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ON GENERALIZED λ_δ^s-SETS AND RELATED TOPICS TOPOLOGY

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Abstract. In this paper, we define and study the concept of Λ_δ^s-closure operator and the associated topology $\tau^{\Lambda_\delta^s}$ on a topological space (X,τ) in terms of $g.\Lambda_\delta^s$-sets.

1. Introduction and preliminaries

Levine [4] defined semiopen sets which are weaker than open sets in topological spaces. After Levine's semiopen sets, mathematicians gave different and interesting new modifications of open sets as well as generalized open sets. In 1968, Veličko [7] introduced δ-open sets, which are stronger than open sets, in order to investigate the characterization of H-closed spaces. In 1997, Park et al. [6] have introduced the notion of δ-semiopen sets which are stronger than semiopen sets but weaker than δ-open sets and investigated the relationships between several types of these open sets. Recently, Caldas et al. ([2], [3]) introduced the concept Λ_δ^s-sets (resp. V_δ^s-sets) which is the intersection of δ-semiopen (resp. union of δ-semiclosed) sets.

In this paper, we define a new closure operator and a new topology $\tau^{\Lambda_\delta^s}$ on a topological space (X,τ) by using generalized Λ_δ^s-sets and generalized V_δ^s-sets as an analogy of the sets introduced by H. Maki [5].

In what follows, (X,τ) and (Y,σ) (or X and Y) denote topological spaces on which no separation axioms are assumed unless explicitly stated. Let A be a subset of X. We denote the interior, the closure and the complement of a set A by $Int(A)$, $Cl(A)$ and $X\backslash A$ or A^c, respectively. A subset A of a topological space X is said to be δ-semiopen [6] if there exists a δ-open set U of X such that $U \subset A \subset Cl(U)$. The complement of a δ-semiopen set is called δ-semiclosed. The intersection (resp. union) of arbitrary collection of δ-semiclosed (resp. δ-semiopen) sets in (X,τ) is δ-semiclosed (resp. δ-semiopen). A point $x \in X$ is called the δ-semicluster point of A if $A \cap U \neq \emptyset$ for every δ-semiopen set U of X containing x. The set of all δ-semicluster points of A

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is called the δ-semiclosure of A, denoted by $\delta Cl_S(A)$ equivalently $\delta Cl_S(A) = \cap\{F \in \delta SC(X,\tau) : A \subset F\}$. We denote the collection of all δ-semiopen (resp. δ-semiclosed) sets by $\delta SO(X,\tau)$ (resp. $\delta SC(X,\tau)$).

Lemma 1.1 (Park et al. [6]). *Let A, B and A_i $(i \in I)$ be subsets of a space (X,τ), the following properties hold*:

(1) $A \subset \delta Cl_S(A)$.

(2) *If* $A \subset B$, *then* $\delta Cl_S(A) \subset \delta Cl_S(B)$.

(3) A *is* δ*-semiclosed if and only* $A = \delta Cl_S(A)$.

(4) $\delta Cl_S(A)$ *is* δ*-semiclosed.*

2. Λ^s_δ-sets and V^s_δ-sets

Recall that in a topological space (X,τ) a subset B is a Λ^s_δ-set (resp. V^s_δ-set) of (X,τ) [3] if $B = B^{\Lambda^s_\delta}$ (resp. $B = B^{V^s_\delta}$), where $B^{\Lambda^s_\delta} = \bigcap\{O : O \supseteq B, O \in \delta SO(X,\tau)\}$ and $B^{V^s_\delta} = \bigcup\{F : F \subseteq B, F^c \in \delta SO(X,\tau)\}$.

By Λ^s_δ (resp. V^s_δ), we denote the family of all Λ^s_δ-sets (resp. V^s_δ-sets) of (X,τ).

Definition 1. In a topological space (X,τ), a subset B is called:

(i) generalized Λ^s_δ-set ($= g.\Lambda^s_\delta$-set) of (X,τ) if $B^{\Lambda^s_\delta} \subseteq F$ whenever $B \subseteq F$ and $F \in \delta SC(X,\tau)$.

(ii) generalized V^s_δ-set ($= g.V^s_\delta$-set) of (X,τ) if B^c is a $g.\Lambda^s_\delta$-set of (X,τ).

By $D^{\Lambda^s_\delta}$ (resp. $D^{V^s_\delta}$), we will denote the family of all $g.\Lambda^s_\delta$-sets (resp. $g.V^s_\delta$-sets) of (X,τ).

Proposition 2.1. *Let A, B and $\{B_\lambda\colon \lambda \in \Omega\}$ be subsets of a topological space (X,τ). Then the following properties are valid*:

(a) $B \subseteq B^{\Lambda^s_\delta}$.

(b) *If* $A \subseteq B$, *then* $A^{\Lambda^s_\delta} \subseteq B^{\Lambda^s_\delta}$.

(c) $(B^{\Lambda^s_\delta})^{\Lambda^s_\delta} = B^{\Lambda^s_\delta}$. (*i.e.*, $B^{\Lambda^s_\delta} \in \Lambda^s_\delta$).

(d) $[\bigcup_{\lambda\in\Omega} B_\lambda]^{\Lambda^s_\delta} = \bigcup_{\lambda\in\Omega} B_\lambda^{\Lambda^s_\delta}$.

(e) *If* $A \in \delta SO(X,\tau)$, *then* $A = A^{\Lambda^s_\delta}$. (*i.e.*, $\delta SO(X,\tau) \subset \Lambda^s_\delta$).

(f) $(B^c)^{\Lambda^s_\delta} = (B^{V^s_\delta})^c$.

(g) $B^{V^s_\delta} \subseteq B$.

(h) *If* $B \in \delta SC(X,\tau)$, *then* $B = B^{V^s_\delta}$. (*i.e.*, $\delta SC(X,\tau) \subset V^s_\delta$).

(i) $[\bigcap_{\lambda\in\Omega} B_\lambda]^{\Lambda^s_\delta} \subseteq \bigcap_{\lambda\in\Omega} B_\lambda^{\Lambda^s_\delta}$.

(j) $[\bigcup_{\lambda\in\Omega} B_\lambda]^{V^s_\delta} \supseteq \bigcup_{\lambda\in\Omega} B_\lambda^{V^s_\delta}$.

Proof. (a) Clear by definition.

(b) Suppose that $x \notin B^{\Lambda^s_\delta}$. Then there exists a subset $O \in \delta SO(X,\tau)$ such that $O \supseteq B$ with $x \notin O$. Since $B \supseteq A$, then $x \notin A^{\Lambda^s_\delta}$ and thus $A^{\Lambda^s_\delta} \subseteq B^{\Lambda^s_\delta}$.

(c) Follows from (a) and definition.

(d) Suppose that there exists a point x such that $x \notin [\bigcup_{\lambda\in\Omega} B_\lambda]^{\Lambda_\delta^s}$. Then, there exists a subset $O \in \delta SO(X,\tau)$ such that $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq O$ and $x \notin O$. Thus, for each $\lambda \in \Omega$ we have $x \notin B_\lambda^{\Lambda_\delta^s}$. This implies that $x \notin \bigcup_{\lambda\in\Omega} B_\lambda^{\Lambda_\delta^s}$.

Conversely, suppose that there exists a point $x \in X$ such that $x \notin \bigcup_{\lambda\in\Omega} B_\lambda^{\Lambda_\delta^s}$. Then by Definition 1, there exist subsets $O_\lambda \in \delta SO(X,\tau)$ (for each $\lambda \in \Omega$) such that $x \notin O_\lambda$, $B_\lambda \subseteq O_\lambda$. Let $O = \bigcup_{\lambda\in\Omega} O_\lambda$. Then we have that $x \notin \bigcup_{\lambda\in\Omega} O_\lambda$, $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq O$ and $O \in \delta SO(X,\tau)$. This implies that $x \notin [\bigcup_{\lambda\in\Omega} B_\lambda]^{\Lambda_\delta^s}$. Thus, the proof of (d) is completed.

(e) By definition and since $A \in \delta SO(X,\tau)$, we have $A^{\Lambda_\delta^s} \subseteq A$. By (a) we have that $A^{\Lambda_\delta^s} = A$.

(f) $(B^{V_\delta^s})^c = \bigcap\{F^c : F^c \supseteq B^c, F^c \in \delta SO(X,\tau)\} = (B^c)^{\Lambda_\delta^s}$.

(g) Clear by definition.

(h) If $B \in \delta SC(X,\tau)$, then $B^c \in \delta SO(X,\tau)$. By (e) and (f): $B^c = (B^c)^{\Lambda_\delta^s} = (B^{V_\delta^s})^c$. Hence $B = B^{V_\delta^s}$.

(i) Suppose that there exists a point x such that $x \notin \bigcap_{\lambda\in\Omega} B_\lambda^{\Lambda_\delta^s}$. Then, there exists $\lambda \in \Omega$ such that $x \notin B_\lambda^{\Lambda_\delta^s}$. Hence there exists $O \in \delta SO(X,\tau)$ such that $O \supseteq B_\lambda$ and $x \notin O$. Thus $x \notin [\bigcap_{\lambda\in\Omega} B_\lambda]^{\Lambda_\delta^s}$.

(j)

$$\begin{aligned}[\bigcup_{\lambda\in\Omega} B_\lambda]^{V_\delta^s} &= [((\bigcup_{\lambda\in\Omega} B_\lambda)^c)^{\Lambda_\delta^s}]^c = [(\bigcap_{\lambda\in\Omega} B_\lambda^c)^{\Lambda_\delta^s}]^c \\ &\supseteq [\bigcap_{\lambda\in\Omega} (B_\lambda^c)^{\Lambda_\delta^s}]^c = [\bigcap_{\lambda\in\Omega} (B_\lambda^{V_\delta^s})^c]^c = \bigcup_{\lambda\in\Omega} B_\lambda^{V_\delta^s} \text{ (by (f) and (i))}.\end{aligned}$$

□

Remark 2.2. In general $(B_1 \bigcap B_2)^{\Lambda_\delta^s} \neq B_1^{\Lambda_\delta^s} \bigcap B_2^{\Lambda_\delta^s}$, as the following example shows.

Example 2.3. Let (X,τ) be a space with $X = \{a,b,c,d\}$ and $\tau = \{\emptyset, \{c\}, \{c,d\}, \{a,b\}, \{a,b,c\}, X\}$. Let $B_1 = \{a\}$ and $B_2 = \{b\}$. Then we have $(B_1 \bigcap B_2)^{\Lambda_\delta^s} = \emptyset$ but $B_1^{\Lambda_\delta^s} \bigcap B_2^{\Lambda_\delta^s} = \{a,b\}$.

Theorem 2.4. (a) *The subsets $\emptyset$ and X are Λ_δ^s-sets and V_δ^s-sets.*

(b) *Every union of Λ_δ^s-sets (resp. V_δ^s-sets) is a Λ_δ^s-set (resp. V_δ^s-set).*

(c) *Every intersection of Λ_δ^s-sets (resp. V_δ^s-sets) is a Λ_δ^s-set (resp. V_δ^s-set).*

(d) *A subset B is a Λ_δ^s-set if and only if B^c is a V_δ^s-set.*

Proof. (a) and (d) are obvious.

(b) Let $\{B_\lambda \colon \lambda \in \Omega\}$ be a family of Λ_δ^s-sets in a topological space (X,τ). Then by definition and Proposition 2.1(d),

$$\bigcup_{\lambda\in\Omega} B_\lambda = \bigcup_{\lambda\in\Omega} B_\lambda^{\Lambda_\delta^s} = [\bigcup_{\lambda\in\Omega} B_\lambda]^{\Lambda_\delta^s}.$$

(c) Let $\{B_\lambda : \lambda \in \Omega\}$ be a family of Λ_δ^s-sets in (X,τ). Then by Proposition 2.1(i) and definition, $[\bigcap_{\lambda\in\Omega} B_\lambda]^{\Lambda_\delta^s} \subseteq \bigcap_{\lambda\in\Omega} B_\lambda^{\Lambda_\delta^s} = \bigcap_{\lambda\in\Omega} B_\lambda$. Hence, by Proposition 2.1(a), $\bigcap_{\lambda\in\Omega} B_\lambda = [\bigcap_{\lambda\in\Omega} B_\lambda]^{\Lambda_\delta^s}$. □

Remark 2.5. By Theorem 2.4, Λ_δ^s (resp. V_δ^s) is a topology on X containing all δ-semiopen (resp. δ-semiclosed) sets. Clearly (X,Λ_δ^s) and (X,V_δ^s) are Alexandroff spaces [1], i.e., arbitrary intersections of open sets are open.

Proposition 2.6. *Let (X,τ) be a topological space. Then*
(a) *Every Λ_δ^s-set is a $g.\Lambda_\delta^s$-set.*
(b) *Every V_δ^s-set is a $g.V_\delta^s$-set.*
(c) *If $B_\lambda \in D^{\Lambda_\delta^s}$ for all $\lambda \in \Omega$, then $\bigcup_{\lambda\in\Omega} B_\lambda \in D^{\Lambda_\delta^s}$.*
(d) *If $B_\lambda \in D^{V_\delta^s}$ for all $\lambda \in \Omega$, then $\bigcap_{\lambda\in\Omega} B_\lambda \in D^{V_\delta^s}$.*

Proof. (a) Follows from definitions.

(b) Let B be a V_δ^s-set subset of X. Then, $B = B^{V_\delta^s}$. By Proposition 2.1(f) $(B^c)^{\Lambda_\delta^s} = (B^{V_\delta^s})^c = B^c$. Therefore, by (a) and Definition 1, B is a $g.V_\delta^s$-set.

(c) Let $B_\lambda \in D^{\Lambda_\delta^s}$ for all $\lambda \in \Omega$. Then, by Proposition 2.1(d) $[\bigcup_{\lambda\in\Omega} B_\lambda]^{\Lambda_\delta^s} = \bigcup_{\lambda\in\Omega} B_\lambda^{\Lambda_\delta^s}$. Hence, by hypothesis and Definition 1, $\bigcup_{\lambda\in\Omega} B_\lambda \in D^{\Lambda_\delta^s}$.

(d) Follows from (c) and Definition 1. □

In the following propositions, we give characterizations of $g.V_\delta^s$-sets by using V_δ^s-operations.

Proposition 2.7. *A subset B of a topological space (X,τ) is a $g.V_\delta^s$-set if and only if $U \subseteq V_\delta^s(B)$ whenever $U \subseteq B$ and $U \in \delta SO(X,\tau)$.*

Proof. Necessity. Let U be a δ-semiopen subset of (X,τ) such that $U \subseteq B$. Then since U^c is δ-semiclosed and $U^c \supseteq B^c$, we have $U^c \supseteq \Lambda_\delta^s(B^c)$ by Definition 1. Hence by Proposition 2.1(f) $U^c \supseteq (V_\delta^s(B))^c$. Thus, $U \subseteq V_\delta^s(B)$.

Sufficiency. Let F be a δ-semiclosed subset of (X,τ) such that $B^c \subseteq F$. Since F^c is δ-semiopen and $F^c \subseteq B$, by assumption we have $F^c \subseteq V_\delta^s(B)$. Then, $F \supseteq (V_\delta^s(B))^c = \Lambda_\delta^s(B^c)$ by Proposition 2.1(f). Thus B^c is a $g.\Lambda_\delta^s$-set, i.e., B is a $g.V_\delta^s$-set. □

Recall that a topological space (X,τ) is said to be δ-semi T_1 [2] if for each pair of distinct points x and y of X, there exist a δ-semiopen set U_x containing x but not y and a δ-semiopen set U_y containing y but not x, equivalently (X,τ) is δ-semi T_1 if and only if for each $x \in X$, the singleton $\{x\}$ is δ-semiclosed.

Theorem 2.8. *Let $\delta SC(X,\tau)$ be closed by unions. Then for a topological space (X,τ), the following properties are equivalent*:
(a) *(X,τ) is δ-semi T_1*;
(b) *Every subset of X is a Λ_δ^s-set*;
(c) *Every subset of X is a V_δ^s-set.*

Proof. It is obvious that (b) $\Leftrightarrow$ (c).

(a) $\Rightarrow$ (c): Let A be any subset of X. Since $A = \bigcup\{\{x\} : x \in A\}$, then A is the union of δ-semiclosed sets. Hence A is a V_δ^s-set.

(c) $\Rightarrow$ (a): Since by (c), we have that every singleton is the union of δ-semiclosed sets, i.e., it is δ-semiclosed, then (X,τ) is a δ-semi T_1 space. □

3. $C^{\Lambda_\delta^s}$-closure operator and the associated topology $\tau^{\Lambda_\delta^s}$

In this section, we define a closure operator $C^{\Lambda_\delta^s}$ and the associated topology $\tau^{\Lambda_\delta^s}$ on the topological spaces (X,τ) by using the family of $g.\Lambda_\delta^s$-sets.

Definition 2. For any subset B of a topological space (X,τ), define $C^{\Lambda_\delta^s}(B) = \bigcap\{U : B \subseteq U, U \in D^{\Lambda_\delta^s}\}$ and $Int^{V_\delta^s}(B) = \bigcup\{F : B \supseteq F, F \in D^{V_\delta^s}\}$.

Proposition 3.1. *Let A, B and $\{B_\lambda : \lambda \in \Omega\}$ be subsets of a topological space (X,τ). Then the following properties are valid*:

(a) $B \subseteq C^{\Lambda_\delta^s}(B)$.
(b) $C^{\Lambda_\delta^s}(B^c) = (Int^{V_\delta^s}(B))^c$.
(c) $C^{\Lambda_\delta^s}(\emptyset) = \emptyset$.
(d) $\bigcup_{\lambda\in\Omega} C^{\Lambda_\delta^s}(B_\lambda) = C^{\Lambda_\delta^s}(\bigcup_{\lambda\in\Omega} B_\lambda)$.
(e) $C^{\Lambda_\delta^s}(C^{\Lambda_\delta^s}(B)) = C^{\Lambda_\delta^s}(B)$.
(f) *If $A \subseteq B$, then $C^{\Lambda_\delta^s}(A) \subseteq C^{\Lambda_\delta^s}(B)$.*
(g) *If B is a $g.\Lambda_\delta^s$-set, then $C^{\Lambda_\delta^s}(B) = B$.*
(h) *If B is a $g.V_\delta^s$-set, then $Int^{V_\delta^s}(B) = B$.*

Proof. (a), (b) and (c): Clear.

(d) Suppose that there exists a point x such that $x \notin C^{\Lambda_\delta^s}(\bigcup_{\lambda\in\Omega} B_\lambda)$. Then, there exists a subset $U \in D^{\Lambda_\delta^s}$ such that $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq U$ and $x \notin U$. Thus, for each $\lambda \in \Omega$ we have $x \notin C^{\Lambda_\delta^s}(B_\lambda)$. This implies that $x \notin \bigcup_{\lambda\in\Omega} C^{\Lambda_\delta^s}(B_\lambda)$.

Conversely, we suppose that there exists a point $x \in X$ such that $x \notin \bigcup_{\lambda\in\Omega} C^{\Lambda_\delta^s}(B_\lambda)$. Then, there exist subsets $U_\lambda \in D^{\Lambda_\delta^s}$ for all $\lambda \in \Omega$ such that $x \notin U_\lambda$ and $B_\lambda \subseteq U_\lambda$. Let $U = \bigcup_{\lambda\in\Omega} U_\lambda$. From this and Proposition 2.6(c), we have that $x \notin U$, $\bigcup_{\lambda\in\Omega} B_\lambda \subseteq U$ and $U \in D^{\Lambda_\delta^s}$. Thus, $x \notin C^{\Lambda_\delta^s}(\bigcup_{\lambda\in\Omega} B_\lambda)$.

(e) Suppose that there exists a point $x \in X$ such that $x \notin C^{\Lambda_\delta^s}(B)$. Then there exists a subset $U \in D^{\Lambda_\delta^s}$ such that $x \notin U$ and $U \supseteq B$. Since $U \in D^{\Lambda_\delta^s}$ we have $C^{\Lambda_\delta^s}(B) \subseteq U$. Thus we have $x \notin C^{\Lambda_\delta^s}(C^{\Lambda_\delta^s}(B))$. Therefore $C^{\Lambda_\delta^s}(C^{\Lambda_\delta^s}(B)) \subseteq C^{\Lambda_\delta^s}(B)$. The converse containment relation is clear by (a).

(f) Clear.

(g) By (a) and Definition 1, the proof is clear.

(h) By Definition 1, by (g) and (b). □

Then we have the following:

Theorem 3.2. *$C^{\Lambda_\delta^s}$ is a Kuratowski closure operator on X.*

Definition 3. Let $\tau^{\Lambda_\delta^s}$ be the topology on X generated by $C^{\Lambda_\delta^s}$ in the usual manner, i.e., $\tau^{\Lambda_\delta^s} = \{B : B \subseteq X, C^{\Lambda_\delta^s}(B^c) = B^c\}$.

We define a family $\rho^{\Lambda_\delta^s}$ by $\rho^{\Lambda_\delta^s} = \{B : B \subseteq X, C^{\Lambda_\delta^s}(B) = B\}$, equivalently $\rho^{\Lambda_\delta^s} = \{B : B \subseteq X, B^c \in \tau^{\Lambda_\delta^s}\}$.

Proposition 3.3. *Let (X,τ) be a topological space. Then,*

(a) $\tau^{\Lambda_\delta^s} = \{B : B \subseteq X, Int^{V_\delta^s}(B) = B\}$.
(b) $\delta SO(X,\tau) \subseteq D^{\Lambda_\delta^s} \subseteq \rho^{\Lambda_\delta^s}$.

(c) $\delta SC(X,\tau) \subseteq D^{V_\delta^s} \subseteq \tau^{\Lambda_\delta^s}$.

(d) *If* $\delta SC(X,\tau) = \tau^{\Lambda_\delta^s}$, *then every* $g.\Lambda_\delta^s$*-set of* (X,τ) *is* δ*-semiopen.*

(e) *If every* $g.\Lambda_\delta^s$*-set of* (X,τ) *is* δ*-semiopen (i.e.,* $D^{\Lambda_\delta^s} \subseteq \delta SO(X,\tau)$*), then* $\tau^{\Lambda_\delta^s} = \{B : B \subseteq X, B = B^{V_\delta^s}\}$.

(f) *If every* $g.\Lambda_\delta^s$*-set of* (X,τ) *is* δ*-semiclosed (i.e.,* $D^{\Lambda_\delta^s} \subseteq \delta SC(X,\tau)$*), then* $\delta SO(X,\tau) = \tau^{\Lambda_\delta^s}$.

Proof. (a) By Definition 3 and Proposition 3.1(b) we have:

If $A \subset X$, then $A \in \tau^{\Lambda_\delta^s}$ if and only if $C^{\Lambda_\delta^s}(A^c) = A^c$ if and only if $(Int^{V_\delta^s}(A))^c = A^c$ if and only if $Int^{V_\delta^s}(A) = A$. Thus, $\tau^{\Lambda_\delta^s} = \{B : B \subseteq X, Int^{V_\delta^s}(B) = B\}$.

(b) Let $B \in \delta SO(X,\tau)$. By Proposition 2.1(e), B is a Λ_δ^s-set. By Proposition 2.6(a), B is a $g.\Lambda_\delta^s$-set, i.e., $B \in D^{\Lambda_\delta^s}$. Let now, B be any element of $D^{\Lambda_\delta^s}$. By Proposition 3.1(g) $B = C^{\Lambda_\delta^s}(B)$, i.e., $B \in \rho^{\Lambda_\delta^s}$. Therefore $\delta SO(X,\tau) \subseteq D^{\Lambda_\delta^s} \subseteq \rho^{\Lambda_\delta^s}$.

(c) Let $B \in \delta SC(X,\tau)$. By Proposition 2.1(h) $B = B^{V_\delta^s}$. Thus B is a V_δ^s-set. By Proposition 2.6(b), B is a $g.V_\delta^s$-set hence $B \in D^{V_\delta^s}$.

Now, if $B \in D^{V_\delta^s}$, then by (a) and Proposition 3.1(h), $B \in \tau^{\Lambda_\delta^s}$.

(d) Let B be any $g.\Lambda_\delta^s$-set, i.e., $B \in D^{\Lambda_\delta^s}$. By (b), $B \in \rho^{\Lambda_\delta^s}$. Thus, $B^c \in \tau^{\Lambda_\delta^s}$. From the assumption, we have $B^c \in \delta SC(X,\tau)$. Hence $B \in \delta SO(X,\tau)$.

(e) Let $A \subset X$ and $A \in \tau^{\Lambda_\delta^s}$. Then by Definition 3 and Definition 2,

$$\begin{aligned} A^c = C^{\Lambda_\delta^s}(A^c) &= \bigcap\{U : U \supseteq A^c, U \in D^{\Lambda_\delta^s}\} \\ &= \bigcap\{U : U \supseteq (A^c), U \in \delta SO(X,\tau)\} = (A^c)^{\Lambda_\delta^s}. \end{aligned}$$

Using Proposition 2.1(f), we have $A = A^{V_\delta^s}$, i.e., $A \in \{B : B \subseteq X, B = B^{V_\delta^s}\}$.

Conversely, if $A \in \{B : B \subset X, B = B^{V_\delta^s}\}$, then by Proposition 2.6(b) A is a $g.V_\delta^s$-set. Thus $A \in D^{V_\delta^s}$. By using (c), $A \in \tau^{\Lambda_\delta^s}$.

(f) Let $A \subset X$ and $A \in \tau^{\Lambda_\delta^s}$. Then

$$\begin{aligned} A = (C^{\Lambda_\delta^s}(A^c))^c &= (\bigcap\{U : A^c \subseteq U, U \in D^{\Lambda_\delta^s}\})^c \\ &= \bigcup\{U^c : U^c \in \delta SO(X,\tau)\} \in \delta SO(X,\tau). \end{aligned}$$

Conversely, if $A \in \delta SO(X,\tau)$, then by Proposition 2.1(e) and Proposition 2.6(a), $A \in D^{\Lambda_\delta^s}$. By assumption $A \in \delta SC(X,\tau)$. By using (c), $A \in \tau^{\Lambda_\delta^s}$. □

Lemma 3.4. *Let* (X,τ) *be a topological space.*

(a) *For each* $x \in X$, $\{x\}$ *is a* δ*-semiopen set or* $\{x\}^c$ *is a* $g.\Lambda_\delta^s$*-set of* (X,τ).

(b) *For each* $x \in X$, $\{x\}$ *is a* δ*-semiopen set or* $\{x\}$ *is a* $g.V_\delta^s$*-set of* (X,τ).

Proof. (a) Suppose that $\{x\}$ is not δ-semiopen. Then the only δ-semiclosed set F containing $\{x\}^c$ is X. Thus $\Lambda_\delta^s(\{x\}^c) \subseteq F = X$ and $\{x\}^c$ is a $g.\Lambda_\delta^s$-set of (X,τ).

(b) It follows from (a) and Definition 2. □

Proposition 3.5. *If $\delta SO(X,\tau) = \tau^{\Lambda_\delta^s}$, then every singleton $\{x\}$ of X is $\tau^{\Lambda_\delta^s}$-open.*

Proof. Suppose that $\{x\}$ is not δ-semiopen in (X,τ). Then by Lemma 3.4, $\{x\}^c$ is a $g.\Lambda_\delta^s$-set. Thus $\{x\} \in \tau^{\Lambda_\delta^s}$ by Definition 3. Suppose that $\{x\}$ is a δ-semiopen in (X,τ). Then $\{x\} \in \delta SO(X,\tau) = \tau^{\Lambda_\delta^s}$. Therefore, every singleton $\{x\}$ is $\tau^{\Lambda_\delta^s}$-open. □

Recall that a subset B of a topological space (X,τ) is said to be δ-semigeneralized closed set (briefly δsg-closed) [3] if $\delta Cl_s(B) \subseteq O$ holds whenever $B \subseteq O$ and $O \in \delta SO(X,\tau)$. Every δ-semiclosed sets is δsg-closed but the converse is not true.

Definition 4. A topological space (X,τ) is said to be a δ-semi $\mathrm{T}_{\frac{1}{2}}$-space [3] if every δsg-closed set in (X,τ) is δ-semiclosed in (X,τ).

We conclude our paper with characterization of δ-semi $\mathrm{T}_{\frac{1}{2}}$-spaces.

Theorem 3.6. *Let (X,τ) be a topological space. Then the following conditions are equivalent*:

(a) *(X,τ) is a δ-semi $T_{\frac{1}{2}}$-space.*
(b) *Every $g.V_\delta^s$-set is a V_δ^s-set.*
(c) *Every $\tau^{\Lambda_\delta^s}$-open set is a V_δ^s-set.*

Proof. (a)⇒(b): Suppose that there exists a $g.V_\delta^s$-set B which is not a V_δ^s-set. Since $B^{V_\delta^s} \subseteq B$ $(B^{V_\delta^s} \neq B)$, then there exists a point $x \in B$ such that $x \notin B^{V_\delta^s}$. Then the singleton $\{x\}$ is not δ-semiclosed. Hence only δ-semiopen set O containing $\{x\}^c$ is X. Thus $\delta Cl_s(\{x\}^c) \subseteq F = X$ and $\{x\}^c$ is a δsg-closed set. On the other hand, we have that $\{x\}$ is not δ-semiopen (since B is a $g.V_\delta^s$-set, $x \notin B^{V_\delta^s}$ and Proposition 2.7). Therefore, we have that $\{x\}^c$ is not δ-semiclosed but it is a δsg-closed set. This contradicts the assumption that (X,τ) is a δ-semi $\mathrm{T}_{\frac{1}{2}}$-space.

(b)⇒(a): Suppose that (X,τ) is not a δ-semi $\mathrm{T}_{\frac{1}{2}}$ space. Then, there exists a δsg-closed set B which is not δ-semiclosed. Since B is not δ-semiclosed, there exists a point x such that $x \notin B$ and $x \in \delta Cl_s(B)$. By Proposition 3.4, we have the singleton $\{x\}$ is a δ-semiopen set or it is a $g.V_\delta^s$-set. When $\{x\}$ is δ-semiopen, we have $\{x\} \bigcap B \neq \emptyset$ because $x \in \delta Cl_s(B)$. This is a contradiction. Let us consider the case: $\{x\}$ is a $g.V_\delta^s$-set. If $\{x\}$ is not δ-semiclosed, we have $\{x\}^{V_\delta^s} = \emptyset$ and hence $\{x\}$ is not a V_δ^s-set. This contradicts to (b). Next, if $\{x\}$ is δ-semiclosed, we have $\{x\}^c \supseteq \delta Cl_s(B)$ (i.e., $x \notin \delta Cl_s(B)$). In fact, the δ-semiopen set $\{x\}^c$ contains the set B which is a δsg-closed set. Then, this also contradicts to the fact that $x \in \delta Cl_s(B)$. Therefore, (X,τ) is a δ-semi $\mathrm{T}_{\frac{1}{2}}$-space.

(b)⇒(c): Let B be a $\tau^{\Lambda_\delta^s}$-open set, i.e., $B = Int^{V_\delta^s}(B)$ (Proposition 3.3(a)). It is enough to show that, $Int^{V_\delta^s}(B)$ is a V_δ^s-set, i.e., $(Int^{V_\delta^s}(B))^{V_\delta^s} = Int^{V_\delta^s}(B)$. Let $\Omega_{V_\delta^s} = \{B : B \text{ is a } V_\delta^s\text{-set}\}$. By Proposition 2.6(b) and assumption (b) we have that $\Omega_{V_\delta^s} = D^{V_s}$. Therefore by Definition 2, Proposition 2.1(j) and the fact that $\Omega_{V_\delta^s} = D^{V_\delta^s}$ we have: $(Int^{V_\delta^s}(B))^{V_\delta^s} = (\bigcup\{F : B \supseteq F, F \in D^{V_\delta^s}\})^{V_\delta^s} = (\bigcup\{F : B \supseteq F, F \in \Omega_{V_\delta^s}\})^{V_\delta^s} \supseteq \bigcup\{F^{V_\delta^s} : B \supseteq F, F \in \Omega_{V_\delta^s}\} = \bigcup\{F : B \supseteq F, F \in D^{V_\delta^s}\} = Int^{V_\delta^s}(B)$. By Proposition 2.1(g), we have $(Int^{V_\delta^s}(B))^{V_\delta^s} = Int^{V_\delta^s}(B)$. Hence $Int^{V_\delta^s}(B)$ is a V_δ^s-set.

(c)⇒(b): Let B be a $g.V_\delta^s$-set. Then $Int^{V_\delta^s}(B) = B$ by Proposition 3.1(h). It follows from the fact in Proposition 3.3(a) that $B \in \tau^{V_\delta^s}$. Hence B is a V_δ^s-set, by (c). □

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