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PERTURBATION ANALYSIS OF THE MOORE-PENROSE INVERSE FOR A CLASS OF BOUNDED OPERATORS IN HILBERT SPACES

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Abstract. Let $\mathcal{H}$ and $\mathcal{K}$ be Hilbert spaces and let $T, \widetilde{T} = T + \delta T$ be bounded operators from $\mathcal{H}$ into $\mathcal{K}$. In this article, two facts related to the perturbation bounds are studied. The first one is to find the upper bound of $\|\widetilde{T}^+ - T^+\|$, which extends the results obtained by the second author and enriches the perturbation theory for the Moore-Penrose inverse. The other one is to develop explicit representations of projectors $\|\widetilde{T}\widetilde{T}^+ - TT^+\|$ and $\|\widetilde{T}^+\widetilde{T} - T^+T\|$. In addition, some spectral cases related to these results are analyzed.

1. Introduction

Let $\mathcal{H}$ and $\mathcal{K}$ be separable complex Hilbert spaces. Denote by $\mathcal{B}(\mathcal{H}, \mathcal{K})$ the set of all bounded linear operators from $\mathcal{H}$ into $\mathcal{K}$. For an operator $A \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, $\mathcal{R}(A)$, $\mathcal{N}(A)$, A^* and $\|A\|$ denote the range, the null space, the adjoint and the spectral norm of A, respectively. The identity onto a closed subspace $\mathcal{M}$ will be denoted by $I_{\mathcal{M}}$ or I if there is no confusion. For $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$, if there exists an operator $T^+ \in \mathcal{B}(\mathcal{K}, \mathcal{H})$ satisfying the following four operator equations

$$TT^+T = T, \quad T^+TT^+ = T^+, \quad TT^+ = (TT^+)^*, \quad T^+T = (T^+T)^*,$$

then T^+ is called the Moore-Penrose inverse of T. It is well known that T has the Moore-Penrose inverse if and only if $\mathcal{R}(T)$ is closed and the Moore-Penrose inverse of T is unique (see [1, 2]).

Let $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ with closed range and let $\widetilde{T} = T + \delta T$ be the perturbation of T by $\delta T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$. The perturbation theory of a generalized inverse is

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concerned with the question that if T has a generalized inverse, when has $\widetilde{T}$ a generalized inverse? What are the upper bounds for $\|\widetilde{T}^+ - T^+\|$?

For the perturbation in Hilbert spaces and Banach spaces, the authors in [21, 3, 5, 4, 6, 7, 8, 12, 9, 10, 11] and [15, 19, 17, 16, 18, 20] have obtained some results. For example, Wei and Ding [19] gave the explicit formula for the generalized inverse of the perturbed operator under some conditions. In this paper, we will generalize the result of [19] to the cases:

Case 1: $(I - TT^+)\delta TT^+T = 0$,

Case 2: $(I - TT^+)(I + \delta TT^+)^{-1}\delta T(I - T^+T) = 0$.

Under these assumptions on perturbation operator δT, upper bounds for $\|\widetilde{T}^+\|$ and $\|\widetilde{T}^+ - T^+\|$ are presented. And the explicit representations of $\widetilde{T}^+$, projectors $\|\widetilde{T}\widetilde{T}^+ - TT^+\|$ and $\|\widetilde{T}^+\widetilde{T} - T^+T\|$ in terms of δT and T are obtained. These not only cover the special cases but also improve over the results of [19].

2. The upper bound of $\|\widetilde{T}^+ - T^+\|$

In this section, we shall consider the problem of the upper bound of $\|\widetilde{T}^+ - T^+\|$, which are based on explicit expressions for $\widetilde{T}^+$. Let T and δT have the form

$$T = \begin{pmatrix} 0 & 0 \\ 0 & T_1 \end{pmatrix}, \quad \delta T = \begin{pmatrix} \delta_3 & \delta_4 \\ \delta_2 & \delta_1 \end{pmatrix}, \tag{1}$$

where T_1 as an operator from $\mathcal{R}(T^*)$ onto $\mathcal{R}(T)$ is invertible. Throughout this paper, we need some notations. Let

$$\begin{aligned} \omega_T &= \Big(I + T^+\delta T\Big)^{-1} T^+\delta T\Big(I - T^+T\Big), \\ \nu_T &= \big((I - TT^+)\delta T\big)^+(I - TT^+)\delta T, \\ M &= T^+T(I + T^+\delta T)^{-1}(I - T^+T), \\ N &= (I - TT^+)(I + \delta TT^+)^{-1}TT^+. \end{aligned} \tag{2}$$

We first present general expressions for $\widetilde{T}^+$ when it is only assumed that $\delta_4 = 0$.

Theorem 1. *Suppose that*

$$\|T^+\delta T\| < 1, \quad (I - TT^+)\delta TT^+T = 0.$$

Then $\widetilde{T}^+$ exists if and only if $\mathcal{R}\Big((I - TT^+)\delta T\Big)$ is closed. In this case,

$$\begin{aligned} \widetilde{T}^+ \quad = \quad & \big(I + (I - \nu_T)^*\omega_T^*\big)T^+T\Big[I + \omega_T(I - \nu_T)(I - \nu_T)^*\omega_T^*\Big]^{-1} \\ & \times(I + T^+\delta T)^{-1}T^+\Big[I - \delta T\big((I - TT^+)\delta T\big)^+\Big] + \big((I - TT^+)\delta T\big)^+ \end{aligned}$$

and

$$\|\widetilde{T}^+ - T^+\| \le \|\big((I-TT^+)\delta T\big)^+\| + \tfrac{\|T^+\delta T\|\|T^+\|}{1-\|T^+\delta T\|}\left(1+\tfrac{\|I-\delta T\big((I-TT^+)\delta T\big)^+\|}{(1-\|T^+\delta T\|)^2}\right).$$

Proof. Let T and δT have the form as Eqn. (1), ω_T and ν_T have the form as Eqn. (2). $\|T^+\delta T\|<1$ implies that $I+T^+\delta T$ is invertible; $(I-TT^+)\delta TT^+T=0$ implies that $\delta_4=0$. Since $T^+=0\oplus T_1^{-1}$, the invertibility of $I+T^+\delta T$ implies that $T_1+\delta_1=T_1(I+T_1^{-1}\delta_1)$ is invertible. From

$$\begin{pmatrix}\delta_3 & 0\\ \delta_2 & T_1+\delta_1\end{pmatrix}\begin{pmatrix} I & 0\\ -(T_1+\delta_1)^{-1}\delta_2 & (T_1+\delta_1)^{-1}\end{pmatrix}=\begin{pmatrix}\delta_3 & 0\\ 0 & I\end{pmatrix},$$

we get that $\mathcal{R}(\widetilde{T})$ is closed if and only if $\mathcal{R}(\delta_3)=\mathcal{R}\left((I-TT^+)\delta T\right)$ is closed. Hence $\widetilde{T}^+$ exists if and only if $\mathcal{R}\left((I-TT^+)\delta T\right)$ is closed.

Since $\mathcal{R}(\delta_3)$ is closed, $\widetilde{T}$ as an operator from $\mathcal{N}(\delta_3)\oplus\mathcal{R}(\delta_3^*)\oplus\mathcal{R}(T^*)$ into $\mathcal{N}(\delta_3^*)\oplus\mathcal{R}(\delta_3)\oplus\mathcal{R}(T)$ has the form

$$\widetilde{T}=\begin{pmatrix}\delta_3 & 0\\ \delta_2 & T_1+\delta_1\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & \delta_{31} & 0\\ \delta_{22} & \delta_{21} & T_1+\delta_1\end{pmatrix},$$

where δ_{31} as an operator from $\mathcal{R}(\delta_3^*)$ onto $\mathcal{R}(\delta_3)$ is invertible. Now, we define $\Delta=\left[I+(T_1+\delta_1)^{-1}\delta_2(I-\delta_3^+\delta_3)\delta_2^*(T_1^*+\delta_1^*)^{-1}\right]^{-1}$,

$$N=\begin{pmatrix}0\\ \delta_{22}\end{pmatrix},\quad M=\begin{pmatrix}\delta_{31} & 0\\ \delta_{21} & T_1+\delta_1\end{pmatrix}$$

and

$$\begin{aligned}\triangle' &= \left[(T_1+\delta_1)(T_1+\delta_1)^*+\delta_{22}\delta_{22}^*\right]^{-1}\\ &= \left[(T_1+\delta_1)(T_1+\delta_1)^*+\delta_2(I-\delta_3^+\delta_3)\delta_2^*\right]^{-1}\\ &= (T_1^*+\delta_1^*)^{-1}\left[I+(T_1+\delta_1)^{-1}\delta_2(I-\delta_3^+\delta_3)\delta_2^*(T_1^*+\delta_1^*)^{-1}\right]^{-1}(T_1+\delta_1)^{-1}\\ &= (T_1^*+\delta_1^*)^{-1}\triangle(T_1+\delta_1)^{-1}.\end{aligned}$$

So

$$\begin{aligned}\widetilde{T}^+ &= \begin{pmatrix}0 & 0\\ N & M\end{pmatrix}^*\left[\begin{pmatrix}0 & 0\\ N & M\end{pmatrix}\begin{pmatrix}0 & 0\\ N & M\end{pmatrix}^*\right]^+\\ &= \begin{pmatrix}0 & N^*(NN^*+MM^*)^{-1}\\ 0 & M^*(NN^*+MM^*)^{-1}\end{pmatrix}\\ &= \begin{pmatrix}0 & -\delta_{22}^*\triangle'\delta_{21}\delta_{31}^{-1} & \delta_{22}^*\triangle'\\ 0 & \delta_{31}^{-1} & 0\\ 0 & -(T_1+\delta_1)^*\triangle'\delta_{21}\delta_{31}^{-1} & (T_1+\delta_1)^*\triangle'\end{pmatrix} \qquad (3)\\ &= \begin{pmatrix}\delta_3^+-(I-\delta_3^+\delta_3)\delta_2^*\triangle'\delta_2\delta_3^+ & (I-\delta_3^+\delta_3)\delta_2^*\triangle'\\ -(T_1+\delta_1)^*\triangle'\delta_2\delta_3^+ & (T_1+\delta_1)^*\triangle'\end{pmatrix}\end{aligned}$$

$$= \begin{pmatrix} \delta_3^+ & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I & (I-\delta_3^+\delta_3)\delta_2^*(T_1^*+\delta_1^*)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix}$$
$$\times \begin{pmatrix} I & 0 \\ 0 & \triangle \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & (T_1+\delta_1)^{-1} \end{pmatrix} \begin{pmatrix} 0 & 0 \\ -\delta_2\delta_3^+ & I \end{pmatrix}.$$

Note that $\delta_3^+ \oplus 0 = \big((I-TT^+)\delta T\big)^+$, $0 \oplus (T_1+\delta_1)^{-1} = (I+T^+\delta T)^{-1}T^+$,

$$\begin{pmatrix} 0 & 0 \\ 0 & (T_1+\delta_1)^{-1} \end{pmatrix}\begin{pmatrix} 0 & 0 \\ -\delta_2\delta_3^+ & I \end{pmatrix} = (I+T^+\delta T)^{-1}T^+\Big[I-\delta T\big((I-TT^+)\delta T\big)^+\Big],$$

$$\begin{aligned}\begin{pmatrix} I & (I-\delta_3^+\delta_3)\delta_2^*(T_1^*+\delta_1^*)^{-1} \\ 0 & I \end{pmatrix} &= \Bigg[I+\Big(I+T^+\delta T\Big)^{-1}T^+\delta T\Big(I-T^+T\Big) \\ &\qquad \times \Big(I-\big((I-TT^+)\delta T\big)^+(I-TT^+)\delta T\Big)\Bigg]^* \\ &= \Big(I+\omega_T(I-\nu_T)\Big)^* \\ &= I+(I-\nu_T)^*\omega_T^*\end{aligned}$$

and

$$\begin{aligned} I\oplus\triangle &= \Bigg\{I+\Bigg[\Big(I+T^+\delta T\Big)^{-1}T^+\delta T\Big(I-T^+T\Big) \\ &\quad \times\Big(I-\big((I-TT^+)\delta T\big)^+(I-TT^+)\delta T\Big)\Bigg] \\ &\quad \times\Bigg[\Big(I+T^+\delta T\Big)^{-1}T^+\delta T\Big(I-T^+T\Big) \\ &\quad \times\Big(I-\big((I-TT^+)\delta T\big)^+(I-TT^+)\delta T\Big)\Bigg]^*\Bigg\}^{-1} \\ &= \Big(I+\omega_T(I-\nu_T)(I-\nu_T)^*\omega_T^*\Big)^{-1}. \end{aligned}$$

So, we have

$$\begin{aligned}\widetilde{T}^+ = {} & \Big[I+\omega_T(I-\nu_T)\Big]^* T^+T\Big[I+\omega_T(I-\nu_T)(I-\nu_T)^*\omega_T^*\Big]^{-1} \\ & \times (I+T^+\delta T)^{-1}T^+\Big[I-\delta T\big((I-TT^+)\delta T\big)^+\Big] + \big((I-TT^+)\delta T\big)^+.\end{aligned}$$

By using the equation that $(I+UV^*)^{-1} = I-U(I+V^*U)^{-1}V^*$ and $U^*(I+UU^*)^{-1} = (I+U^*U)^{-1}U^*$ we can prove that

$$\widetilde{T}^+ = \Big[I+\omega_T(I-\nu_T)\Big]^* T^+T\Big[I+\omega_T(I-\nu_T)(I-\nu_T)^*\omega_T^*\Big]^{-1}$$

$$
\begin{aligned}
&\times (I+T^+\delta T)^{-1}T^+\Big[I-\delta T\big((I-TT^+)\delta T\big)^+\Big]+\big((I-TT^+)\delta T\big)^+\\
&=(I+T^+\delta T)^{-1}\Big[I+(I-\nu_T)^*\omega_T^*\omega_T(I-\nu_T)\Big]^{-1}(I-T^+T)\\
&\quad\Big[(I+T^+\delta T)^{-1}T^+\delta T\Big(I-\nu_T\Big)\Big]^*(I+T^+\delta T)^{-1}T^+\Big[I-\delta T\big((I-TT^+)\delta T\big)^+\Big]\\
&\quad+\big((I-TT^+)\delta T\big)^+ +(I+T^+\delta T)^{-1}T^+.
\end{aligned}
$$

It follows from $(I+T^+\delta T)^{-1}T^+=T^+-T^+\delta T(I+T^+\delta T)^{-1}T^+$ that

$$
\begin{aligned}
\widetilde{T}^+-T^+=&\big((I-TT^+)\delta T\big)^+-T^+\delta T(I+T^+\delta T)^{-1}T^++(I+T^+\delta T)^{-1}\\
&\times\Big[I+(I-\nu_T)^*\omega_T^*\omega_T(I-\nu_T)\Big]^{-1}(I-T^+T)\\
&\times\Big[(I+T^+\delta T)^{-1}T^+\delta T\Big(I-\nu_T\Big)\Big]^*(I+T^+\delta T)^{-1}T^+\\
&\times\Big[I-\delta T\big((I-TT^+)\delta T\big)^+\Big].
\end{aligned}
$$

Since $\|(I+A^*A)^{-1}\|\le 1$ for arbitrary operator $A\in\mathcal{B}(\mathcal{H},\mathcal{K})$ and $\|I-\big((I-TT^+)\delta T\big)^+(I-TT^+)\delta T\|\le 1$, we arrive at

$$
\begin{aligned}
&\|\widetilde{T}^+-T^+\|\\
&\le\|\big((I-TT^+)\delta T\big)^+\|+\frac{\|T^+\delta T\|\|T^+\|}{1-\|T^+\delta T\|}+\frac{1}{1-\|T^+\delta T\|}\\
&\quad\times\frac{\|T^+\delta T\|}{1-\|T^+\delta T\|}\frac{\|T^+\|\|I-\delta T\big((I-TT^+)\delta T\big)^+\|}{1-\|T^+\delta T\|}\\
&=\|\big((I-TT^+)\delta T\big)^+\|+\frac{\|T^+\delta T\|\|T^+\|}{1-\|T^+\delta T\|}\left(1+\frac{\|I-\delta T\big((I-TT^+)\delta T\big)^+\|}{(1-\|T^+\delta T\|)^2}\right).
\end{aligned}
$$
□

In addition, if $\delta_4=0$ and $\delta_2\delta_3^+=0$, we can get the following corollary.

Corollary 2. *Suppose that*

$$\|T^+\delta T\|<1,\quad (I-TT^+)\delta TT^+T=0,\quad TT^+\delta T[(I-T^+T)\delta T]^+=0.$$

Then $\widetilde{T}^+$ *exists if and only if* $\mathcal{R}\left((I-TT^+)\delta T\right)$ *is closed. In this case,*

$$\widetilde{T}^+=(I+\omega_T^*)T^+T(I+\omega_T\omega_T^*)^{-1}(I+T^+\delta T)^{-1}T^++\big((I-TT^+)\delta T\big)^+$$

with

$$\frac{\|\widetilde{T}^+-T^+\|}{\|T^+\|}\le\frac{\|\big((I-TT^+)\delta T\big)^+\|}{\|T^+\|}+\frac{\|T^+\delta T\|}{1-\|T^+\delta T\|}\left[1+\frac{1}{(1-\|T^+\delta T\|)^2}\right].$$

Proof. Let T and δT have the form as Eqn. (1). Then

$$\delta TT^+T = TT^+\delta TT^+T \Longrightarrow \delta_4 = 0,$$
$$TT^+\delta T[(I-T^+T)\delta T]^+ = 0 \Longrightarrow \delta_2\delta_3^+ = 0.$$

So by Eqn. (3) we have

$$\begin{aligned} \widetilde{T}^+ &= \begin{pmatrix} \delta_3^+ & \delta_2^*(T_1^*+\delta_1^*)^{-1}\triangle_0(T_1+\delta_1)^{-1} \\ 0 & \triangle_0(T_1+\delta_1)^{-1} \end{pmatrix} \\ &= \begin{pmatrix} \delta_3^+ & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} I & \delta_2^*(T_1^*+\delta_1^*)^{-1} \\ 0 & I \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & I \end{pmatrix} \\ &\quad \times \begin{pmatrix} I & 0 \\ 0 & \triangle_0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 0 & (T_1+\delta_1)^{-1} \end{pmatrix}, \end{aligned} \tag{4}$$

where

$$\triangle_0 = \left[I + (T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\right]^{-1}.$$

Note that $\delta_3^+ \oplus 0 = \big((I-TT^+)\delta T\big)^+$, $0 \oplus (T_1+\delta_1)^{-1} = (I+T^+\delta T)^{-1}T^+$,
$\begin{pmatrix} I & \delta_2^*(T_1^*+\delta_1^*)^{-1} \\ 0 & I \end{pmatrix} = \left[I + \Big(I+T^+\delta T\Big)^{-1} T^+\delta T\Big(I-T^+T\Big)\right]^* = I+\omega_T^*$,
and

$$\begin{aligned} I \oplus \triangle_0 &= \left\{ I + \left[\Big(I+T^+\delta T\Big)^{-1} T^+\delta T\Big(I-T^+T\Big)\right] \right. \\ &\quad \left. \times \left[\Big(I+T^+\delta T\Big)^{-1} T^+\delta T\Big(I-T^+T\Big)\right]^* \right\}^{-1} \\ &= (I+\omega_T\omega_T^*)^{-1}. \end{aligned}$$

The result then follows from these expressions. This completes the proof. □

Comparing our Theorem 1 and Corollary 2 with Theorem 2 in [19], we can see that

$$(I-TT^+)\delta T = 0 \Longleftrightarrow \mathcal{R}(\delta T) \subset \mathcal{R}(T)$$

and Theorem 2 in [19] become a particular case of Theorem 1 and Corollary 2. So Theorem 1 gives an improvement over that of [19].

Corollary 3 ([19])**.** *Suppose that*

$$\mathcal{R}(\delta T) \subset \mathcal{R}(T), \quad \|T^+\delta T\| < 1.$$

Then $\widetilde{T}^+$ *exists,*

$$\widetilde{T}^+ = (I+\omega_T^*)T^+T(I+\omega_T\omega_T^*)^{-1}(I+T^+\delta T)^{-1}T^+,$$

with

$$\frac{\|\widetilde{T}^+ - T^+\|}{\|T^+\|} \le \frac{\|T^+\delta T\|}{1-\|T^+\delta T\|}\left[1 + \frac{1}{(1-\|T^+\delta T\|)^2}\right].$$

Moreover, if $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ is injective with closed range, then $T^+T = I$. The following corollary are the special case of Theorem 1, Corollary 2, and Corollary 3.

Corollary 4 ([11])**.** *Suppose that $T \in \mathcal{B}(\mathcal{H}, \mathcal{K})$ is injective with closed range. If $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$ and $\|T^+\delta T\| < 1$, then $\widetilde{T}^+$ is injective with closed range. Moreover, $\mathcal{R}(\widetilde{T}) = \mathcal{R}(T)$, $\widetilde{T}^+ = (I + T^+\delta T)^{-1}T^+ = T^+(I + \delta TT^+)^{-1}$ and*

$$\frac{\|\widetilde{T}^+ - T^+\|}{\|T^+\|} \le \frac{\|T^+\delta T\|}{1 - \|T^+\delta T\|}.$$

Similarly, if $\delta_2 = 0$, then $\widetilde{T}^*$ and T^* satisfy the conditions of Theorem 1. It is easy to see the following result holds.

Theorem 5. *Suppose that*

$$\|\delta TT^+\| < 1, \quad TT^+\delta T(I - T^+T) = 0.$$

Then $\widetilde{T}^+$ exists if and only if $\mathcal{R}\big(\delta T(I - T^+T)\big)$ is closed. In this case,

$$\begin{aligned}\widetilde{T}^+ = \big(\delta T(I - T^+T)\big)^+ &+ \Big[I - \big(\delta T(I - T^+T)\big)^+\delta T\Big]T^+(I + \delta TT^+)^{-1}\\ &\times \Big[I + {\omega'_T}^*(I - \nu'_T)^*(I - \nu'_T)\omega'_T\Big]^{-1} TT^+\big(I + {\omega'_T}^*(I - \nu'_T)^*\big)\end{aligned}$$

and

$$\|\widetilde{T}^+ - T^+\| \le \|\big(\delta T(I - T^+T)\big)^+\| + \frac{\|\delta TT^+\|\|T^+\|}{1 - \|\delta TT^+\|}\left(1 + \frac{\|I - \big(\delta T(I - T^+T)\big)^+\delta T\|}{(1 - \|\delta TT^+\|)^2}\right),$$

where $\omega'_T = (I - TT^+)\,\delta TT^+\,(I + \delta TT^+)^{-1}$ and $\nu'_T = \big(\delta T(I - T^+T)\big)\big(\delta T(I - T^+T)\big)^+$.

3. The bound of $\|\widetilde{T}\widetilde{T}^+ - TT^+\|$

In this section, we mainly study the perturbation on the case that $\delta_3 = \delta_4(T_1 + \delta_1)^{-1}\delta_2$. The explicit representations of projectors $\|\widetilde{T}\widetilde{T}^+ - TT^+\|$ and $\|\widetilde{T}^+\widetilde{T} - T^+T\|$ will be establish. As we know, if $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed, then the rule $(AB)^+ = B^+A^+$ is called the reverse order rule for the Moore-Penrose inverse (and it does not hold in general). In [1], it is shown that if $\mathcal{R}(A), \mathcal{R}(B)$ and $\mathcal{R}(AB)$ are closed, then the following statements are equivalent:

(a) $(AB)^+ = B^+A^+$;

(b) $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$.

Let ω_T, ν_T, M and N be defined as Eqn. (2). From above result, we can deduce the following perturbation result.

Theorem 6. *Suppose that*

$$\|\delta TT^+\| < 1, \quad (I - TT^+)(I + \delta TT^+)^{-1}\delta T(I - T^+T) = 0.$$

Then $\widetilde{T}^+$ exists. In this case,

$$\widetilde{T}^+ = (T^+T - M^*)(I + MM^*)^{-1}T^+(I + \delta TT^+)^{-1}(I + N^*N)^{-1}(TT^+ - N^*)$$

and

$$\|\widetilde{T}\| \le \frac{\|T^+\|}{1-\|\delta T T^+\|}.$$

Proof. Let T and δT have the representations as Eqn. (1). $\|T^+\delta T\| < 1$ implies that $T_1+\delta_1$ is invertible. If $(I-TT^+)(I+\delta TT^+)^{-1}\delta T(I-T^+T)=0$, then $\delta_3=\delta_4(T_1+\delta_1)^{-1}\delta_2$ and

$$\widetilde{T}=\begin{pmatrix} \delta_4(T_1+\delta_1)^{-1}\delta_2 & \delta_4 \\ \delta_2 & T_1+\delta_1 \end{pmatrix}=\begin{pmatrix} 0 & \delta_4 \\ 0 & T_1+\delta_1 \end{pmatrix}\begin{pmatrix} 0 & 0 \\ (T_1+\delta_1)^{-1}\delta_2 & I \end{pmatrix}.$$

Note that

$$M=T^+T(I+T^+\delta T)^{-1}(I-T^+T)=\begin{pmatrix} 0 & 0 \\ -(T_1+\delta_1)^{-1}\delta_2 & 0 \end{pmatrix},$$
$$N=(I-TT^+)(I+\delta TT^+)^{-1}TT^+=\begin{pmatrix} 0 & -\delta_4(T_1+\delta_1)^{-1} \\ 0 & 0 \end{pmatrix}.$$

Let

$$A=\begin{pmatrix} 0 & \delta_4 \\ 0 & T_1+\delta_1 \end{pmatrix}, \quad B=\begin{pmatrix} 0 & 0 \\ (T_1+\delta_1)^{-1}\delta_2 & I \end{pmatrix}. \tag{5}$$

Then

$$\begin{aligned} A^+ &= (A^*A)^+A^* \\ &= \begin{pmatrix} 0 & \delta_4(T_1+\delta_1)^{-1}\left[I+(T_1^*+\delta_1^*)^{-1}\delta_4^*\delta_4(T_1+\delta_1)^{-1}\right]^{-1}(T_1^*+\delta_1^*)^{-1} \\ 0 & \left[I+(T_1^*+\delta_1^*)^{-1}\delta_4^*\delta_4(T_1+\delta_1)^{-1}\right]^{-1}(T_1^*+\delta_1^*)^{-1} \end{pmatrix}^* \\ &= T^+(I+\delta TT^+)^{-1}(I+N^*N)^{-1}(TT^+-N^*). \end{aligned}$$

and

$$\begin{aligned} \|A^+\|^2 &= \|A^+A^{+*}\| \\ &= \left\|(T_1+\delta_1)^{-1}\left[I+(T_1^*+\delta_1^*)^{-1}\delta_4^*\delta_4(T_1+\delta_1)^{-1}\right]^{-1}(T_1^*+\delta_1^*)^{-1}\right\| \\ &\le \|(T_1+\delta_1)^{-1}\|^2 = \|T^+(I+\delta TT^+)^{-1}\|^2 \\ &\le \left(\frac{\|T^+\|}{1-\|\delta TT^+\|}\right)^2. \end{aligned}$$

Similarly, we have $B^+=B^*(BB^*)^+=(T^+T-M^*)(I+MM^*)^{-1}$ and

$$\|B^+\|^2=\|B^{+*}B^+\|=\left\|\left[I+(T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\right]^{-1}\right\|\le 1.$$

By a direct calculation, we obtain that

$$A^+A=0\oplus I, \quad BB^+=0\oplus I$$

and

$$A^+ABB^*A^* = BB^*A^*, \quad BB^+A^*AB = A^*AB.$$

It implies that $\mathcal{R}(A^*AB) \subset \mathcal{R}(B)$ and $\mathcal{R}(BB^*A^*) \subset \mathcal{R}(A^*)$. Hence the reverse order rule holds,

$$\begin{aligned}\widetilde{T}^+ &= (AB)^+ = B^+A^+ \\ &= (T^+T - M^*)(I + MM^*)^{-1}T^+(I + \delta TT^+)^{-1}(I + N^*N)^{-1}(TT^+ - N^*)\end{aligned}$$

and $\|\widetilde{T}\| \le \|B^+\|\|A^+\| \le \frac{\|T^+\|}{1-\|\delta TT^+\|}$. □

Next, we will give explicit expressions for $\|\widetilde{T}\widetilde{T}^+ - TT^+\|$ and $\|\widetilde{T}^+\widetilde{T} - T^+T\|$. A auxiliary result is summarized in the following lemma.

Lemma 7 ([1]). *If P and Q are two orthogonal projections, then*

$$\|P - Q\| = \max\{\|P(I - Q)\|, \|Q(I - P)\|\}.$$

Theorem 8. *Suppose that*

$$\|T^+\delta T\| < 1, \quad (I - TT^+)\delta TT^+T = 0.$$

Then $\widetilde{T}^+$ exists if and only if $\mathcal{R}\big((I - TT^+)\delta T\big)$ is closed. In this case,

$$\widetilde{T}\widetilde{T}^+ - TT^+ = P_{\mathcal{R}\left((I-TT^+)\delta T\right)}$$

and

$$\begin{aligned}&\|\widetilde{T}^+\widetilde{T} - T^+T\|^2 \\ =&\begin{cases} \max\left\{1 + \left\|\kappa^*\kappa(I + \kappa^*\kappa)^{-1}\right\|, \left\|\kappa\kappa^*(I + \kappa\kappa^*)^{-1}\right\|\right\} & \text{if } (I - TT^+)\delta T \neq \{0\}, \\ \max\left\{\left\|\kappa^*\kappa(I + \kappa^*\kappa)^{-1}\right\|, \left\|\kappa\kappa^*(I + \kappa\kappa^*)^{-1}\right\|\right\}, & \text{if } (I - TT^+)\delta T = \{0\},\end{cases}\end{aligned}$$

where $\kappa = \omega_T(I - \nu_T)$.

Proof. Let $T, \delta T$ and $\widetilde{T}^+$ have the representations as Eqn. (1) and Eqn. (3). Then a direct calculation can show that $\widetilde{T}\widetilde{T}^+ = \delta_3\delta_3^+ \oplus I = P_{\mathcal{R}((I-TT^+)\delta T)} \oplus I$ and $TT^+ = 0 \oplus I$. So the first result holds. From Eqn. (3) again we have $T^+T = 0 \oplus I$ and $\widetilde{T}^+\widetilde{T}$ has the matrix representation as:

$$\begin{pmatrix} \delta_3^+\delta_3 + (I - \delta_3^+\delta_3)\delta_2^*(T_1^* + \delta_1^*)^{-1}\triangle(T_1 + \delta_1)^{-1}\delta_2(I - \delta_3^+\delta_3) & (I - \delta_3^+\delta_3)\delta_2^*(T_1^* + \delta_1^*)^{-1}\triangle \\ \triangle(T_1 + \delta_1)^{-1}\delta_2(I - \delta_3^+\delta_3) & \triangle \end{pmatrix}$$

So we have

$$\widetilde{T}^+\widetilde{T}(I - T^+T) = \begin{pmatrix} \delta_3^+\delta_3 + (I - \delta_3^+\delta_3)\delta_2^*(T_1^* + \delta_1^*)^{-1}\triangle(T_1 + \delta_1)^{-1}\delta_2(I - \delta_3^+\delta_3) & 0 \\ \triangle(T_1 + \delta_1)^{-1}\delta_2(I - \delta_3^+\delta_3) & 0 \end{pmatrix}$$

and

$$T^+T(I - \widetilde{T}^+\widetilde{T}) = \begin{pmatrix} 0 & 0 \\ -\triangle(T_1 + \delta_1)^{-1}\delta_2(I - \delta_3^+\delta_3) & I - \triangle \end{pmatrix},$$

where $\triangle = \left[I + (T_1 + \delta_1)^{-1}\delta_2(I - \delta_3^+\delta_3)\delta_2^*(T_1^* + \delta_1^*)^{-1}\right]^{-1}$.

Let $X = (T_1+\delta_1)^{-1}\delta_2(I-\delta_3^+\delta_3)$ and $\kappa = \omega_T(I-\nu_T)$. By the proof of Theorem 1, we have $\|X\| = \|\kappa\|$. Hence

$$\begin{aligned}
&\|\widetilde{T}^+\widetilde{T}(I-T^+T)\|^2\\
&= \left\| \begin{pmatrix} \delta_3^+\delta_3 + (I-\delta_3^+\delta_3)\delta_2^*(T_1^*+\delta_1^*)^{-1}\triangle(T_1+\delta_1)^{-1}\delta_2(I-\delta_3^+\delta_3) & 0 \\ \triangle(T_1+\delta_1)^{-1}\delta_2(I-\delta_3^+\delta_3) & 0 \end{pmatrix} \right\|^2\\
&= \left\| \begin{pmatrix} \delta_3^+\delta_3 + X^*(I+XX^*)^{-1}X & 0 \\ (I+XX^*)^{-1}X & 0 \end{pmatrix} \right\|^2\\
&= \left\|(\delta_3^+\delta_3 + X^*(I+XX^*)^{-1}X)^2 + X^*(I+XX^*)^{-2}X\right\|\\
&= \left\|\delta_3^+\delta_3 + X^*(I+XX^*)^{-1}XX^*(I+XX^*)^{-1}X + X^*(I+XX^*)^{-2}X\right\|\\
&= \left\|\delta_3^+\delta_3 + X^*(I+XX^*)^{-1}X\right\|\\
&= \left\|\delta_3^+\delta_3\right\| + \left\|X^*(I+XX^*)^{-1}X\right\|\\
&= \left\|\delta_3^+\delta_3\right\| + \left\|X^*X(I+X^*X)^{-1}\right\|\\
&= \begin{cases} 1 + \left\|\kappa^*\kappa(I+\kappa^*\kappa)^{-1}\right\|, & \text{if } (I-TT^+)\delta T \neq \{0\}, \\ \left\|\kappa^*\kappa(I+\kappa^*\kappa)^{-1}\right\|, & \text{if } (I-TT^+)\delta T = \{0\}, \end{cases}
\end{aligned}$$

and

$$\begin{aligned}
\|T^+T(I-\widetilde{T}^+\widetilde{T})\|^2 &= \left\| \begin{pmatrix} 0 & 0 \\ -\triangle(T_1+\delta_1)^{-1}\delta_2(I-\delta_3^+\delta_3) & I-\triangle \end{pmatrix} \right\|^2\\
&= \left\| \begin{pmatrix} 0 & 0 \\ (I+XX^*)^{-1}X & I-(I+XX^*)^{-1} \end{pmatrix} \right\|^2\\
&= \left\|(I+XX^*)^{-1}XX^*(I+XX^*)^{-1} + (I-(I+XX^*)^{-1})^2\right\|\\
&= \left\|I-(I+XX^*)^{-1}\right\|\\
&= \left\|XX^*(I+XX^*)^{-1}\right\|\\
&= \left\|\kappa\kappa^*(I+\kappa\kappa^*)^{-1}\right\|.
\end{aligned}$$

By Lemma 7, the result holds. □

In particular, we can obtain the following corollary.

Corollary 9. *Suppose that $\mathcal{R}(\delta T) \subset \mathcal{R}(T)$ and $\|T^+\delta T\| < 1$. Then $\widetilde{T}\widetilde{T}^+ - TT^+ = 0$ and*

$$\|\widetilde{T}^+\widetilde{T} - T^+T\|^2 = \max\left\{ \left\|\omega_T^*\omega_T(I+\omega_T^*\omega_T)^{-1}\right\|, \left\|\omega_T\omega_T^*(I+\omega_T\omega_T^*)^{-1}\right\| \right\}.$$

From Theorem 8, a continuity of the Moore-Penrose inverse can be developed.

Corollary 10. *Let $T \in \mathcal{B}(\mathcal{H},\mathcal{K})$ with the Moore-Penrose inverse T^+. Let $T_n \in \mathcal{B}(\mathcal{H},\mathcal{K})$ satisfy $T_n \to T$, $(T_n - T)T^+T = TT^+(T_n - T)T^+T$ and $\mathcal{R}((I-TT^+)(T_n - T))$ closed for n large enough. Then $T_n^+ \to T^+$ if and only if $(I-TT^+)(T_n - T) = 0$.*

Proof. Since $T_n \to T$ and $T_n^+ \to T^+$, we get $\|P_{\mathcal{R}((I-TT^+)(T_n-T))}\| = \|T_nT_n^+ - TT^+\| < 1$ by Theorem 8. It implies that $P_{\mathcal{R}((I-TT^+)(T_n-T))} = 0$. Hence $(I-TT^+)(T_n-T) = 0$.

Conversely, if $(I-TT^+)(T_n-T)=0$, then Theorem 8 implies that

$$T_nT_n^+ - TT^+ = 0, \quad T_n^+T_n - T^+T \to 0 \text{ as } n \to \infty.$$

Since

$$T_n^+ - T^+ = T_n^+(T-T_n)T^+ + T_n^+(T_nT_n^+ - TT^+) + (T_n^+T_n - T^+T)T^+,$$

we get $T_n^+ \to T^+$ if $T_n \to T$. □

Theorem 11. *Suppose that*

$$\|\delta TT^+\| < 1, \quad (I-TT^+)(I+\delta TT^+)^{-1}\delta T(I-T^+T) = 0.$$

Then

$$\begin{aligned}\|\widetilde{T}^+\widetilde{T} - T^+T\|^2 &= \max\Big\{\|M^*M(I+M^*M)^{-1}\|, \|(I+MM^*)^{-1}MM^*\|\Big\},\\ \|\widetilde{T}\widetilde{T}^+ - TT^+\|^2 &= \max\Big\{\|N^*N(I+N^*N)^{-1}\|, \|(I+NN^*)^{-1}NN^*\|\Big\}.\end{aligned}$$

Proof. Let A and B be defined as Eqn. (5). From Theorem 6 we get

$$\begin{aligned}&\widetilde{T}^+\widetilde{T}(I-T^+T) = B^+A^+AB(I-T^+T) = B^+B(I-T^+T)\\ &= \begin{pmatrix} \delta_2^*(T_1^*+\delta_1^*)^{-1}\Big[I+(T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\Big]^{-1}(T_1+\delta_1)^{-1}\delta_2 & 0\\ \Big[I+(T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\Big]^{-1}(T_1+\delta_1)^{-1}\delta_2 & 0\end{pmatrix}\end{aligned}$$

and

$$\begin{aligned}&T^+T(I-\widetilde{T}^+\widetilde{T}) = T^+T(I-B^+B)\\ &= \begin{pmatrix} 0 & -\delta_2^*(T_1^*+\delta_1^*)^{-1}\Big[I+(T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\Big]^{-1}\\ 0 & I-\Big[I+(T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\Big]^{-1}\end{pmatrix}^*.\end{aligned}$$

Similar to the proof of Theorem 8, we get

$$\begin{aligned}&\|\widetilde{T}^+\widetilde{T}(I-T^+T)\|^2\\ &= \left\|\delta_2^*(T_1^*+\delta_1^*)^{-1}\Big[I+(T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\Big]^{-1}(T_1+\delta_1)^{-1}\delta_2\right\|\\ &= \|M^*(I+MM^*)^{-1}M\| = \|M^*M(I+M^*M)^{-1}\|\end{aligned}$$

and

$$\begin{aligned}&\|T^+T(I-\widetilde{T}^+\widetilde{T})\|^2\\ &= \left\|\Big[I+(T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\Big]^{-1}(T_1+\delta_1)^{-1}\delta_2\delta_2^*(T_1^*+\delta_1^*)^{-1}\right\|\end{aligned}$$

$$= \quad \|(I+MM^*)^{-1}MM^*\|.$$

Hence

$$\|\widetilde{T}^+\widetilde{T} - T^+T\|^2 = \max\left\{\|M^*M(I+M^*M)^{-1}\|, \|(I+MM^*)^{-1}MM^*\|\right\}.$$

In a similar vein, we can show that

$$\|\widetilde{T}\widetilde{T}^+ - TT^+\|^2 = \max\left\{\|N^*N(I+N^*N)^{-1}\|, \|(I+NN^*)^{-1}NN^*\|\right\}. \qquad \square$$

4. Conclusion remarks

In this paper, we present some perturbation bounds of $\|\widetilde{T}^+ - T^+\|$, $\|\widetilde{T}\widetilde{T}^+ - TT^+\|$ and $\|\widetilde{T}^+\widetilde{T} - T^+T\|$ under some conditions. It is natural to ask if we can remove these restrictions, which will be the future research topic.

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