J. Korean Math. Soc. **47** (2010), No. 4, pp. 845–860
DOI 10.4134/JKMS.2010.47.4.845

THE NEHARI MANIFOLD APPROACH FOR DIRICHLET PROBLEM INVOLVING THE $p(x)$-LAPLACIAN EQUATION

Rabil A. Mashiyev, Sezai Ogras, Zehra Yucedag, and Mustafa Avci

Abstract. In this paper, using the Nehari manifold approach and some variational techniques, we discuss the multiplicity of positive solutions for the $p(x)$-Laplacian problems with non-negative weight functions and prove that an elliptic equation has at least two positive solutions.

1. Introduction

In this paper, we study the multiplicity of positive solutions for the following elliptic equation

$$(E_\lambda) \qquad \begin{cases} -\Delta_{p(x)}u(x) = \lambda a(x)\,|u|^{q(x)-2}\,u + b(x)\,|u|^{h(x)-2}\,u & \text{in } \Omega \\ u(x) = 0 & \text{on } \partial\Omega, \end{cases}$$

where the following conditions are satisfied:

Ω is a bounded domain with smooth boundary in $\mathbb{R}^N$, $N \geq 2$, $q, p, h \in C(\overline{\Omega})$ such that $1 < q(x) < p(x) < h(x) < p^*(x)$ ($p^*(x) = \frac{Np(x)}{N-p(x)}$ if $N > p(x)$, $p^*(x) = \infty$ if $N \leq p(x)$), $1 < p^- :=\operatorname{ess}\inf\limits_{x\in\Omega} p(x) \leq p(x) \leq p^+ := \operatorname{ess}\sup\limits_{x\in\Omega} p(x) < \infty$, $1 < q^- \leq q^+ < p^- \leq p^+ < h^- \leq h^+$, $\lambda > 0 \in \mathbb{R}$ and $a, b \in C\left(\overline{\Omega}\right)$ are non-negative weight functions with compact support in Ω.

Over the last decade, the variable exponent Lebesgue spaces $L^{p(x)}$ and the corresponding Sobolev space $W^{1,p(x)}$ have been a subject of active research area (we refer to [8, 11, 12, 18, 19] for the fundamental properties of these spaces). These investigations are stimulated mainly by the development of the studies of problems in Elasticity, Electrorheological fluids, Image Processing, Flow in Porous Media, Calculus of Variations, Differential Equations with $p(x)$-growth (see Acerbi and Mingione [1], Diening [7], Buhrii and Mashiyev [5], Halsey [15], Mihăilescu and Rădulescu [21], Růžička [24], Zhikov [28]). Among these problems, the study of $p(x)$-Laplacian problems via variational methods is an

Received October 7, 2008; Revised November 4, 2008.
2000 *Mathematics Subject Classification.* 35B38, 35D05, 35J60, 35J70, 58E05.
Key words and phrases. variable exponent Lebesgue-Sobolev spaces, $p(x)$-Laplacian, variational methods, Nehari manifold, multiple positive solutions.
This Research Project was supported by DUAPK-2008-59-74, Dicle University, Turkey.

interesting topic. A lot of researchers have devoted their works to this area (see Chabrowski and Fu [6], Fan [10], Fan and Zhang [13, 14], Mihăilescu [21], Mihăilescu and Rădulescu [22], Harjulehto, Hästö, Koskenoja and S. Varonen [16], Hästö [17], Ogras, Mashiyev, Avci and Yucedag [23]). We refer to the $p(x)$-Laplace operator $\Delta_{p(x)}u := \operatorname{div}\left(|\nabla u|^{p(x)-2}\nabla u\right)$, where p is a continuous non-constant function. This differential operator is a natural generalization of the p-Laplace operator $\Delta_p u := \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right)$, where $p > 1$ is a real constant. However, the $p(x)$-Laplace operator possesses more complicated nonlinearity than p-Laplace operator, due to the fact that $\Delta_{p(x)}$ is not homogeneous. This fact implies some difficulties; for example, we can not use the Lagrange Multiplier Theorem in many problems involving this operator.

In recent years, the similar problems of the form (E_λ) have been studied by many authors using various methods. In [20] for the case $p(x) > 1$ and $1 < q < \inf_\Omega p < \sup_\Omega p < h < \min\left\{N, \frac{N.p^-}{N-p^-}\right\}$ and $a(x) \equiv a, b(x) \equiv b > 0$, using Ekeland's variational principle and the mountain-pass lemma Mihăilescu proved that, if a and b small enough then there are two distinct solutions for the problem; in [22] under the assumptions $1 < \min_{x\in\overline{\Omega}} q(x) < \min_{x\in\overline{\Omega}} p(x) < \max_{x\in\overline{\Omega}} q(x)$, where $p(x), q(x)$ are continuous on $\overline{\Omega}$, $h(x) = 0$, $a(x) \equiv 1, b(x) = 0$, Mihăilescu and Rădulescu showed that there exists λ^* such that any $\lambda \in (0, \lambda^*)$ is an eigenvalue for the problem by using Ekeland's variational principle and the mountain-pass lemma; in [14] for the case $p(x) = q(x) > 1, h(x) = 0$, where $p(x)$ is continuous on $\overline{\Omega}$, and $a(x) \equiv 1, b(x) = 0$, Fan, Zhang, and Zhao obtained that, $\Lambda = \Lambda_{p(x)}$, the set of eigenvalues, is a nonempty infinite set such that $\sup\Lambda = +\infty$. In addition, they present some sufficient conditions for $\inf\Lambda = 0$ and for $\inf\Lambda > 0$, respectively; in [4] under the conditions $p(x) = 2, q(x) = 2$ and $1 < h < \frac{N+2}{N-2}$, where h is constant, and $a(x), b(x) : \overline{\Omega} \subset \mathbb{R}^N \to \mathbb{R}$ are smooth functions which may change sign in $\overline{\Omega}$, Brown and Zhang used the relationship between the Nehari manifold and fibrering maps to show how existence and non-existence results for positive solutions of the equation are linked to properties of the Nehari manifold; in [2] Afrouzi, Mahdavi, and Naghizadeh dealt with the similar problem for the case $p(x) = q(x) = p, h(x) = h$, $1 < h < p$, $a(x) = 1$ and $b(x) : \Omega \subset \mathbb{R}^N \to \mathbb{R}$ is a smooth function which may change sign and they discussed the existence and multiplicity of non-negative solutions of the problem from a variational viewpoint by making use of the Nehari manifold. Under the conditions $p(x) = p$, $q(x) = q$, $h(x) = h$, $1 < q < p < h < p^*$ $\left(p^* = \frac{Np}{N-p} \text{ if } N > p,\ p^* = \infty \text{ if } N \le p\right)$ and the weight functions $a(x) \equiv b(x) \equiv 1$, the authors Ambrosetti-Brezis-Cerami [3] have investigated equation (E_λ). They found that there exists λ_0 such that equation (E_λ) admits at least two positive solutions for $\lambda \in (0, \lambda_0)$, has a positive solution for $\lambda = \lambda_0$ and no positive solution exists for $\lambda > \lambda_0$, and also in [26] under the same assumptions and with sign-changing weight functions $a(x)$ and $b(x)$, Wu gave a variational proof of the existence of at least two positive solutions of equation (E_λ) for $p \in (1, p^*)$, using Palais-Smale and decomposition of the Nehari manifold.

In this paper, we have generalized the articles of Ambrosetti-Brezis-Cerami [3] and Wu [26], to the $p(x)$-Laplacian by using the Nehari manifold under the similar conditions. We shall discuss the multiplicity of positive solutions for the problem (E_λ) and prove the existence of at least two positive solutions.

If we consider all above mentioned papers use of the Nehari manifold approach for the case $p(x)$-growth condition makes our study quite different and very interesting.

2. Notations and preliminaries

We will discuss our problem (E_λ) in the variable exponent Sobolev space $W_0^{1,p(x)}(\Omega)$, so we need some theories and basic properties on spaces $L^{p(x)}(\Omega)$ and $W^{1,p(x)}(\Omega)$.

Write

$$L_+^\infty(\Omega) = \left\{p \in L^\infty(\Omega) : \ p^- > 1\right\}.$$

Let's define by $\mathcal{U}(\Omega)$ the set of all measurable real functions defined on Ω. For any $p \in L_+^\infty(\Omega)$, we denote the variable exponent Lebesgue space by

$$L^{p(x)}(\Omega) = \left\{ u \in \mathcal{U}(\Omega) \ : \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\},$$

which is equipped with the norm, so-called Luxemburg norm [11, 12, 18]

$$|u|_{p(x)} = \inf \left\{ \delta > 0 : \int_\Omega \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\}$$

and $\left(L^{p(x)}(\Omega), |\cdot|_{p(x)}\right)$ becomes a Banach space, we call it as variable exponent Lebesgue space.

Let c is a measurable real-valued function and $c(x) > 0$ for $x \in \Omega$. Then the weighted variable exponent Lebesgue space $L_{c(x)}^{p(x)}(\Omega)$ is defined by

$$L_{c(x)}^{p(x)}(\Omega) = \left\{ u \in \mathcal{U}(\Omega) : \int_\Omega c(x) |u(x)|^{p(x)}\, dx < \infty; \ c(x) > 0 \right\}$$

which is equipped with the norm

$$|u|_{(p(x),c(x))} = \inf \left\{ \delta > 0 : \int_\Omega c(x) \left| \frac{u(x)}{\delta} \right|^{p(x)} dx \leq 1 \right\}.$$

Proposition 2.1 ([11, 18])**.** *The conjugate space of $L^{p(x)}(\Omega)$ is $L^{p'(x)}(\Omega)$, where $\frac{1}{p'(x)}+\frac{1}{p(x)}=1$. For any $u\in L^{p(x)}(\Omega)$ and $v\in L^{p'(x)}(\Omega)$, we have*

$$\left|\int_{\Omega} uvdx\right| \leq \left(\frac{1}{p^-}+\frac{1}{(p')^-}\right)|u|_{p(x)}|v|_{p'(x)} \leq 2\,|u|_{p(x)}|v|_{p'(x)}.$$

Proposition 2.2 ([11, 18])**.** *Denote $\rho(u)=\int_{\Omega}|u(x)|^{p(x)}dx$, $\forall u\in L^{p(x)}\Omega$, then we have*

i) $|u|_{p(x)}<1\,(=1;>1)\Leftrightarrow \rho(u)<1\,(=1;>1)$,

ii) $|u|_{p(x)}>1 \implies |u|_{p(x)}^{p^-}\leq \rho(u)\leq |u|_{p(x)}^{p^+}$,

iii) $|u|_{p(x)}<1 \implies |u|_{p(x)}^{p^+}\leq \rho(u)\leq |u|_{p(x)}^{p^-}$.

Proposition 2.3 ([11, 18])**.** *If $u,u_n\in L^{p(x)}(\Omega)$, $n=1,2,\dots$, then the following statements are equivalent to each other*:

(1) $\lim\limits_{n\to\infty}|u_n-u|_{p(x)}=0$;

(2) $\lim\limits_{n\to\infty}\rho(u_n-u)=0$;

(3) *$u_n\to u$ in measure in Ω and $\lim\limits_{n\to\infty}\rho(u_n)=\rho(u)$.*

Define the variable exponent Sobolev space $W^{1,p(x)}(\Omega)$ by

$$W^{1,p(x)}(\Omega)=\{u\in L^{p(x)}(\Omega);\ \ |\nabla u|\in L^{p(x)}(\Omega)\}$$

and it can be equipped with the norm

$$\|u\|=|u|_{p(x)}+|\nabla u|_{p(x)}\quad,\forall u\in W^{1,p(x)}(\Omega).$$

The space $W_0^{1,p(x)}(\Omega)$ is denoted by the closure of $C_0^{\infty}(\Omega)$ in $W^{1,p(x)}(\Omega)$. We will use $\|u\|=|\nabla u|_{p(x)}$ for $u\in W_0^{1,p(x)}(\Omega)$ in the following discussions.

Proposition 2.4 ([11, 18])**.** *If $p^->1$ and $p^+<\infty$, then the spaces $L^{p(x)}(\Omega)$, $L^{p(x)}_{c(x)}(\Omega)$, $W^{1,p(x)}(\Omega)$ and $W_0^{1,p(x)}(\Omega)$ are separable and reflexive Banach spaces.*

Given two Banach spaces X and Y the symbol $X\hookrightarrow Y$ means that X is *continuously* embedded in Y and also the symbol $X\hookrightarrow\hookrightarrow Y$ means that there is a *compact* embedding of X in Y.

Proposition 2.5 ([11, 18])**.** (i) *Assume that the boundary of Ω possesses the cone property and $p\in C(\overline{\Omega})$. If $q\in C(\overline{\Omega})$ and $1\leq q(x)<p^*(x)$ for any $x\in\overline{\Omega}$, then $W^{1,p(x)}(\Omega)\hookrightarrow\hookrightarrow L^{q(x)}(\Omega)$.*

(ii) *If $p,q\in C(\overline{\Omega})$ and $p(x)\leq q(x)\leq p^*(x)$ for any $x\in\overline{\Omega}$, then $W^{1,p(x)}(\Omega)\hookrightarrow L^{q(x)}(\Omega)$ and also there is a constant $c>0$ such that*

$$|u|_{q(x)}\leq c\,\|u\|\quad\forall\ u\in W_0^{1,p(x)}(\Omega).$$

Proposition 2.6 ([8]). *Let $p(x)$ and $q(x)$ be measurable functions such that $p(x) \in L^{\infty}(\Omega)$ and $1 \leq p(x)q(x) \leq \infty$ for a.e. $x \in \Omega$. Let $u \in L^{q(x)}(\Omega)$, $u \neq 0$. Then*

$$|u|_{p(x)q(x)} \leq 1 \implies |u|^{p^+}_{p(x)q(x)} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|^{p^-}_{p(x)q(x)}$$

$$|u|_{p(x)q(x)} \geq 1 \implies |u|^{p^-}_{p(x)q(x)} \leq \left| |u|^{p(x)} \right|_{q(x)} \leq |u|^{p^+}_{p(x)q(x)}.$$

In particular, if $p(x) = p$ is constant, then

$$\left| |u|^p \right|_{q(x)} = |u|^p_{pq(x)}.$$

Theorem 2.7. *Assume that the boundary of Ω possesses the cone property and $p \in C\left(\overline{\Omega}\right)$. Suppose that $b \in L^{\beta(x)}(\Omega)$, $b(x) > 0$ for $x \in \Omega$, $\beta \in C\left(\overline{\Omega}\right)$ and $\beta^- > 1$, $\beta_0^- \leq \beta_0(x) \leq \beta_0^+$ $\left(\frac{1}{\beta(x)} + \frac{1}{\beta_0(x)} = 1\right)$. If $h \in C\left(\overline{\Omega}\right)$ and*

$$1 < h(x) < \frac{\beta(x) - 1}{\beta(x)} p^*(x), \ \forall x \in \overline{\Omega} \tag{2.1}$$

or

$$1 < \beta(x) < \frac{Np(x)}{Np(x) - h(x)\left(N - p(x)\right)},$$

then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{h(x)}_{b(x)}(\Omega)$ is compact. Moreover, there is a constant $c_5 > 0$ such that the inequality

$$\int_{\Omega} b(x) |u|^{h(x)} dx \leq c_5 \left(\|u\|^{h^-} + \|u\|^{h^+} \right) \tag{2.2}$$

holds.

Proof. We must remark that our proof of the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{h(x)}_{b(x)}(\Omega)$ is similar to Fan [10]. Let $u \in W^{1,p(x)}(\Omega)$ and set $r(x) = \frac{\beta(x)}{\beta(x)-1} h(x) = \beta_0(x) h(x)$. Then (2.1) implies $r(x) < p^*(x)$. Hence, by Proposition 2.5 we have the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{r(x)}(\Omega)$. So, for $u \in W^{1,p(x)}(\Omega)$, we have $|u|^{h(x)} \in L^{\beta_0(x)}(\Omega)$. By Proposition 2.1,

$$\int_{\Omega} b(x) |u|^{h(x)} dx \leq c_1 |b|_{\beta(x)} \left| |u|^{h(x)} \right|_{\beta_0(x)} < \infty.$$

This implies that $W^{1,p(x)}(\Omega) \subset L^{h(x)}_{b(x)}(\Omega)$. Now let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ and

$$u_n \rightharpoonup 0 \text{ (weakly) in } W^{1,p(x)}(\Omega).$$

Then, we have

$$u_n \to 0 \text{ (strongly) in } L^{r(x)}(\Omega).$$

So, it follows that $\left| |u_n|^{h(x)} \right|_{\beta_0(x)} \to 0$. Thus, we have

$$\int_{\Omega} b(x) |u_n|^{h(x)} dx \leq c_1 |b|_{\beta(x)} \left| |u_n|^{h(x)} \right|_{\beta_0(x)} \to 0,$$

which implies $|u_n|_{(h(x),b(x))} \to 0$. Hence, we have the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{h(x)}_{b(x)}(\Omega)$.

Now let's show the inequality (2.2) holds. By the above inequity we know that

$$\int_\Omega b(x)\,|u|^{h(x)}\,dx \le c_1\,|b|_{\beta(x)}\left||u|^{h(x)}\right|_{\beta_0(x)} < \infty.$$

Since $h^- \le h(x) \le h^+$and $|u|^{h(x)} \le |u|^{h^-} + |u|^{h^+}$ it follows that

$$\int_\Omega b(x)\,|u|^{h(x)}\,dx \le \int_\Omega b(x)\,|u|^{h^-}\,dx + \int_\Omega b(x)\,|u|^{h^+}\,dx.$$

Moreover, by Propositions 2.1, 2.2, 2.5, 2.6 and the condition $p(x) < h^-\beta_0(x) \le h^+\beta_0(x) < p^*(x)$, we have

(2.3)

$$\int_\Omega b(x)\,|u|^{h^-}\,dx \le c_2\,|b|_{\beta(x)}\left||u|^{h^-}\right|_{\beta_0(x)} = c_2\,|b|_{\beta(x)}\,|u|^{h^-}_{h^-\beta_0(x)} \le c_3\,\|u\|^{h^-}.$$

Similarly, we can obtain

$$\int_\Omega b(x)\,|u|^{h^+}\,dx \le c_4\,\|u\|^{h^+}. \tag{2.4}$$

As a result, from (2.3) and (2.4) it follows that

$$\int_\Omega b(x)\,|u|^{h(x)}\,dx \le c_5\left(\|u\|^{h^-} + \|u\|^{h^+}\right).$$

The proof is complete. $\square$

Theorem 2.8. *Assume that the boundary of Ω possesses the cone property and $p \in C(\overline{\Omega})$. Suppose that $a \in L^{\alpha(x)}(\Omega)$, $a(x) > 0$ for $x \in \Omega$, $\alpha \in C(\overline{\Omega})$ and $\alpha^- > 1$, $\alpha_0^- \le \alpha_0(x) \le \alpha_0^+$ $\left(\frac{1}{\alpha(x)} + \frac{1}{\alpha_0(x)} = 1\right)$. If $q \in C(\overline{\Omega})$, $p(x) < \frac{\alpha(x)}{\alpha(x)-1} q(x)$ and*

$$1 < q(x) < \frac{\alpha(x)-1}{\alpha(x)} p^*(x),\ \forall x \in \overline{\Omega} \tag{2.5}$$

or

$$\frac{Np(x)}{Np(x) - q(x)\,(N - p(x))} < \alpha(x) < \frac{p(x)}{p(x) - q(x)} \quad \textit{with,}$$

then the embedding from $W^{1,p(x)}(\Omega)$ to $L^{q(x)}_{a(x)}(\Omega)$ is compact. Moreover, there is a constant $c_7 > 0$ such that the inequality

$$\int_\Omega a(x)\,|u|^{q(x)}\,dx \le c_7\left(\|u\|^{q^-} + \|u\|^{q^+}\right) \tag{2.6}$$

holds.

Proof. Let $u \in W^{1,p(x)}(\Omega)$. Set $m(x) = \frac{\alpha(x)}{\alpha(x)-1} q(x) = \alpha_0(x) q(x)$. Then (2.5) implies $m(x) < p^*(x)$. Hence, by Proposition 2.5 there is the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{m(x)}(\Omega)$. For $u \in W^{1,p(x)}(\Omega)$ we have $|u|^{q(x)} \in L^{\alpha_0(x)}(\Omega)$. By Proposition 2.1,

$$\int_\Omega a(x) |u|^{q(x)} dx \le c_6 |a|_{\alpha(x)} \left| |u|^{q(x)} \right|_{\alpha_0(x)} < \infty.$$

This implies that $W^{1,p(x)}(\Omega) \subset L^{q(x)}_{a(x)}(\Omega)$. Now let $\{u_n\} \subset W^{1,p(x)}(\Omega)$ and

$$u_n \rightharpoonup 0 \text{ in } W^{1,p(x)}(\Omega).$$

Then, we have

$$u_n \to 0 \quad \text{in} \quad L^{m(x)}(\Omega).$$

So, it follows that $\left| |u_n|^{q(x)} \right|_{\alpha_0(x)} \to 0$. Thus, we have

$$\int_\Omega a(x) |u|^{q(x)} dx \le c_6 |a|_{\alpha(x)} \left| |u|^{q(x)} \right|_{\alpha_0(x)} \to 0,$$

which implies $|u_n|_{(q(x),a(x))} \to 0$. Hence, we have the embedding $W^{1,p(x)}(\Omega) \hookrightarrow\hookrightarrow L^{q(x)}_{a(x)}(\Omega)$.

Now, let's show that the inequality (2.6) holds. By the above inequality we know that

$$\int_\Omega a(x) |u|^{q(x)} dx \le c_6 |a|_{\alpha(x)} \left| |u|^{q(x)} \right|_{\alpha_0(x)} < \infty.$$

Considering the condition $p(x) < q^-\alpha_0(x) \le q^+\alpha_0(x) < p^*(x)$ and applying the similar steps as we did in proof of Theorem 2.7, we have

$$\int_\Omega a(x) |u|^{q(x)} dx \le c_7 \left(\|u\|^{q^-} + \|u\|^{q^+} \right).$$

The proof is complete. □

Proposition 2.9. *Assume that the conditions of Theorem 2.7 and Theorem 2.8 hold, respectively. Let $u \in W^{1,p(x)}(\Omega)$, then there are positive constants c_8, c_9, c_{10}, $c_{11} > 0$ such that the following inequalities hold*

i)

$$\int_\Omega b(x) |u|^{h(x)} dx \le \begin{cases} c_8 \|u\|^{h^+} & \text{if } \|u\| > 1 \\ c_9 \|u\|^{h^-} & \text{if } \|u\| < 1 \end{cases}$$

ii)

$$\int_\Omega a(x) |u|^{q(x)} dx \le \begin{cases} c_{10} \|u\|^{q^+} & \text{if } \|u\| > 1 \\ c_{11} \|u\|^{q^-} & \text{if } \|u\| < 1. \end{cases}$$

Proof. It follows immediately from the conclusions of Theorem 2.7 and Theorem 2.8, respectively. □

3. The main results

Let $J \in C^1(X, \mathbb{R})$ be the Euler functional associated with an elliptic problem on Banach space X. If J is bounded below and has a minimizer on X, then this minimizer is a critical point of J. Hence, it is a solution of the corresponding elliptic problem. However, in many problems J is not bounded below on the whole space X, but is bounded below on an appropriate subset of X, and minimizer on this set (if it exists) may give rise to solutions of the corresponding elliptic problem. A good candidate for an appropriate subset of X is the Nehari manifold.

If we consider our problem (E_λ), then, the corresponding Euler functional is defined by

$$J_\lambda(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx - \lambda \int_\Omega \frac{1}{q(x)} a(x) |u|^{q(x)}\, dx - \int_\Omega \frac{1}{h(x)} b(x) |u|^{h(x)}\, dx.$$

Then, by Theorems 2.7, 2.8, and Proposition 2.2, we have

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+} \int_\Omega |\nabla u|^{p(x)}\, dx - \frac{\lambda}{q^-} \int_\Omega a(x) |u|^{q(x)}\, dx - \frac{1}{h^-} \int_\Omega b(x) |u|^{h(x)}\, dx, \\ &\geq \frac{1}{p^+} \|u\|^{p^-} - \frac{\lambda}{q^-} c_7 \left(\|u\|^{q^-} + \|u\|^{q^+} \right) - \frac{1}{h^-} c_5 \left(\|u\|^{h^-} + \|u\|^{h^+} \right). \end{aligned}$$

Since $q^+ < p^- \leq p^+ < h^- \leq h^+$, this shows J_λ is not bounded below on whole $W_0^{1,p(x)}(\Omega)$. However, we shall show it is bounded on the Nehari manifold $M_\lambda(\Omega)$ which is given by

$$M_\lambda\ (\Omega) = \left\{ u \in W_0^{1,p(x)}(\Omega) \setminus \{0\} : \langle J_\lambda'(u), u \rangle = 0 \right\},$$

where $\langle ., . \rangle$ denotes the usual duality between $W_0^{1,p(x)}(\Omega)$ and $W^{-1,p'(x)}(\Omega)$. It is clear that all critical points of J_λ must lie on $M_\lambda(\Omega)$ and local minimizers on $M_\lambda(\Omega)$ are usually critical points of J_λ.

Thus, $u \in M_\lambda(\Omega)$ if and only if

(3.1)

$$\begin{aligned} I_\lambda(u) := \langle J_\lambda'(u), u \rangle = \int_\Omega |\nabla u|^{p(x)}\, dx - \lambda \int_\Omega a(x) |u|^{q(x)}\, dx - \int_\Omega b(x) |u|^{h(x)}\, dx \\ = 0. \end{aligned}$$

Then for $u \in M_\lambda(\Omega)$, we have

$$\begin{aligned}\langle I'_\lambda(u), u\rangle &= \int\limits_\Omega p(x)\,|\nabla u|^{p(x)}\,dx - \lambda \int\limits_\Omega q(x)\,a(x)\,|u|^{q(x)}\,dx \\ &\quad - \int\limits_\Omega h(x)\,b(x)\,|u|^{h(x)}\,dx \\ &\le \left(p^+ - q^-\right)\lambda \int\limits_\Omega a(x)\,|u|^{q(x)}\,dx + \left(p^+ - h^-\right)\int\limits_\Omega b(x)\,|u|^{h(x)}\,dx.\end{aligned}$$

Now, we split $M_\lambda(\Omega)$ into three parts:

$$\begin{aligned} M_\lambda^+ &= \{u \in M_\lambda(\Omega) : \langle I'_\lambda(u), u\rangle > 0\}, \\ M_\lambda^- &= \{u \in M_\lambda(\Omega) : \langle I'_\lambda(u), u\rangle < 0\}, \\ M_\lambda^0 &= \{u \in M_\lambda(\Omega) : \langle I'_\lambda(u), u\rangle = 0\}. \end{aligned}$$

Theorem 3.1. *Suppose that u_0 is a local maximum or minimum for J_λ on $M_\lambda(\Omega)$. If $u_0 \notin M_\lambda^0(\Omega)$, then u_0 is a critical point of J_λ.*

Proof. The proof of Theorem 3.1 can be obtained directly from the following lemmas (particularly Lemma 3.3). □

Lemma 3.2. *The energy functional J_λ is coercive and bounded below on $M_\lambda(\Omega)$.*

Proof. Let $u \in M_\lambda(\Omega)$ and $\|u\| > 1$. Then using (3.1), Propositions 2.2 and 2.9, we have

$$\begin{aligned} J_\lambda(u) &= \int\limits_\Omega \frac{1}{p(x)}\,|\nabla u|^{p(x)}\,dx - \lambda\int\limits_\Omega \frac{1}{q(x)}a(x)\,|u|^{q(x)}\,dx - \int\limits_\Omega \frac{1}{h(x)}b(x)\,|u|^{h(x)}\,dx \\ &\ge \frac{1}{p^+}\int\limits_\Omega |\nabla u|^{p(x)}\,dx - \frac{\lambda}{q^-}\int\limits_\Omega a(x)\,|u|^{q(x)}\,dx \\ &\quad - \frac{1}{h^-}\left(\int\limits_\Omega |\nabla u|^{p(x)}\,dx - \lambda\int\limits_\Omega a(x)\,|u|^{q(x)}\,dx\right) \\ &\ge \left(\frac{1}{p^+} - \frac{1}{h^-}\right)\int\limits_\Omega |\nabla u|^{p(x)}\,dx + \lambda\left(\frac{1}{h^-} - \frac{1}{q^-}\right)\int\limits_\Omega a(x)\,|u|^{q(x)}\,dx \\ &\ge \left(\frac{h^- - p^+}{h^- p^+}\right)\|u\|^{p^-} - c_{10}\lambda\left(\frac{h^- - q^-}{h^- q^-}\right)\|u\|^{q^+}. \end{aligned}$$

Since, $p^- > q^+$ so, $J_\lambda(u) \to \infty$ as $\|u\| \to \infty$. This implies J_λ is coercive and bounded below on $M_\lambda(\Omega)$. □

Lemma 3.3. *There exists $\lambda_1 > 0$ such that for $0 < \lambda < \lambda_1$ we have $M_\lambda^0(\Omega) = \varnothing$.*

Proof. Suppose otherwise, that is, $M_\lambda^0(\Omega) \neq \varnothing$ for all $\lambda \in \mathbb{R} \setminus \{0\}$. Let $u \in M_\lambda^0(\Omega)$ such that $\|u\| > 1$. Then using (3.1), (2.4) and definition of $M_\lambda^0(\Omega)$, we have

$$\begin{aligned}
0 &= \langle I_\lambda^{\prime}(u), u\rangle \\
&= \int_\Omega p(x)|\nabla u|^{p(x)}\,dx - \lambda\int_\Omega q(x)a(x)|u|^{q(x)}\,dx - \int_\Omega h(x)b(x)|u|^{h(x)}\,dx \\
&\geq p^-\int_\Omega |\nabla u|^{p(x)}\,dx - q^+\left(\int_\Omega |\nabla u|^{p(x)}\,dx - \int_\Omega b(x)|u|^{h(x)}\,dx\right) \\
&\quad - h^+\int_\Omega b(x)|u|^{h(x)}\,dx \\
&\geq \left(p^- - q^+\right)\int_\Omega |\nabla u|^{p(x)}\,dx + \left(q^+ - h^+\right)\int_\Omega b(x)|u|^{h(x)}\,dx.
\end{aligned}$$

By Proposition 2.9,

$$0 \geq \left(p^- - q^+\right)\|u\|^{p^-} + c_8\left(q^+ - h^+\right)\|u\|^{h^+},$$

$$\|u\| \geq c_{12}\left(\frac{p^- - q^+}{h^+ - q^+}\right)^{\frac{1}{h^+ - p^-}}. \tag{3.2}$$

Similarly,

$$\begin{aligned}
0 &= \langle I_\lambda'(u), u\rangle \\
&\leq p^+\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda q^-\int_\Omega a(x)|u|^{q(x)}\,dx - h^-\int_\Omega b(x)|u|^{h(x)}\,dx \\
&\leq p^+\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda q^-\int_\Omega a(x)|u|^{q(x)}\,dx \\
&\quad - h^-\left(\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda\int_\Omega a(x)|u|^{q(x)}\,dx\right).
\end{aligned}$$

By Proposition 2.9,

$$0 \leq \left(p^+ - h^-\right)\|u\|^{p^-} + \lambda c_{10}\left(h^- - q^-\right)\|u\|^{q^+},$$

$$\|u\| \leq c_{13}\left(\lambda\frac{h^- - q^-}{h^- - p^+}\right)^{\frac{1}{p^- - q^+}}. \tag{3.3}$$

If λ is sufficiently small (e.g. $\lambda = \left(\frac{h^- - p^+}{h^- - q^-}\right)\left(\frac{p^- - q^+}{h^+ - q^+}\right)^{\frac{p^- - q^+}{h^+ - p^-}}$), then from (3.2) and (3.3) we get $\|u\| < 1$ which contradicts with our assumption. Hence, we conclude $M_\lambda^0(\Omega) = \varnothing$. $\square$

By Lemma 3.3, for $0 < \lambda < \lambda_1$, we can write $M_\lambda(\Omega) = M_\lambda^+(\Omega) \cup M_\lambda^-(\Omega)$. Therefore, we can let

$$\alpha_\lambda^+ = \inf_{u \in M_\lambda^+(\Omega)} J_\lambda(u) \quad \text{and} \quad \alpha_\lambda^- = \inf_{u \in M_\lambda^-(\Omega)} J_\lambda(u).$$

Lemma 3.4. *If $0 < \lambda < \lambda_1$, then for all $u \in M_\lambda^+(\Omega)$, $J_\lambda(u) < 0$.*

Proof. Let $u \in M_\lambda^+(\Omega)$. By definition of $J_\lambda(u)$, we can write

$$J_\lambda(u) \le \frac{1}{p^-}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{\lambda}{q^+}\int_\Omega a(x)\,|u|^{q(x)}\,dx - \frac{1}{h^+}\int_\Omega b(x)\,|u|^{h(x)}\,dx. \tag{3.4}$$

Since $u \in M_\lambda^+(\Omega)$, we have

$$p^+\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda q^-\int_\Omega a(x)\,|u|^{q(x)}\,dx - h^-\int_\Omega b(x)\,|u|^{h(x)}\,dx > 0. \tag{3.5}$$

Now, if we multiply (3.1) by $(-q^-)$ and add with (3.5), we get

$$\int_\Omega b(x)\,|u|^{h(x)}\,dx < \frac{p^+ - q^-}{h^- - q^-}\int_\Omega |\nabla u|^{p(x)}\,dx. \tag{3.6}$$

Moreover, using (3.1) together with (3.4)

$$J_\lambda(u) \le \left(\frac{1}{p^-} - \frac{1}{q^+}\right)\int_\Omega |\nabla u|^{p(x)}\,dx + \left(\frac{1}{q^+} - \frac{1}{h^+}\right)\int_\Omega b(x)\,|u|^{h(x)}\,dx, \tag{3.7}$$

and applying (3.6) in (3.7), it follows

$$J_\lambda(u) < -\frac{(p^- - q^+)(h^+ - p^-)}{h^+ p^- q^+}\,\|u\|^{p^-} < 0.$$

Hence, we have $\alpha_\lambda^+ = \inf_{u \in M_\lambda^+(\Omega)} J_\lambda(u) < 0$. $\square$

Theorem 3.5. *If $0 < \lambda < \lambda_1$, there exists a minimizer of J_λ on $M_\lambda^+(\Omega)$.*

Proof. Since J_λ is bounded below on $M_\lambda(\Omega)$ and so on $M_\lambda^+(\Omega)$. Then, there exits a minimizing sequence $\{u_n^+\} \subseteq M_\lambda^+(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n^+) = \inf_{u \in M_\lambda^+(\Omega)} J_\lambda(u) = \alpha_\lambda^+ < 0.$$

Since J_λ is coercive, u_n^+ is bounded in $W_0^{1,p(x)}(\Omega)$. Thus, we may assume that, without loss of generality, $u_n^+ \rightharpoonup u_0^+$ in $W_0^{1,p(x)}(\Omega)$ and by the compact embeddings we have

$$u_n^+ \to u_0^+ \text{ in } L^{q(x)}_{a(x)}(\Omega),$$

and

$$u_n^+ \to u_0^+ \text{ in } L_{b(x)}^{h(x)}(\Omega).$$

Now, we shall prove $u_n^+ \to u_0^+$ in $W_0^{1,p(x)}(\Omega)$. Otherwise, suppose $u_n^+ \nrightarrow u_0^+$ in $W_0^{1,p(x)}(\Omega)$. Then

$$\int_\Omega \left|\nabla u_0^+\right|^{p(x)} dx < \lim_{n\to\infty} \inf \int_\Omega \left|\nabla u_n^+\right|^{p(x)} dx.$$

Moreover, by the compact embeddings we have

$$\int_\Omega a(x) \left|u_0^+\right|^{q(x)} dx = \lim_{n\to\infty} \inf \int_\Omega a(x) \left|u_n^+\right|^{q(x)} dx,$$
$$\int_\Omega b(x) \left|u_0^+\right|^{h(x)} dx = \lim_{n\to\infty} \inf \int_\Omega b(x) \left|u_n^+\right|^{h(x)} dx.$$

Using the fact that $\langle J'_\lambda(u_n^+), u_n^+\rangle = 0$ and Theorem 2.8, we can write the followings

$$J_\lambda(u_n^+) \geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right) \int_\Omega \left|\nabla u_n^+\right|^{p(x)} dx + \lambda \left(\frac{1}{h^-} - \frac{1}{q^-}\right) \int_\Omega a(x) \left|u_n^+\right|^{q(x)} dx,$$

$$\begin{aligned} \lim_{n\to\infty} J_\lambda(u_n^+) &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right) \lim_{n\to\infty} \int_\Omega \left|\nabla u_n^+\right|^{p(x)} dx \\ &\quad + \lambda \left(\frac{1}{h^-} - \frac{1}{q^-}\right) \lim_{n\to\infty} \int_\Omega a(x) \left|u_n^+\right|^{q(x)} dx, \end{aligned}$$

$$\begin{aligned} \alpha_\lambda^+ &= \inf_{u\in M_\lambda^+} J_\lambda(u) \\ &> \left(\frac{1}{p^+} - \frac{1}{h^-}\right) \left\|u_0^+\right\|^{p^-} + c_7 \lambda \left(\frac{1}{h^-} - \frac{1}{q^-}\right) \left(\left\|u_0^+\right\|^{q^-} + \left\|u_0^+\right\|^{q^+}\right), \end{aligned}$$

since $p^- > q^+$, for $\left\|u_0^+\right\| > 1$, we have

$$\alpha_\lambda^+ = \inf_{u\in M_\lambda^+} J_\lambda(u) > 0.$$

However, in Lemma 3.4 it was showed that for any $u \in M_\lambda^+(\Omega)$, $J_\lambda(u) < 0$. So, this is a contradiction. Hence, $u_n^+ \to u_0^+$ in $W_0^{1,p(x)}(\Omega)$ and

$$J_\lambda(u_0^+) = \lim_{n\to\infty} J_\lambda(u_n^+) = \inf_{u\in M_\lambda^+(\Omega)} J_\lambda(u).$$

Thus, u_0^+ is a minimizer for J_λ on $M_\lambda^+(\Omega)$. □

Lemma 3.6. *If* $0 < \lambda < \lambda_1$, *then for all* $u \in M_\lambda^-(\Omega)$, $J_\lambda(u) > 0$.

Proof. Let $u \in M_\lambda(\Omega)$. By definition of $J_\lambda(u)$ and (3.1), we have

$$J_\lambda(u) \geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{\lambda}{q^-}\int_\Omega a(x)\,|u|^{q(x)}\,dx - \frac{1}{h^-}\int_\Omega b\,(x)\,|u|^{h(x)}\,dx,$$

and

$$\int_\Omega b\,(x)\,|u|^{h(x)}\,dx = \int_\Omega |\nabla u|^{p(x)}\,dx - \lambda\int_\Omega a\,(x)\,|u|^{q(x)}\,dx.$$

Using the two expressions above, it follows

$$\begin{aligned} J_\lambda(u) &\geq \frac{1}{p^+}\int_\Omega |\nabla u|^{p(x)}\,dx - \frac{\lambda}{q^-}\int_\Omega a(x)\,|u|^{q(x)}\,dx \\ &\quad - \frac{1}{h^-}\left(\int_\Omega |\nabla u|^{p(x)}\,dx - \lambda\int_\Omega a\,(x)\,|u|^{q(x)}\,dx\right) \\ &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right)\int_\Omega |\nabla u|^{p(x)}\,dx + \lambda\left(\frac{1}{h^-} - \frac{1}{q^-}\right)\int_\Omega a(x)\,|u|^{q(x)}\,dx. \end{aligned}$$

By Propositions 2.2, 2.9, and the condition $p^- > q^+$, it follows

$$\begin{aligned} J_\lambda(u) &\geq \left(\frac{1}{p^+} - \frac{1}{h^-}\right)\|u\|^{p^-} + c_{10}\lambda\left(\frac{1}{h^-} - \frac{1}{q^-}\right)\|u\|^{q+} \\ &\geq \left(\frac{h^- - p^+}{p^+h^-} + c_{10}\lambda\frac{q^- - h^-}{q^-h^-}\right)\|u\|^{p^-}. \end{aligned}$$

So, if we choose $\lambda < \frac{q^-\left(h^- - p^+\right)}{c_{10}p^+(h^- - q^-)}$, we get $J_\lambda(u) > 0$. Moreover, if we consider the facts $M_\lambda(\Omega) = M_\lambda^+(\Omega) \cup M_\lambda^-(\Omega)$ (see Lemma 3.3), $M_\lambda^+(\Omega) \cap M_\lambda^-(\Omega) = \varnothing$, and Lemma 3.4, we must have $u \in M_\lambda^-(\Omega)$. □

Theorem 3.7. *If* $0 < \lambda < \lambda_1$, *there exists a minimizer of* J_λ *on* $M_\lambda^-(\Omega)$.

Proof. Since J_λ is bounded below on $M_\lambda(\Omega)$ and so on $M_\lambda^-(\Omega)$, then there exits a minimizing sequence $\{u_n^-\} \subseteq M_\lambda^-(\Omega)$ such that

$$\lim_{n\to\infty} J_\lambda(u_n^-) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u) = \alpha_\lambda^- > 0.$$

Since J_λ is coercive, u_n^- is bounded in $W_0^{1,p(x)}(\Omega)$. Thus, we may assume that, without loss of generality, $u_n^- \rightharpoonup u_0^-$ in $W_0^{1,p(x)}(\Omega)$ and by the compact embeddings we have

$$u_n^- \to u_0^- \text{ in } L_{a(x)}^{q(x)}(\Omega),$$

and

$$u_n^- \to u_0^- \text{ in } L_{b(x)}^{h(x)}(\Omega).$$

Moreover, if $u_0^- \in M_\lambda^-(\Omega)$, then there is a constant $t > 0$ such that $tu_0^- \in M_\lambda^-(\Omega)$ and $J_\lambda(u_0^-) \geq J_\lambda(tu_0^-)$. Indeed, since

$$I'_\lambda(u) = \int_\Omega p(x)\,|\nabla u|^{p(x)}\,dx - \lambda \int_\Omega q(x)\,a(x)\,|u|^{q(x)}\,dx - \int_\Omega h(x)\,b(x)\,|u|^{h(x)}\,dx,$$

then,

$$\begin{aligned} I'_\lambda(tu_0^-) &= \int_\Omega p(x)\,|\nabla tu_0^-|^{p(x)}\,dx - \lambda \int_\Omega q(x)\,a(x)\,|tu_0^-|^{q(x)}\,dx - \int_\Omega h(x)\,b(x)\,|tu_0^-|^{h(x)}\,dx \\ &\leq t^{p^+} p^+ \int_\Omega |\nabla u_0^-|^{p(x)}\,dx - \lambda t^{q^-} q^- \int_\Omega a(x)\,|u_0^-|^{q(x)}\,dx - t^{h^-} h^- \int_\Omega b(x)\,|u_0^-|^{h(x)}\,dx. \end{aligned}$$

Since $q^- < p^+ < h^-$, and by the assumptions on a and b, it follows $I'_\lambda(tu_0^-) < 0$. Hence, by the definition of $M_\lambda^-(\Omega)$, $tu_0^- \in M_\lambda^-(\Omega)$.

Now, we shall show $u_n^- \to u_0^-$ in $W_0^{1,p(x)}(\Omega)$. Otherwise, suppose $u_n^- \nrightarrow u_0^-$ in $W_0^{1,p(x)}(\Omega)$. Then using the fact that

$$\int_\Omega |\nabla u_0^-|^{p(x)}\,dx < \lim_{n\to\infty} \inf \int_\Omega |\nabla u_n^-|^{p(x)}\,dx,$$

we have,

$$\begin{aligned} J_\lambda(tu_0^-) &\leq \frac{t^{p^+}}{p^-}\int_\Omega |\nabla u_0^-|^{p(x)}\,dx - \lambda \frac{t^{q^-}}{q^+}\int_\Omega a(x)\,|u_0^-|^{q(x)}\,dx - \frac{t^{h^-}}{h^+}\int_\Omega b(x)\,|u_0^-|^{h(x)}\,dx \\ &< \lim_{n\to\infty}\left[\frac{t^{p^+}}{p^-}\int_\Omega |\nabla u_n^-|^{p(x)}\,dx - \lambda \frac{t^{q^-}}{q^+}\int_\Omega a(x)\,|u_n^-|^{q(x)}\,dx - \frac{t^{h^-}}{h^+}\int_\Omega b(x)\,|u_n^-|^{h(x)}\,dx\right] \\ &\leq \lim_{n\to\infty} J_\lambda(tu_n^-) \leq \lim_{n\to\infty} J_\lambda(u_n^-) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u) = \alpha_\lambda^-. \end{aligned}$$

This implies that $J_\lambda(tu_0^-) < \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u) = \alpha_\lambda^-$, which is a contradiction. Hence, $u_n^- \to u_0^-$ in $W_0^{1,p(x)}(\Omega)$ and so

$$J_\lambda(u_0^-) = \lim_{n\to\infty} J_\lambda(u_n^-) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u).$$

Thus, u_0^- is a minimizer for J_λ on $M_\lambda^-(\Omega)$. □

Corollary 3.8. *By Theorems 3.5 and 3.7, we conclude that there exists $u_0^+ \in M_\lambda^+(\Omega)$ and $u_0^- \in M_\lambda^-(\Omega)$ such that $J_\lambda(u_0^+) = \inf_{u\in M_\lambda^+(\Omega)} J_\lambda(u)$ and $J_\lambda(u_0^-) = \inf_{u\in M_\lambda^-(\Omega)} J_\lambda(u)$. Moreover, since $J_\lambda(u_0^\pm) = J_\lambda(|u_0^\pm|)$ and $|u_0^\pm| \in M_\lambda^\pm(\Omega)$, we may assume $u_0^\pm \geq 0$. By Theorem 3.1, $u_0^\pm$ are critical points of J_λ on $W_0^{1,p(x)}(\Omega)$ and hence are weak solutions (and so by standard regularity results classical* *solutions) of* (E_λ). *Finally, by the Harnack inequality due to* [25, 27], *we obtain that* $u_0^\pm$ *are positive solutions of* (E_λ).

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Rabil A. Mashiyev
Department of Mathematics
Faculty of Science and Arts
Dicle University
21280-Diyarbakir, Turkey
E-mail address: mrabil@dicle.edu.tr

Sezai Ogras
Department of Mathematics
Faculty of Science and Arts
Dicle University
21280-Diyarbakir, Turkey
E-mail address: seazio@dicle.edu.tr

Zehra Yucedag
Department of Mathematics
Faculty of Science and Arts
Dicle University
21280-Diyarbakir, Turkey
E-mail address: zehra@dicle.edu.tr

Mustafa Avci
Department of Mathematics
Faculty of Science and Arts
Dicle University
21280-Diyarbakir, Turkey
E-mail address: mustafaxavci@hotmail.com