n+i} = e_i$ for all $i \geq 1$. Thus

$$\begin{aligned} x &= \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{1}{q^n}\left(\frac{a_{n+1}}{q} + \frac{a_{n+2}}{q^2} + \cdots\right) \\ &= \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{1}{q^n}\left(\frac{e_1}{q} + \frac{e_2}{q^2} + \cdots\right) \\ &= \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{a_n}{q^n} + \frac{y}{q^n} \\ &\geq \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n}, \end{aligned}$$

again contradicting the definition of the greedy q-expansion of x (because $a_n = 0$).

(b) Assume on the contrary that the greedy q-expansion (a_i) of some $x \in [0, 1/(q-1)]$ is eventually periodic with period $e_1 e_2 \cdots e_{k-1}(e_k - 1)$. Since

$$y - \frac{1}{q^k} = \frac{e_1}{q^1} + \cdots + \frac{e_k - 1}{q^k},$$

we have

$$\begin{aligned} x &= \left(\frac{a_1}{q} + \cdots + \frac{a_r}{q^r}\right) + \frac{1}{q^r}\left(\frac{e_1}{q} + \cdots + \frac{e_k - 1}{q^k}\right) \\ &\quad + \left(\frac{e_1}{q} + \cdots + \frac{e_k - 1}{q^k}\right)\left(\frac{1}{q^{r+k}} + \frac{1}{q^{r+2k}} + \cdots\right) \\ &= \left(\frac{a_1}{q} + \cdots + \frac{a_r}{q^r}\right) + \frac{1}{q^r}\left(\frac{e_1}{q} + \cdots + \frac{e_k - 1}{q^k}\right) + \left(y - \frac{1}{q^k}\right)\left(\frac{\frac{1}{q^{r+k}}}{1 - \frac{1}{q^k}}\right) \\ &\geq \left(\frac{a_1}{q^1} + \cdots + \frac{a_r}{q^r}\right) + \frac{1}{q^r}\left(\frac{e_1}{q^1} + \cdots + \frac{e_k - 1}{q^k}\right) + \left(1 - \frac{1}{q^k}\right)\left(\frac{\frac{1}{q^{r+k}}}{1 - \frac{1}{q^k}}\right) \\ &= \left(\frac{a_1}{q^1} + \cdots + \frac{a_r}{q^r}\right) + \frac{1}{q^r}\left(\frac{e_1}{q^1} + \cdots + \frac{e_k}{q^k}\right) = \sum_{i=1}^{r+k-1} \frac{a_i}{q^i} + \frac{1}{q^{r+k}}, \end{aligned}$$

contradicting the definition of the q-greedy expansion of x (because $a_{r+k} = 0$). □

Remarks. 1) The case where $y = 1$ is Lemmas 2(a) in [2] and Lemma 1.4(a) in [5].

2) The converse of Theorem 2.1(a) is not true, i.e., there exists an $x \in [0, 1/(q-1)]$, whose q-expansion, (a_i), satisfies the condition (2.1), but this expansion is not the greedy q-expansion of x, as seen in the following example.

Example. Take $q = \frac{9}{5}$ and $x = y = \frac{27199387096045}{22876792454961} = \frac{27199387096045}{9^{14}} \geq 1$. Here

$$(e_i) = (1, 1, 1, 1, 1, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, \ldots),$$

the expression holding up to the first eighteen digits, is the greedy q-expansion of $x = y$ and

$$(a_i) = (1, 1, 1, 1, 1, 0, 0, 0, 0, 1, 1, 0, 1, 1)$$

is a finite q-expansion of $x = y$ which satisfies the condition (2.1), but (a_i) is not a greedy q-expansion.

Next we derive more characterizations of greedy q-expansions.

Theorem 2.2. *Let $q \in (1, 2]$. A sequence (a_i) is the greedy q-expansion of x if and only if $\sum_{i=1}^{\infty} a_{k+i}/q^i < 1$ whenever $a_k = 0$.*

Proof. Let (a_i) be the greedy q-expansion of x and assume $a_k = 0$. By definition,

$$\sum_{i=1}^{k-1} \frac{a_i}{q^i} + \frac{1}{q^k} > x,$$

and so

$$\frac{1}{q^k} > \sum_{i=k}^{\infty} \frac{a_i}{q^i} = \sum_{i=k+1}^{\infty} \frac{a_i}{q^i} = \sum_{i=1}^{\infty} \frac{a_{k+i}}{q^{k+i}}.$$

The required inequality follows after multiplying by q^k.

Assume $\sum_{i=1}^{\infty} a_{k+i}/q^i < 1$ whenever $a_k = 0$. If $a_n = 1$, then

$$x = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n} + \sum_{i>n} \frac{a_i}{q^i} \geq \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n}.$$

If $a_n = 0$, then $\sum_{i=1}^{\infty} a_{n+i}/q^i < 1$, and so $\sum_{i=1}^{\infty} a_{n+i}/q^{n+i} < 1/q^n$. Thus

$$x = \sum_{i=1}^{\infty} \frac{a_i}{q^i} = \sum_{i \neq n} \frac{a_i}{q^i} = \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \sum_{i=1}^{\infty} \frac{a_{n+i}}{q^{n+i}} < \sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n},$$

i.e., (a_i) is the greedy q-expansion of x. □

Remark. Theorem 2.2 is Lemma 1(a) in [2], but the proof here is different.

Theorem 2.3. *Let* $q \in (1, 2]$.

(a) *Let* (e_i) *be an infinite q-expansion of* $y \in [0, 1]$ *and let* (a_i) *be a q-expansion of* $x \in [0, 1/(q-1)]$. *If the condition* (2.1) *holds, then* (a_i) *is the greedy q-expansion of* x.

(b) *Let* (e_i) *be a q-expansion of* $y \in [0, 1]$ *and let* (a_i) *be a finite q-expansion of* $x \in [0, 1/(q-1)]$. *If the condition* (2.1) *holds, then* (a_i) *is the greedy q-expansion of* x.

(c) *Let* (e_i) *be a finite q-expansion of* $y \in [0, 1]$ *and denote by* e_k *its last nonzero element. Let* (a_i) *be a q-expansion of* $x \in [0, 1/(q-1)]$. *Assume* (2.1) *holds.*

(c.1) *If* $y < 1$, *then* (a_i) *is the greedy q-expansion of* x.

(c.2) *If* $y = 1$ *and assume that* (a_i) *is not eventually periodic with period* $e_1 \cdots e_{k-1}(e_k - 1)$, *then* (a_i) *is the greedy q-expansion of* x.

Proof. There is nothing to prove if $a_n = 1$, while for those n with $a_n = 0$, the results follow from Theorem 2.2 if we can show that

$$\sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} < 1. \tag{2.2}$$

From (2.1), there is a sequence of integers $n = k_0 < k_1 < \cdots$ satisfying the conditions: with $j \in \mathbb{N}$,

$$a_{k_{j-1}+i} = e_i \text{ for all } 1 \leq i < k_j - k_{j-1} \text{ and } a_{k_j} < e_{k_j - k_{j-1}}.$$

(a) If the sequence (e_i) is infinite, then

$$\frac{1}{q^n} \sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{a_{k_{j-1}+i}}{q^{k_{j-1}+i}} \right) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{e_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}} \right)$$

$$< \sum_{j=1}^{\infty} \left(\frac{y}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) \leq \frac{1}{q^{k_0}} \tag{2.3}$$

proving (2.2).

(b) If the sequence (a_i) is finite, assume that there exits a positive integer m satisfying $a_i = 0$ for all $i > k_m$. Now

$$\begin{aligned} \frac{1}{q^n} \sum_{i=1}^{\infty} \frac{a_{n+i}}{q^i} &= \sum_{j=1}^{m} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{a_{k_{j-1}+i}}{q^{k_{j-1}+i}} \right) = \sum_{j=1}^{m} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{e_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}} \right) \\ &\leq \sum_{j=1}^{m} \left(\frac{y}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) \leq \frac{1}{q^{k_0}} - \frac{1}{q^m} < \frac{1}{q^{k_0}}, \end{aligned}$$

proving (2.2).

(c) If the sequence (e_i) is finite, proceeding as in the proof of (a) leads to (2.3) with strict inequality being now non-strict. Observe that $e_{k_j - k_{j-1}} = 1$ so $k_j - k_{j-1} \leq k$. A closer inspection of the proof reveals that we obtain equality exactly when $y = 1$ and $k_j - k_{j-1} = k$ for every j, i.e., when the sequence (a_{n+i}) is periodic with period $e_1 \cdots e_{k-1}(e_k - 1)$. This contradicts the fact that (a_i) is not eventually periodic with period $e_1 \cdots e_{k-1}(e_k - 1)$. Hence, (a_i) is the greedy q-expansion of x. □

Remarks. 1) Theorem 2.3 (a), (b) is Lemma 3 in [2] and the proofs given here are the same. Lemma 1.5 (a) in [5] is a special case of Theorem 2.3 (c.2) above.

2) The converse of Theorem 2.3 (c.1) is not true, i.e., there exist y with finite q-expansion (e_i), and x with greedy q-expansion (a_i), such that (a_i) does not satisfy the condition (2.1) as seen in the following example.

Example. Take $q = 4/3$. We have

$$(a_i) = (1,0,0,0,0,0,1,0,0,0,0,0,0,0,1,0,0,0,0,0,1,0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,1)$$

is the greedy expansion of $x = \frac{68001788610072039914841}{75557863725914323419136}$. Taking $y = x < 1$, we get $(a_i) = (e_i)$. Note that (a_i) does not satisfy the condition (2.1).

Theorem 2.4. *Let $q, q' \in (1, 2]$, $x \in [0, 1/(q-1)] \cap [0, 1/(q'-1)]$. Let (e_i) and (e'_i) be the greedy q-expansion, respectively, greedy q'-expansion of x. If $q < q'$, then $(e_i) \prec (e'_i)$.*

Proof. Suppose that the conclusion is false. We have two possible cases.
Case 1: $(e_i) = (e'_i)$. Thus $x = \sum_{i=1}^{\infty} e'_i/(q')^i < \sum_{i=1}^{\infty} e_i/q^i = x$, a contradiction.
Case 2: $(e_i) \succ (e'_i)$. Thus there exists an integer n such that $e_i = e'_i$ for all $1 \leq i < n$ but $e_n > e'_n$. We must have $e_n = 1$ and $e'_n = 0$. By the definition of the greedy q-expansion,

$$\sum_{i=1}^{n-1} \frac{e'_i}{q'^i} + \frac{1}{q'^n} \leq \sum_{i=1}^{n-1} \frac{e_i}{q^i} + \frac{1}{q'^n} < \sum_{i=1}^{n-1} \frac{e_i}{q^i} + \frac{1}{q^n} \leq x,$$

contradicting the definition of greedy q'-expansion of x as $e'_n = 0$. □

3. Lazy expansions

Let $q \in (1,2],\ y \in [0, 1/(q-1)]$. The *lazy q-expansion* (b_i) of y is defined as follows: if for some positive integer n the numbers b_i are defined for all $i < n$, then set $b_n = 0$ whenever

$$\sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i>n} \frac{1}{q^i} \geq y,$$

and set $b_n = 1$ otherwise, where the summation is taken as 0 if $n = 1$.

Lazy q-expansions enjoy two simple properties which we now describe.

Property L1. A real number $y \in [0, 1/(q-1)]$ has (b_i) as its lazy q-expansion if and only if the sequence $(a_i) := (1-b_i)$ is the greedy q-expansion of $\frac{1}{q-1} - y$ (This "duality"property implies that every $y \in [0, 1/(q-1)]$ has a lazy q-expansion).

Proof. First observe that

(b_i) is a q-expansion of $y \Leftrightarrow \sum_{i=1}^{\infty} b_i/q^i = y \ \Leftrightarrow \sum_{i=1}^{\infty}(1-b_i)/q^i = \frac{1}{q-1} - y \ \Leftrightarrow$ $(1-b_i)$ is a q-expansion of $\frac{1}{q-1} - y$.

Assume that (b_i) is the lazy q-expansion of y. If $1 - b_n = 0$, then

$$y > \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i>n} \frac{1}{q^i},$$

and so

$$\frac{1}{q-1} - y < \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{b_i}{q^i} - \sum_{i>n} \frac{1}{q^i} = \sum_{i=1}^{n-1} \frac{1-b_i}{q^i} + \frac{1}{q^n}.$$

If $1 - b_n = 1$, then $y \leq \sum_{i=1}^{n-1} b_i/q^i + \sum_{i>n} 1/q^i$, and so

$$\frac{1}{q-1} - y \geq \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{b_i}{q^i} - \sum_{i>n} \frac{1}{q^i} = \sum_{i=1}^{n-1} \frac{1-b_i}{q^i} + \frac{1}{q^n}.$$

Thus $(1-b_i)$ is the greedy q-expansion of $\frac{1}{q-1} - y$.

Assume that $(1-b_i)$ is the greedy q-expansion of $\frac{1}{q-1} - y$. If $b_n = 0$, then $\frac{1}{q-1} - y \geq \sum_{i=1}^{n-1}(1-b_i)/q^i + 1/q^n$, and so

$$y \leq \frac{1}{q-1} - \sum_{i=1}^{n-1} \frac{1-b_i}{q^i} - \frac{1}{q^n} = \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i>n} \frac{1}{q^i}.$$

If $b_n = 1$, then $\frac{1}{q-1} - y < \sum_{i=1}^{n-1}(1-b_i)/q^i + 1/q^n$, and so $y > \sum_{i=1}^{n-1} b_i/q^i + \sum_{i>n} 1/q^i$. Thus (b_i) is the lazy q-expansion of y. □

Property L2. If (a_i) and (b_i) are the greedy and lazy q-expansions, respectively, of x, and if there exists another q-expansion (c_i) of x, then

$$(b_i) \preceq (c_i) \preceq (a_i)$$

(In other words, the greedy q-expansion is the greatest q-expansion and the lazy q-expansion is the smallest q-expansion of x lexicographically).

Proof. Let (a_i) and (b_i) be the greedy, respectively, lazy q-expansions of x and let (c_i) be another q-expansion of x.

To show that $(b_i) \preceq (c_i)$, assume $(b_i) \succ (c_i)$. Then there exists an integer n such that $b_i = c_i$ for all $1 \leq i < n$ but $b_n > c_n$. Thus $b_n = 1$ and $c_n = 0$. By the definition of lazy q-expansion, we have

$$\sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i \geq n+1} \frac{1}{q^i} < x = \sum_{i=1}^{\infty} \frac{c_i}{q^i} = \sum_{i=1}^{n-1} \frac{c_i}{q^i} + \sum_{i \geq n} \frac{c_i}{q^i}.$$

Thus

$$\sum_{i \geq n+1} \frac{1}{q^i} < \sum_{i \geq n} \frac{c_i}{q^i} = \sum_{i \geq n+1} \frac{c_i}{q^i},$$

contradicting the definition of the sequence $(c_i) \subseteq \{0, 1\}$.

To show that $(c_i) \preceq (a_i)$, assume $(c_i) \succ (a_i)$. Then there exists an integer n such that $c_i = a_i$ for all $1 \leq i < n$ but $c_n > a_n$. Thus $c_n = 1$ and $a_n = 0$. By the definition of greedy q-expansion, we have

$$\sum_{i=1}^{n-1} \frac{a_i}{q^i} + \frac{1}{q^n} > x = \sum_{i=1}^{\infty} \frac{c_i}{q^i} = \sum_{i=1}^{n-1} \frac{c_i}{q^i} + \sum_{i \geq n} \frac{c_i}{q^i},$$

which implies $0 > \sum_{i \geq n+1} \frac{c_i}{q^i}$, again contradicting the definition of the sequence $(c_i) \subseteq \{0, 1\}$. □

Remark. Properties L1 and L2 are well known and have appeared in several articles and with quite short proofs, e.g. [6] and [1], where in the latter paper simple and short dynamical proofs are given. We give the above proofs for two reasons; first, they are elementary and second, to make this exposition self-contained.

We next derive further characterizations of lazy q-expansions.

Theorem 3.1. *Let $q \in (1, 2]$, $x \in [0, 1/(q-1)]$. Then (b_i) is the lazy q-expansion of x if and only if $\sum_{i=1}^{\infty}(1 - b_{k+i})/q^i < 1$ whenever $b_k = 1$.*

Proof. Let (b_i) be the lazy q-expansion of x. Assuming $b_k = 1$, we get

$$\sum_{i=1}^{k} \frac{b_i}{q^i} + \sum_{i \geq k+1} \frac{1}{q^i} < x + \frac{1}{q^k} = \sum_{i=1}^{\infty} \frac{b_i}{q^i} + \frac{1}{q^k},$$

and so

$$\sum_{i=1}^{\infty} \frac{1 - b_{k+i}}{q^i} < 1.$$

Conversely, assume $\sum_{i=1}^{\infty}(1-b_{k+i})/q^i < 1$ whenever $b_k = 1$. If $b_n = 0$, then

$$x = \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i \geq n+1} \frac{b_i}{q^i} \leq \sum_{i=1}^{n-1} \frac{b_i}{q^i} + \sum_{i \geq n+1} \frac{1}{q^i}.$$

If $b_n = 1$, then from the assumption we have $\sum_{i=1}^{\infty}(1-b_{n+i})/q^{n+i} < 1/q^n$, and so $\sum_{i=1}^{\infty} 1/q^{n+i} + \sum_{i=1}^{n} b_i/q^i < x + 1/q^n$, i.e., $\sum_{i=1}^{n-1} b_i/q^i + \sum_{i \geq n+1} 1/q^i < x$, showing that the q-expansion is lazy. □

Remark. Theorem 3.1 is Lemma 1(b) in [2], but the proof here is different.

Theorem 3.2. *Let (e_i) be an infinite q-expansion of $y \leq 1$. If another infinite q-expansion (b_i) of $x \in [0, 1/(q-1)]$ satisfies the condition*

$$(1 - b_{n+i}) \prec (e_i) \text{ whenever } b_n = 1, \tag{3.1}$$

then (b_i) is the lazy q-expansion of x.

Proof. By Theorem 3.1, it suffices to show that if $b_k = 1$, then

$$\sum_{i=1}^{\infty} \frac{1 - b_{k+i}}{q^i} < 1. \tag{3.2}$$

Let $b_k = 1$. By hypothesis, there is a sequence of integers $k = k_0 < k_1 < \cdots$ satisfying for each $j = 1, 2, \ldots$ the conditions

$$1 - b_{k_{j-1}+i} = e_i \text{ when } 1 \leq i < k_j - k_{j-1}, \text{ and } 1 - b_{k_j} < e_{k_j - k_{j-1}}.$$

We have

$$\frac{1}{q^k} \sum_{i=1}^{\infty} \frac{1 - b_{k+i}}{q^i} = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{1 - b_{k_{j-1}+i}}{q^{k_{j-1}+i}} \right) = \sum_{j=1}^{\infty} \left(\sum_{i=1}^{k_j - k_{j-1}} \frac{e_i}{q^{k_{j-1}+i}} - \frac{1}{q^{k_j}} \right)$$

$$= \sum_{j=1}^{\infty} \left(\frac{1}{q^{k_{j-1}}} \sum_{i=1}^{k_j - k_{j-1}} \frac{e_i}{q^i} - \frac{1}{q^{k_j}} \right) < \sum_{j=1}^{\infty} \left(\frac{y}{q^{k_{j-1}}} - \frac{1}{q^{k_j}} \right) \leq \frac{1}{q^{k_0}}, \tag{3.3}$$

and the desired result follows at once. □

Remark. Proposition 2.1 in [5] is a special case of Theorem 3.2 when $y = 1$.

Theorem 3.3. *Let $q \in (1, 2]$, (e_i) be a finite q-expansion of $y \leq 1$ and denote by e_L its last nonzero digit. If an infinite q-expansion (b_i) of $x \in [0, 1/(q-1)]$ satisfies the condition* (3.1) *and*

$$L > \min\{k; \text{ for each } i \in \mathbb{N}, \text{ if } b_i = 1, \text{ then } b_{i+j} \neq e_j \text{ when } 1 \leq j < k \text{ and } b_{i+k} = e_k = 1\}, \tag{3.4}$$

then (b_i) is the lazy q-expansion of x.

Proof. Proceeding exactly as in the proof of Theorem 3.2, we end up at (3.3) but the strict inequality now becomes non-strict. If (3.3) is an equality, then $y = 1$ and more importantly, $k_j - k_{j-1} = L$ for each j but the condition (3.4) prevents this from happening. □

Remarks. Theorem 3.3 is new and complements Theorem 3.2. The condition (3.1) is not needed when $x = 0$. For then x has only a unique q-expansion which must then be (0) violating (3.1).

4. Numbers with unique *q*-expansion and smallest base

In this section, we first find conditions for which the greedy and lazy q-expansions of a fixed real number coincide, i.e., conditions for which the q-expansion is unique.

Theorem 4.1. *If the number $\sigma \geq 1$ has a unique q-expansion, (ε_i), for a given $q \in (1, 2]$, then this unique q-expansion is a U-sequence.*

Proof. Let $\sigma \geq 1$ and (ε_i) be unique, and so is a greedy q-expansion. We deduce from Theorem 2.1, using $x = y = \sigma$, that $(\varepsilon_{n+i}) \prec (\varepsilon_i)$ whenever $\varepsilon_n = 0$. Since (ε_i) is also the lazy q-expansion of σ, by Property L1, the q-expansion $(1-\varepsilon_i)$ is the greedy q-expansion of $\frac{1}{q-1} - \sigma$. Taking $x = \frac{1}{q-1} - \sigma$, $y = \sigma$ in Theorem 2.1, we get $(1 - \varepsilon_{n+i}) \prec (\varepsilon_i)$ whenever $1 - \varepsilon_n = 0$, which shows that (ε_i) is U-sequence. □

Remark. Theorem 4.1 is Lemma 2(b) in [2], but the proof here is different.

Theorem 4.2. *If the greedy q-expansion (ε_i) of $\sigma \in [0, 1]$ with $q \in (1, 2]$ is an U-sequence, then σ has a unique q-expansion for this given q .*

Proof. Assume the q-expansion (ε_i) is a U-sequence. Then $(1 - \varepsilon_{n+i}) \prec (\varepsilon_i)$ whenever $1 - \varepsilon_n = 0$. Since (ε_i) is a q-expansion of σ, by the first part of the proof of Property L1, $(1-\varepsilon_i)$ is a q-expansion of $\frac{1}{q-1} - \sigma$. Being a U-sequence, (ε_i) is infinite. Taking $y = \sigma \in [0, 1]$, $x = \frac{1}{q-1} - \sigma$ in Theorem 2.3(a), we deduce that $(1-\varepsilon_i)$ is the greedy q-expansion of $\frac{1}{q-1} - \sigma$. By Property L1, (ε_i) is the lazy q-expansion of σ. Since (ε_i) is both greedy and lazy, the number σ has a unique q-expansion. □

Remark. Taking $\sigma = 1$ in Theorems 4.1 and 4.2, we get Theorem 2.2 in [5], which shows how special the number 1 is.

For certain real number $y \leq 1$, among base numbers q for which y has unique q-expansions, it is possible to determine the smallest such base q, which we now show.

Theorem 4.3. *Let $(\delta_i) \subseteq \{0, 1\}$ be defined recursively as follows*:
- *First set $\delta_1 = 1$.*
- *If $n \geq 0$ and if $\delta_1, \ldots, \delta_{2^n}$ are already defined, set $\delta_{2^n+k} = 1 - \delta_k$ for $1 \leq k < 2^n$ and $\delta_{2^{n+1}} = 1$.*

If $y \in \left[\sum_{i=1}^{\infty} \delta_i/2^i, 1\right]$, then there is a smallest base $q \in (1,2]$ for which y has a unique U-sequence q-expansion. This q is the unique positive solution of the equation

$$y = \sum_{i=1}^{\infty} \frac{\delta_i}{q^i}.$$

Proof. From Theorem 3 in [3], (δ_i) is the smallest U-sequence. For fixed $y \in \left[\sum_{i=1}^{\infty} \delta_i/2^i, 1\right]$, by Theorem 1.1, using $(\delta_i) = (e_i)$, there exists a unique $q \in (1,2]$ for which $y = \sum_{i=1}^{\infty} \delta_i/q^i$. Using Theorem 2.3 (a) with $x = y$, $(e_i) = (a_i) = (\delta_i)$, it follows that (δ_i) is the greedy q-expansion and so by Theorem 4.2, y has a unique q-expansion.

If y has another U-sequence q'-expansion (e_i), which is also unique by the previous arguments, since (δ_i) is the smallest U-sequence, then $(e_i)\succeq(\delta_i)$ and Theorem 2.4 implies $q' \geq q$. □

5. Numbers with exactly two q-expansions and smallest sequence

We now proceed to find conditions for which there are exactly two q-expansions, which must then be greedy and lazy, of a positive number $y \leq 1$. Let (e_i) be an infinite T-sequence. Since (e_i) is also a D-sequence, then $e_1 = 1$; for otherwise applying (1.1) we would get $(e_i) \equiv (0)$, contradicting (1.1). From Theorem 1.1, for $y \in \left[\sum_{i=1}^{\infty} e_i/2^i,\ \sum_{i=1}^{\infty} e_i\right)$, there exists a unique $q \in (1,2]$ satisfying

$$\sum_{i=1}^{\infty} \frac{e_i}{q^i} = y. \tag{5.1}$$

By Theorem 2.3 (a), (e_i) is the greedy q-expansion of y. Let m, (ε_i), (δ_i) be as defined in the definition of T-sequence. Assume further that (ε_i) is a q-expansion of 1. Thus

$$\sum_{i=1}^{\infty} \frac{\delta_i}{q^i} = \sum_{i<m} \frac{e_i}{q^i} + \sum_{i>m} \frac{e_i + \varepsilon_{i-m}}{q^i} = \sum_{i\neq m} \frac{e_i}{q^i} + \frac{1}{q^m} \sum_{i=1}^{\infty} \frac{\varepsilon_i}{q^i} = \sum_{i=1}^{\infty} \frac{e_i}{q^i} = y, \tag{5.2}$$

showing that (δ_i) is also a q-expansion of y. Notice that the q-expansions (e_i) and (δ_i) are different because $e_m = 1$ but $\delta_m = 0$.

Theorem 5.1. *Let (e_i) be an infinite T-sequence with corresponding m, (ε_i), (δ_i). For $y \in \left[\sum_{i=1}^{\infty} e_i/2^i, 1\right]$, let $q \in (1,2]$ be the unique base, as guaranteed by Theorem 1.1, such that $y = \sum_{i=1}^{\infty} e_i/q^i$. Assume (ε_i) is a q-expansion of* 1. *Then y has exactly two different q-expansions, given by* (5.1) *and* (5.2).

Proof. From what mentioned above, (e_i) is the greedy q-expansion of y. On the other hand, from (1.3) and Theorem 3.2, we see that (δ_i) is the lazy q-expansion of $y \leq 1$. It remains to verify that if a sequence $(\rho_i) \subseteq \{0,1\}$ satisfies the strict inequalities $(\delta_i) \prec (\rho_i) \prec (e_i)$, then

$$\sum_{i=1}^{\infty} \frac{\rho_i}{q^i} \neq y. \tag{5.3}$$

Fix such a sequence (ρ_i). Then $\rho_i = \delta_i = e_i$ for all $i < m$. Since $\delta_m = 0$ and $e_m = 1$, we have either $\rho_m = \delta_m = 0$ or $\rho_m = e_m = 1$.

Case 1: $\rho_m = 0$. Then there is an integer $n > m$ such that $\rho_i = \delta_i$ for all $i < n$ and $\delta_n = 0 < 1 = \rho_n$. Using the properties of T-sequence and the same arguments as in the proof of Theorem 2.3 (a) up to equation (2.3), we deduce $\sum_{i=1}^{\infty} \delta_{n+i}/q^i < 1$. Thus

$$\sum_{i=1}^{\infty} \frac{\rho_i}{q^i} - y = \frac{1}{q^n} + \sum_{i>n} \frac{\rho_i - \delta_i}{q^i} \geq \frac{1}{q^n} - \sum_{i>n} \frac{\delta_i}{q^i} = \frac{1}{q^n}\left(1 - \sum_{i=1}^{\infty} \frac{\delta_{n+i}}{q^i}\right) > 0,$$

proving (5.3).

Case 2: $\rho_m = 1$. Then there is an integer $n > m$ such that $\rho_i = e_i$ for all $i < n$ and $\rho_n = 0 < 1 = e_n$. Using the properties of T-sequence and the same arguments as in the first case, we have $\sum_{i=1}^{\infty} \overline{e_{n+i}}/q^i < 1$. Hence

$$\sum_{i=1}^{\infty} \frac{\rho_i}{q^i} - y = -\frac{1}{q^n} + \sum_{i>n} \frac{\rho_i - e_i}{q^i} \leq -\frac{1}{q^n} + \sum_{i>n} \frac{\overline{e_i}}{q^i} = -\frac{1}{q^n}\left(1 - \sum_{i=1}^{\infty} \frac{\overline{e_{n+i}}}{q^i}\right) < 0,$$

again implying (5.3). □

Remark. Theorem 3.1 in [5] is a special case of Theorem 5.1 above when $y = 1$.

Theorem 5.2. *Let (e_i) be a finite T-sequence with e_L being its last nonzero digit and corresponding m, (ε_i), (δ_i). For $y \in \left[\sum_{i=1}^{\infty} e_i/2^i, 1\right]$, let $q \in (1, 2]$ be the unique base such that $y = \sum_{i=1}^{\infty} e_i/q^i$. Assume that (ε_i) is the greedy q-expansion of 1 and that*

$$L > \min\{k;\ \textit{for each } i \in \mathbb{N},\ \textit{if } \delta_i = 1,\ \textit{then } \delta_{i+j} \neq e_j \textit{ when } 1 \leq j < k \textit{ and } \delta_{i+k} = e_k = 1\}.$$

Then y has exactly two different q-expansions, given by (5.1) *and* (5.2).

Proof. The proof proceeds exactly as in Theorem 5.1, except that now, at the beginning of the proof, we appeal to Theorem 3.3 instead of Theorem 3.2. □

Remark. Theorem 5.2 is new and complements Theorem 5.1.

As an example for Theorem 5.2, take

$$y = \frac{390256388822139544981725106156190566398241267049 0}{391414433390307379180896260679628095791663279244 1}$$

and $q = 1.9$. Here

$$(e_i) = (1, 1, 1, 0, 1, 0, 0, 1, 0, 1, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1)$$

is a q-expansion of y,

$(\varepsilon_i) = (1,1,1,0,1,0,0,1,1,0,1,1,0,1,1,0,0,0,1,0,1,1,1,0,0,1,0,0,1,1,0,0,1,0,0,1,1,0,0,0,1,0,1,1,0,0,1,1,0,1,0,1,\ldots)$

is the greedy q-expansion of 1, $m = 10$ and $e_{i+m} + \varepsilon_i \in \{0,1\}$ $(i \geq 1)$.

Let (e_i) be an infinite T-sequence with corresponding m, (ε_i) and (δ_i). For a given real number y in an appropriate range, under the hypotheses of Theorem 5.1, y has exactly two q-expansions, namely the greedy (e_i) and the lazy (δ_i). The corresponding base q is then a T-base number. We now ask the question: given the real number y in an appropriate range what is its smallest, with respect to lexicographic order, T-sequence? An answer is given in the next theorem.

Theorem 5.3. *Let* $(e'_i) = 111\ \underline{\underline{001}}$*, the symbol* $\underline{\underline{001}}$ *denoting the period* 001 *of a periodic sequence. If* (e_i) *is an infinite T-sequence q-expansion of* $y \in \left[\sum_{i=1}^{\infty} e_i/2^i, 1\right] \cap \left[\sum_{i=1}^{\infty} e'_i/2^i, 1\right]$ *which begins with* 111 *with corresponding* $m > 3$ *not a multiple of* 3*, and* (ε_i) *being the greedy q-expansion of* 1*, then* $q \geq q'$*, where* $q' \in (1,2]$ *is the unique real number satisfying* $\sum_{i=1}^{\infty} e'_i/(q')^i = y$.

Proof. By Theorem 2.4, it suffices to show that $(e_i) \succeq (e'_i)$. Assume

$$(e_i) \prec (e'_i) = 111\ \underline{\underline{001}}. \tag{5.4}$$

Thus (e_i) takes the form

$$\underbrace{111}_{1}\ \underbrace{001}_{2} \cdots \underbrace{001}_{k}\ 000 \cdots \quad \text{for some } k \geq 2 \tag{5.5}$$

or 111000 which may be treated as (5.5) with $k = 1$. From (1.2) the sequence (δ_i) also begins with (5.5), i.e., δ_i : $111001 \cdots 001000$. Applying (1.3), we conclude that $\delta_{3k+4} = \delta_{3k+5} = 1$ (because $\delta_{3k} = 1$). Therefore, the sequence (e_i) also begins with 111 001 $\cdots$ 001 000 11. We distinguish two cases.

Case 1: If $e_{3k+6} = 1$, then $\delta_{3k+6} = 1$ (since m cannot is a multiple of 3). From (1.1), $e_{3k+7} = e_{3k+8} = 0$ (because $e_{3k+3} = 0$).

Subcase 1.1: If $e_{3k+9} = 1$, then $\delta_{3k+9} = 1$ (since m cannot is a multiple of 3). From (1.1), $e_{3k+10} = e_{3k+11} = 0$ (because $e_{3k+3} = 0$). The step now repeats as in Case 1.

Subcase 1.2: If $e_{3k+9} = 0$, then $\delta_{3k+9} = 0$ (because of $3k+9 < m$ and (5.2)) The step then repeats as in Case 2.

Case 2: If $e_{3k+6} = 0$, then $\delta_{3k+6} = 0$ (because of $3k+6 < m$ and (5.2)). From (1.3), $\delta_{3k+7} = \delta_{3k+8} = 1$ (because $\delta_{3k} = 1$). Thus $e_{3k+7} = e_{3k+8} = 1$ (because of $3k+8 < m$ and (5.2)). The step now repeats as in Case 1.

Subcase 2.1: If $e_{3k+9} = 1$, then $\delta_{3k+9} = 1$ (since m cannot is a multiple of 3).

Subcase 2.2: If $e_{3k+9} = 0$, then $\delta_{3k+9} = 0$ (because of $3k+9 < m$ and (5.2)). From (1.3), $\delta_{3k+10} = \delta_{3k+11} = 1$ (because $\delta_{3k} = 1$). The step repeats as in Case 2.

Continuing in the same manner, we deduce that m must be arbitrarily large, which is impossible. ∎

Remark. Theorem 4.1 in [5] is a special case of Theorem 5.3 when $y = 1$.

As an example of Theorem 5.3, let

$$y = \frac{390256388822139544981725106156190566398241267049 0}{391414433390307379180896260679628095791663279244 1}$$

and $q = 1.9$. The unique positive solution of the equation $\sum_{i=1}^{\infty} e_i'/(q')^i = y$ is $q' \approx 1.874535175$. From Theorem 4.1 in [5], when $y = 1$ we have $q' = 1.871349313$.

The last two results show that for certain $y \leq 1$, the sequence (e_i') with base q' yields a unique q'-expansion whose base is an accumulation point of, yet smaller than, other T-base numbers q of y with exactly two q-expansions.

Theorem 5.4. *Let $(e_i') = 111\ \underline{\underline{001}}$. For $y \in \left[\sum_{i=1}^{\infty} e_i'/2^i, 1\right]$, there is a unique $q' \in (1, 2]$ such that (e_i') is a q'-expansion of y and this q'-expansion is always unique.*

Proof. Taking both sequences to be (e_i') in Theorem 2.3 (a), we have that (e_i') is the greedy q'-expansion of y. Since (e_i') is also a U-sequence, Theorem 4.2 infers that y has a unique q'-expansion. □

Theorem 5.5. *Let $(e_i') = 111\ \underline{\underline{001}}$, $y \in \left[\sum_{i=1}^{\infty} e_i'/2^i, 1\right]$. By Theorem 1.1, there exists a unique $q' \in (1, 2]$ such that $y = \sum_{i=1}^{\infty} e_i'/(q')^i$. Let $k \in \mathbb{N}$ and let*

$$\left(e_i^{(k)}\right) := 111\ \overbrace{001}^{1}\ \cdots\ \overbrace{001}^{k}\ 1\underbrace{0\cdots0}_{3k+4}\ 001\ 001\ 001\ 001\ \cdots$$

be the sequence obtained by inserting the block $10\cdots0$ (one 1 followed by $(3k+4)$ 0's) between the k^{th} and $(k+1)^{th}$ block of 001 of (e_i'). By Theorem 1.1, there exists a unique $q_k \in (1, 2]$ such that $y = \sum_{i=1}^{\infty} e_i^{(k)}/q_k^i$. Let $\varepsilon_i^{(k)}$ be the greedy q_k-expansion of 1. Assume there are infinitely many k such that

$$\left(\overline{\varepsilon_{n+i}^{(k)}}\right) \prec (e_i^{(k)}) \text{ whenever } \varepsilon_n^{(k)} = 1 \text{ and } 1 \leq n \leq 3k+3, \tag{5.6}$$

$$(\varepsilon_{n+i}^{(k)}) \prec (e_i^{(k)}) \text{ whenever } \varepsilon_n^{(k)} = 0 \text{ and } 1 \leq n \leq 3k+4, \tag{5.7}$$

$$\varepsilon_{3k+4+3\mathbb{N}}^{(k)} = 0 \text{ where } 3\mathbb{N} = \{3t; t \in \mathbb{N}\}, \tag{5.8}$$

$$\varepsilon_{3k+2+3\mathbb{N}}^{(k)}\ \varepsilon_{3k+3+3\mathbb{N}}^{(k)} \neq 11, \tag{5.9}$$

$$\varepsilon_{3k+2+3\mathbb{N}}^{(k)}\ \varepsilon_{3k+3+3\mathbb{N}}^{(k)}, \varepsilon_{3k+5+3\mathbb{N}}^{(k)} \varepsilon_{3k+6+3\mathbb{N}}^{(k)} \neq 01, 10. \tag{5.10}$$

Then q' is an accumulation point of the set of T-base numbers.

Proof. We start by verifying that $(e_i^{(k)})$ is a T-sequence with $m = 3k+4$ and $(\delta_i^{(k)}) \subseteq \{0, 1\}$ so constructed as in (1.2) with corresponding $(e_i^{(k)})$ and $(\varepsilon_i^{(k)})$; such construction is valid by (5.8).

From the shape of the sequence $(e_i^{(k)})$, we see that $(e_i^{(k)})$ is a D-sequence and $e_{3k+4}^{(k)} = 1$. There remains to check the requirements (1.3), (1.4) and (1.5). The requirement (1.4) follows immediately from the shape of the sequence $(e_i^{(k)})$.

When $i < m = 3k+4$, since $\delta_i^{(k)} = e_i^{(k)}$, the requirement (1.3) holds for these i. From (5.6), respectively (5.7), together with the definition (1.2), $(\delta_i^{(k)})$ satisfies (1.3), respectively (1.5), when $m+1 \le n \le m+3k+4$.

For $n \ge m+3k+5$, (1.3) holds by the definition (1.2) and the shape of $(e_i^{(k)})$. As for (1.5), we distinguish four cases.

Case 1: $\delta_{m+3k+2+3\mathbb{N}}^{(k)}\delta_{m+3k+3+3\mathbb{N}}^{(k)} = 00$. From (5.9),

$$\delta_{m+3k+5+3\mathbb{N}}^{(k)}\delta_{m+3k+6+3\mathbb{N}}^{(k)} \ne 11,$$

i.e., $(\delta_i^{(k)})$ satisfies (1.5).

Case 2: $\delta_{m+3k+2+3\mathbb{N}}^{(k)}\delta_{m+3k+3+3\mathbb{N}}^{(k)} = 01$. From (5.9),

$$\delta_{m+3k+5+3\mathbb{N}}^{(k)}\delta_{m+3k+6+3\mathbb{N}}^{(k)} \ne 11,$$

while from (5.10), $\delta_{m+3k+5+3\mathbb{N}}^{(k)}\delta_{m+3k+6+3\mathbb{N}}^{(k)} \ne 10$, i.e., $(\delta_i^{(k)})$ satisfies (1.5).

Case 3: $\delta_{m+3k+2+3\mathbb{N}}^{(k)}\delta_{m+3k+3+3\mathbb{N}}^{(k)} = 10$. From (5.9),

$$\delta_{m+3k+5+3\mathbb{N}}^{(k)}\delta_{m+3k+6+3\mathbb{N}}^{(k)} \ne 11,$$

i.e., $(\delta_i^{(k)})$ satisfies (1.5).

Case 4: $\delta_{m+3k+2+3\mathbb{N}}^{(k)}\delta_{m+3k+3+3\mathbb{N}}^{(k)} = 11$. That $(\delta_i^{(k)})$ satisfies (1.5) follows at once from (5.9).

Since $(e_i^{(k)})$ is a T-sequence, taking $k \to \infty$, we have $(e_i^{(k)}) \to (e_i')$ and the corresponding base numbers $q_k \to q'$, which completes the proof. □

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