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A STRUCTURE ON COEFFICIENTS OF NILPOTENT POLYNOMIALS

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Abstract. We observe a structure on the products of coefficients of nilpotent polynomials, introducing the concept of *n-semi-Armendariz* that is a generalization of Armendariz rings. We first obtain a classification of reduced rings, proving that a ring R is reduced if and only if the n by n upper triangular matrix ring over R is n-semi-Armendariz. It is shown that n-semi-Armendariz rings need not be $(n+1)$-semi-Armendariz and vice versa. We prove that a ring R is n-semi-Armendariz if and only if so is the polynomial ring over R. We next study interesting properties and useful examples of n-semi-Armendariz rings, constructing various kinds of counterexamples in the process.

1. n-semi-Armendariz rings

Throughout this paper all rings are associative with identity unless otherwise stated. The polynomial ring with an indeterminate x over a ring R is denoted by $R[x]$.

A ring is called *reduced* if it has no nonzero nilpotent elements. For a reduced ring R Armendariz [3, Lemma 1] proved that

$$(*) \qquad a_ib_j = 0 \text{ for all } i, j \text{ whenever } f(x)g(x) = 0,$$

where $f(x) = \sum_{i=0}^{m} a_ix^i$, $g(x) = \sum_{j=0}^{n} b_jx^j$ are in $R[x]$. Rege et al. [14] called a ring (not necessarily reduced) *Armendariz* if it satisfies $(*)$. Reduced rings are Armendariz by [3, Lemma 1]. The structure of Armendariz rings was observed by many authors containing Anderson et al. [1], Hirano [4], Huh et al. [6], Kim et al. [8], Lee et al. [11], Rege et al. [14], etc. A ring is called *abelian* if every idempotent is central. Armendariz rings are abelian by the proof of [1, Theorem 6] or [6, Corollary 8].

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A ring R is called *n-semi-Armendariz* provided that if $f(x) = a_0 + a_1x + \cdots + a_mx^m$ in $R[x]$ satisfies $f(x)^n = 0$, then $a_{i_1}a_{i_2}\cdots a_{i_n} = 0$ for any choice of a_{i_j}'s in $\{a_0, \ldots, a_m\}$, where $j = 1, \ldots, n$ (of course $n \geq 2$). A ring is called *semi-Armendariz* if it is n-semi-Armendariz for all $n \geq 2$. Armendariz rings are semi-Armendariz by [1, Proposition 1], but the converse need not hold since the 2 by 2 upper triangular matrix ring over a reduced ring is semi-Armendariz by Theorem 1.2 below and it is non-abelian.

Let R be a ring and n be a positive integer. Let $\mathrm{Mat}_n(R)$ denote the n by n matrix ring over R and I_n be the identity of $\mathrm{Mat}_n(R)$. We use $U_n(R)$ (resp. $L_n(R)$) to denote the n by n upper (resp. lower) triangular matrix ring over R. E_{ij} denotes the n by n matrix with (i, j)-entry 1 and zero elsewhere. Next define

$$D_n(R) = \{M \in U_n(R) \mid \text{ the diagonal entries of } M \text{ are equal}\}.$$

According to [12], define $RA = \{rA \mid r \in R\}$ for $A \in \mathrm{Mat}_n(R)$ and $V = \sum_{i=1}^{n-1} E_{i(i+1)} \in U_n(R)$. $N(R)$ denotes the set of all nilpotent elements in R.

Lemma 1.1. (1) *Let R be a reduced ring, n be any positive integer and $r_i \in R$ for $i = 1, \ldots, n$. Then $r_1r_2\cdots r_n = 0$ implies $r_{\sigma(1)}Rr_{\sigma(2)}R\cdots Rr_{\sigma(n)} = 0$ for any permutation σ of the set $\{1, 2, \ldots, n\}$.*

(2) *The class of (n-semi-)Armendariz rings is closed under subrings.*

(3) *Any direct product of n-semi-Armendariz rings is n-semi-Armendariz.*

(4) *Any direct sum of n-semi-Armendariz rings is n-semi-Armendariz.*

(5) *Let R be a ring and $n \geq 2$. Then R is reduced if and only if $R[x]/(x^n)$ is Armendariz if and only if $RI_n + RV + \cdots + RV^{n-1}$ (a subring of $D_n(R)$) is Armendariz if and only if $D_3(R)$ is Armendariz if and only if $D_2(R)$ is Armendariz, where (x^n) is the ideal of $R[x]$ generated by x^n.*

(6) *A ring R is Armendariz if and only if $f_1\cdots f_n = 0$ implies $a_1\cdots a_n = 0$, where $f_1, \ldots, f_n \in R[x]$ and a_i is any coefficient of f_i.*

(7) *If a ring R is semi-Armendariz, then $N(R[x]) \subseteq N(R)[x]$.*

Proof. (1) From the reducedness of R we obtain by [2, Theorem I.3] that $r_1r_2r_3 = 0$ implies $r_{\sigma(1)}r_{\sigma(2)}r_{\sigma(3)} = 0$ for any permutation σ of the set $\{1, 2, 3\}$, and that $ab = 0$ implies $aRb = 0$ for $a, b \in R$. Thus the result is proved by [10, Proposition 1] or [2, Theorem I.1].

(2) is obtained from the definition.

(3) For a polynomial $f(x) \in R[x]$, C_f denotes the set of coefficients of $f(x)$. Let R_i be an n-semi-Armendariz ring for $i \in I$ and let $T = \prod_{i\in I} R_i$ be the direct product of R_i's. Consider $f(x) = \sum_{j=0}^{m} a_jx^j$ in $T[x]$ such that $f(x)^n = 0$, where $a_j = (\alpha_{ij}) \in T$ and $\alpha_{ij} \in R_i$. And we put $f_i(x) = \sum_{j=0}^{m} \alpha_{ij}x^j$ in $R_i[x]$ for $i \in I$. Then, from $f(x)^n = 0$, $f_i(x)^n = 0$ for all $i \in I$. Since every R_i is n-semi-Armendariz, $\alpha_{i_1}\alpha_{i_2}\cdots\alpha_{i_n} = 0$ for any α_{i_j}'s in C_{f_i}. Thus $a_{s_1}a_{s_2}\cdots a_{s_n} = 0$ for all a_{s_k}'s in $\{a_0, \ldots, a_m\}$, where $k = 1, \ldots, n$.

(4) is obtained by (2) and (3).

(5) It is proved by [1, Theorem 5] that R is reduced if and only if $R[x]/(x^n)$ is Armendariz for any $n \geq 2$. It is obtained from the corresponding $x + (x^n) \mapsto V$ that $R[x]/(x^n)$ is Armendariz if and only if $RI_n + RV + \cdots + RV^{n-1}$ is Armendariz. It is proved by [8, Proposition 2] that $D_3(R)$ is Armendariz when R is reduced. $D_2(R)$ is Armendariz by (2) when so is $D_3(R)$. R is reduced by [11, Theorem 2.3] when $D_2(R)$ is Armendariz.

(6) is obtained from [1, Proposition 1] and the definition.

(7) Let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ with $f(x)^n = 0$. Since R is n-semi-Armendariz, $a_i^n = 0$ for all i. □

Due to Marks [13], a ring R is called *NI* if $N(R)$ forms an ideal of R. Note that R is NI if and only if $R/N^*(R)$ is a reduced ring, where $N^*(R)$ is the upper nilradical of R. If R is NI, then $R/N(R)[x] \cong R[x]/N(R)[x]$ implies $N(R[x]) \subseteq N(R)[x]$. The converse of Lemma 1.1(7) need not be true by Example 1.6 below as can be seen by $U_n(R)$ $(n \geq 4)$ over a reduced ring R since $U_n(R)$ is NI. [7, Example 1.2] shows that there is an NI ring that is not n-semi-Armendariz for any n, with the help of Example 1.6 below. If a ring R is Armendariz and NI, then $N(R)$ is the sum of all nilpotent ideals in R by [9, Lemma 2.3(5)]. While if a ring R is semi-Armendariz and NI, then $N(R[x])$ is contained in a proper ideal of $R[x]$ by Lemma 1.1(7).

For a ring R and a positive integer n define

$$N_n(R) = \{A \in U_n(R) \mid \text{ each diagonal entry of } A \text{ is zero}\}.$$

Theorem 1.2. *Let R be a ring and n be a positive integer. Then the following conditions are equivalent*:

(1) *R is reduced*;

(2) *$U_h(R)$ is n-semi-Armendariz for $h = 1, 2, \ldots, n+1$*;

(3) *$U_n(R)$ is n-semi-Armendariz*;

(4) *$L_h(R)$ is n-semi-Armendariz for $h = 1, 2, \ldots, n+1$*;

(5) *$L_n(R)$ is n-semi-Armendariz.*

Proof. (1)⇒(2): Suppose that R is reduced. It suffices to prove that $U_{n+1}(R)$ is n-semi-Armendariz by Lemma 1.1(2). Let $f(x) = A_0 + A_1 x + \cdots + A_m x^m \in U_{n+1}(R)[x]$ with $f(x)^n = 0$ $(n \geq 2)$. Write

$$A_i = (a(i)_{uv}) \text{ for } i = 0, 1, \ldots, m \text{ with } a(i)_{uv} = 0 \text{ for } u > v.$$

We will use the reducedness of R freely. From $f(x)^n = 0$, we have the system of equations

$$\sum_{s_1+s_2+\cdots+s_n=k} A_{s_1} A_{s_2} \cdots A_{s_n} = 0 \text{ for } k = 0, 1, \ldots, mn.$$

From $A_0^n = 0$ and $A_m^n = 0$, we have $a(0)_{11} = \cdots = a(0)_{(n+1)(n+1)} = 0$ and $a(m)_{11} = \cdots = a(m)_{(n+1)(n+1)} = 0$, entailing $A_0, A_m \in N_{n+1}(R)$. In $\sum_{s_1+\cdots+s_n=n} A_{s_1} A_{s_2} \cdots A_{s_n}$, any term, except A_1^n, contains A_0 as a factor, and so it is contained in $N_{n+1}(R)$ from $A_0 \in N_{n+1}(R)$. Consequently $A_1^n \in$

$N_{n+1}(R)$ and so we get $A_1 \in N_{n+1}(R)$. We proceed by induction on $i = 0, 1, \ldots, m-1$. In $\sum_{s_1+\cdots+s_n=in} A_{s_1}A_{s_2}\cdots A_{s_n}$, any term (except A_i^n) contains A_j with $j < i$ as a factor, and so it is contained in $N_{n+1}(R)$ by the induction hypothesis. Consequently $A_i^n \in N_{n+1}(R)$ and then $A_i \in N_{n+1}(R)$. Therefrom we have

$$a(i)_{11} = a(i)_{22} = \cdots = a(i)_{(n+1)(n+1)} = 0$$

for $i = 0, 1, \ldots, m$ and it follows that

$$A_{s_1}A_{s_2}\cdots A_{s_n} = (a(s_1)_{12}a(s_2)_{23}\cdots a(s_n)_{n(n+1)})E_{1(n+1)}$$

for any choice of s_i's. This result implies the system of equations

$$\sum_{s_1+s_2+\cdots+s_n=k} a(s_1)_{12}a(s_2)_{23}\cdots a(s_n)_{n(n+1)} = 0 \text{ for } k = 0, 1, \ldots, mn.$$

If we multiply the equation $\sum_{s_1+s_2+\cdots+s_n=1} a(s_1)_{12}a(s_2)_{23}\cdots a(s_n)_{n(n+1)} = 0$ on the right side by $a(0)_{12}\cdots a(0)_{(i-1)i}a(0)_{(i+1)(i+2)}\cdots a(0)_{n(n+1)}$, then from $a(0)_{12}\cdots\ a(0)_{n(n+1)} = 0$ and Lemma 1.1(1) we obtain

$$(a(0)_{12}\cdots a(0)_{(i-1)i}a(1)_{i(i+1)}a(0)_{(i+1)(i+2)}\cdots a(0)_{n(n+1)})$$
$$(a(0)_{12}\cdots a(0)_{(i-1)i}a(0)_{(i+1)(i+2)}\cdots a(0)_{n(n+1)}) = 0$$

for $i = 1, \ldots, n$ since every other term contains $a(0)_{i(i+1)}$ for $i = 1, 2, \ldots, n$ as factors. It then follows that

$$(a(0)_{12}\cdots a(0)_{(i-1)i}a(1)_{i(i+1)}a(0)_{(i+1)(i+2)}\cdots a(0)_{n(n+1)})^2 = 0$$

by Lemma 1.1(1) and then

$$a(0)_{12}\cdots a(0)_{(i-1)i}a(1)_{i(i+1)}a(0)_{(i+1)(i+2)}\cdots a(0)_{n(n+1)} = 0.$$

We proceed by induction on $k = 0, 1, \ldots, mn-1$. Let v be maximal in the set of s_i's satisfying $s_1+s_2+\cdots+s_n=k$. Consider a term $a(s_1)_{12}a(s_2)_{23}\cdots a(s_n)_{n(n+1)}$ with $s_i = v$ and $s_1 + s_2 + \cdots + s_n = k$. Note that not all s_j's are equal. Multiplying $\sum_{s_1+s_2+\cdots+s_n=k} a(s_1)_{12}a(s_2)_{23}\cdots a(s_n)_{n(n+1)} = 0$ on the right side by

$$a(s_1)_{12}\cdots a(s_{i-1})_{(i-1)i}a(s_{i+1})_{(i+1)(i+2)}\cdots a(s_n)_{n(n+1)},$$

then we have

$$(a(s_1)_{12}\cdots a(s_{i-1})_{(i-1)i}a(s_i)_{i(i+1)}a(s_{i+1})_{(i+1)(i+2)}\cdots a(s_n)_{n(n+1)})$$
$$(a(s_1)_{12}\cdots a(s_{i-1})_{(i-1)i}a(s_{i+1})_{(i+1)(i+2)}\cdots a(s_n)_{n(n+1)}) = 0$$

by the induction hypothesis and Lemma 1.1(1) since every other term (after multiplying) contains $a(t_1)_{12}, \ldots, a(t_n)_{n(n+1)}$, with $t_1 + \cdots + t_n \leq k-1$, as factors. Thus we have

$$(a(s_1)_{12}\cdots a(s_{i-1})_{(i-1)i}a(s_i)_{i(i+1)}a(s_{i+1})_{(i+1)(i+2)}\cdots a(s_n)_{n(n+1)})^2 = 0$$

by Lemma 1.1(1), entailing $a(s_1)_{12}\cdots a(s_n)_{n(n+1)} = 0$. Next take such v in the remaining terms and apply the same computation method. Proceeding in this manner we finally get to $a(u_1)_{12}\ a(u_2)_{23}\cdots a(u_n)_{n(n+1)} = 0$ for any choice of

u_i's such that $u_1 + u_2 + \cdots + u_n = k$ and not all u_i's are equal. In this situation, if k is divisible by n, then we have $a(\frac{k}{n})_{12}a(\frac{k}{n})_{23}\cdots a(\frac{k}{n})_{n(n+1)} = 0$ as a consequence. Thus all terms in $\sum_{s_1+s_2+\cdots+s_n=k} a(s_1)_{12}a(s_2)_{23}\cdots a(s_n)_{n(n+1)}$ are zero, and consequently $a(s_1)_{12}a(s_2)_{23}\cdots a(s_n)_{n(n+1)} = 0$ for any $k \in \{1, 2, \ldots, mn-1\}$ and any choice of s_i's with $s_1 + s_2 + \cdots + s_n = k$.

Seeing that $a(s_1)_{12}\cdots a(s_n)_{n(n+1)} = 0$ is equivalent to $A_{s_1}\cdots A_{s_n} = 0$, we get $A_{s_1}\cdots A_{s_n} = 0$ for any $k \in \{0, 1, 2, \ldots, mn\}$ and any choice of s_i's with $s_1 + \cdots + s_n = k$. Therefore $U_{n+1}(R)$ is n-semi-Armendariz.

(3)⇒(1): Assume on the contrary that there is $0 \neq a \in R$ with $a^2 = 0$. Let $A = (a_{ij}) \in N_n(R)$ with $a_{i(i+1)} = 1$ for all i and elsewhere zero, and $B = (b_{ij}) \in U_n(R)$ with $b_{11} = a, b_{nn} = -a$ and elsewhere zero. Then we have the following computation:

$$(\dagger) \qquad ABA = BA^hB = B^2 = 0, A^{n-k}B = (-a)E_{kn}, BA^k = aE_{1(k+1)}$$

for $k = 1, \ldots, n-1$ and all h. Consider $f(x) = A + Bx \in U_n(R)[x]$. Then

$$f(x)^n = (A^{n-1}B + BA^{n-1})x = ((-a)E_{1n} + aE_{1n})x = 0$$

by (†) but $A^{n-1}B, BA^{n-1}$ are both nonzero. Thus $U_n(R)$ is not n-semi-Armendariz, a contradiction.

(2)⇒(3) is obtained from Lemma 1.1(2) and the proofs of (1)⇒(4)⇒(5)⇒(1) are similar to the case of $U_n(R)$. □

By Theorem 1.2 and Lemma 1.1(2),(5) we get the following.

Corollary 1.3. *A ring R is reduced if and only if $U_3(R)$ is semi-Armendariz if and only if $U_2(R)$ is semi-Armendariz if and only if $D_3(R)$ is Armendariz if and only if $D_2(R)$ is Armendariz.*

Proof. Let R be reduced. Then by Theorem 1.2, $U_k(R)$ is $(k-1)$-semi-Armendariz for $k \geq 3$. Since $U_k(R)$ is a subring of $U_{k+1}(R)$, $U_k(R)$ is ℓ-semi-Armendariz for all $\ell \geq k-1$ by Lemma 1.1(2), entailing that $U_3(R)$ is semi-Armendariz. It then by Lemma 1.1(2) follows that if $U_3(R)$ is semi-Armendariz, then so is $U_2(R)$. Remaining directions are obtained from Lemma 1.1(2), (5). □

Actually let $U_2(R)$ be semi-Armendariz and assume on the contrary that there is $0 \neq a \in R$ with $a^2 = 0$. Consider $A = \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$ and $B = \left(\begin{smallmatrix} a & 0 \\ 0 & -a \end{smallmatrix}\right)$ in $U_2(R)$. Then $A^2 = B^2 = 0$ and $AB + BA = 0$. So letting $f(x) = A + Bx \in U_2(R)[x]$ we get $f(x)^2 = 0$, but AB and BA are both nonzero. Thus $U_2(R)$ is not semi-Armendariz, a contradiction.

If $D_2(R)$ is Armendariz, then R is Armendariz (hence semi-Armendariz) by Lemma 1.1(2) or Corollary 1.3. In the following we see a non-semi-Armendariz $D_2(R)$ when the given ring R is Armendariz but not reduced.

Example 1.4. Let $\mathbb{Z}_2$ be the ring of integers modulo 2. Then $R = D_3(\mathbb{Z}_2)$ is an Armendariz ring by Corollary 1.3. Let $S = \{\left(\begin{smallmatrix} A & B \\ 0 & A \end{smallmatrix}\right) \mid A, B \in R\}$. Consider

$$f(x) = \left(\begin{smallmatrix} C & 0 \\ 0 & C \end{smallmatrix}\right) + \left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right) x \text{ with } C = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

in $S[x]$. Then $f(x)^2 = 0$, but $\left(\begin{smallmatrix} C & 0 \\ 0 & C \end{smallmatrix}\right)\left(\begin{smallmatrix} 0 & D \\ 0 & 0 \end{smallmatrix}\right) \neq 0$. Thus S is not 2-semi-Armendariz and so not semi-Armendariz.

From Theorem 1.2 one may ask whether a ring R is reduced when $D_n(R)$ is n-semi-Armendariz. We do not know the answer, but have an affirmative situation when the characteristic of given rings are $n \geq 2$.

Proposition 1.5. *Let R be a ring of characteristic $n \geq 2$. Then R is reduced if and only if $D_n(R)$ is n-semi-Armendariz.*

Proof. It suffices, by Theorem 1.2 and Lemma 1.1(2), to show that R is reduced when $D_n(R)$ is n-semi-Armendariz. Assume on the contrary that there is $0 \neq a \in R$ with $a^2 = 0$. Consider $f(x) = A + Bx$ in $D_n(R)[x]$ where $A = (a_{ij}) \in N_n(R)$ with $a_{i(i+1)} = 1$ for all i and elsewhere zero, and $B = (b_{ij}) \in D_n(R)$ with $b_{ii} = a$ and elsewhere zero. Then $A^n = B^2 = 0$ and $AB = BA$, so that $f(x)^n = nA^{n-1}Bx = naE_{1n}x = 0$ because R is of characteristic n. But $A^{n-1}B = aE_{1n} \neq 0$ and so $D_n(R)$ is not n-semi-Armendariz, a contradiction. □

Also from Theorem 1.2, one may conjecture that $U_{n+2}(R)$ is n-semi-Armendariz over a reduced ring R. However there exists a counterexample as follows.

Example 1.6. Consider $U_{n+2}(R)$ $(n \geq 3)$ over any ring R and set

$$A = E_{12} + \cdots + E_{(n-2)(n-1)} + E_{(n-1)(n+1)} + E_{n(n+2)}$$

and

$$B = E_{(n-1)n} + E_{(n-1)(n+1)} + E_{n(n+2)} - E_{(n+1)(n+2)}$$

in $N_{n+2}(R)$. Then we have the following computation:

$$\begin{aligned} AB &= E_{(n-2)n} + E_{(n-2)(n+1)} - E_{(n-1)(n+2)}, BA = E_{(n-1)(n+2)}, \\ B^2 &= BA^2 = BAB = 0, \\ A^kBA &= E_{(n-k-1)(n+2)} \neq 0 \text{ for } k = 1, \ldots, n-2, \\ A^tB &= E_{(n-t-1)n} + E_{(n-t-1)(n+1)} - E_{(n-t)(n+2)} \neq 0 \text{ for } t = 1, \ldots, n-2, \\ A^{n-1}B &= -E_{1(n+2)}. \end{aligned}$$

Thus we get

$$(A + Bx)^n = (A^{n-2}BA + A^{n-1}B)x = (E_{1(n+2)} + (-E_{1(n+2)}))x = 0.$$

However $U_{n+2}(R)$ is not n-semi-Armendariz from $A^{n-2}BA \neq 0$ and $A^{n-1}B \neq 0$.

By Lemma 1.1(2), Theorem 1.2 and Example 1.6 we now can say that

m-semi-Armendariz rings need not be n-semi-Armendariz for $m \geq n+1$.

For, assuming that m-semi-Armendariz rings are n-semi-Armendariz for some $m \geq n+1$, then $U_{n+2}(R)$ is n-semi-Armendariz by Lemma 1.1(2) and Theorem 1.2 over any reduced ring R, a contradiction to Example 1.6.

Next conversely one may ask whether n-semi-Armendariz rings are $(n+1)$-semi-Armendariz. However the answer is also negative by the following example.

Example 1.7. Consider a positive integer $v \geq 3$ with the primary decomposition $v = p_1^{r_1} \cdots p_\alpha^{r_\alpha}$, where p_i's are distinct prime numbers and r_i's are positive integers. Put $w = p_1 \cdots p_\alpha$.

Let $\mathbb{Z}_w$ be the ring of integers modulo w, and $\mathbb{Z}_w[x, y]$ be the polynomial ring with commuting indeterminates x, y over $\mathbb{Z}_w$. Note that $\mathbb{Z}_w$ is reduced (hence so are the polynomial rings over $\mathbb{Z}_w$). Set $R = \mathbb{Z}_w[x, y]/I$, where I is the ideal of $\mathbb{Z}_w[x, y]$ generated by x^v, x^2y^2 and y^v. We will show that R is $(v-1)$-semi-Armendariz but not v-semi-Armendariz. For simplicity, we use x and y for $x + I$ and $y + I$, respectively. Next let $R[t]$ be the polynomial ring with an indeterminate t over R.

Since the characteristic of R is w and w divides v, we have $(x + yt)^v = x^v + vx^{v-1}y + vxy^{v-1} + y^v t^v = 0$. But $x^{v-1}y \neq 0$, so that R is not v-semi-Armendariz.

Now put $(f_0 + f_1 t + \cdots + f_m t^m)^{v-1} = 0$ in $R[t]$ with

$$\begin{aligned} f_i = {} & a_{i0} + a_{i1}x + \cdots + a_{i(v-1)}x^{v-1} + b_{i1}y + \cdots + b_{i(v-1)}y^{v-1} \\ & + c_{i1}xy + \cdots + c_{i(v-1)}x^{v-1}y + d_{i2}xy^2 + \cdots + d_{i(v-1)}xy^{v-1} \end{aligned}$$

for $i = 0, 1, \ldots, m$. Then we can convert $(f_0 + f_1 t + \cdots + f_m t^m)^{v-1} = 0$ into

$$\begin{aligned} (g_0 + g_1 x + \cdots + g_{v-1}x^{v-1} + h_1 y + \cdots + h_{v-1}y^{v-1} \\ + k_1 xy + \cdots + k_{v-1}x^{v-1}y + q_2 xy^2 + \cdots + q_{v-1}xy^{v-1})^{v-1} = 0 \end{aligned}$$

with $g_j = \sum_{i=0}^m a_{ij}t^i$, $h_\ell = \sum_{i=0}^m b_{i\ell}t^i$, $k_\ell = \sum_{i=0}^m c_{i\ell}t^i$ and $q_s = \sum_{i=0}^m d_{is}t^i \in \mathbb{Z}_w[t]$ and for $j = 0, \ldots, v-1$, $\ell = 1, \ldots, v-1$, $s = 2, \ldots, v-1$. We concentrate on the expansion of the preceding equality. Since $\mathbb{Z}_w[t]$ is reduced, $g_0^{v-1} = 0$ implies $g_0 = 0$. In the coefficients of x^{v-1} (resp. y^{v-1}) any term except g_1^{v-1} (resp. h_1^{v-1}) contains g_0 as a factor, so that $g_1^{v-1} = h_1^{v-1} = 0$; hence $g_1 = h_1 = 0$ since $\mathbb{Z}_w[t]$ is reduced. Consequently each monomial occurring in f_i (for $i = 0, 1, \ldots, m$) has degree ≥ 2. Let $f_{\sigma_1} f_{\sigma_2} \cdots f_{\sigma_{v-1}}$ be any product of $(v-1)$-number of f_{σ_i}'s taken in $\{f_0, f_1, \ldots, f_m\}$. Notice that any term in the expansion of $f_{\sigma_1} f_{\sigma_2} \cdots f_{\sigma_{v-1}}$ contains the product of $(v-1)$-number of monomials taken in $\{x^2, y^2, xy\}$ by the preceding result, so that $f_{\sigma_1} f_{\sigma_2} \cdots f_{\sigma_{v-1}} = 0$. Therefore R is $(v-1)$-semi-Armendairz.

With the help of Example 1.7 we can also conclude that

n-semi-Armendariz rings need not be $(n+1)$-semi-Armendariz,

letting $n+1 = p_1^{r_1}\cdots p_\alpha^{r_\alpha}$.

It is also natural to ask whether Theorem 1.2 holds for the full matrix ring case. However the following answers negatively.

Example 1.8. Let S be any ring and $R = \mathrm{Mat}_n(S)$. We first compute the cases of $n = 2, 3$. Take

$$f(x) = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix} x^2 \in \mathrm{Mat}_2(S)[x].$$

Then $f(x)^2 = 0$. But $f(x) \notin N(R)[x]$ and so by Lemma 1.1(7) $\mathrm{Mat}_2(S)$ is not 2-semi-Armendariz.

Take

$$f(x) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} x \in \mathrm{Mat}_3(S)[x] \text{ and } g(x) = f(x)^2.$$

Then we have

$$f(x)^3 = 0, g(x) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} + \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} x + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix} x^2, g(x)^2 = 0.$$

But

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix}^2 \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix} = -E_{12} \text{ and } \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix}^2$$

are both nonzero. So $\mathrm{Mat}_3(S)$ is neither 2-semi-Armendariz (by $g(x)$ and Lemma 1.1(7)) nor 3-semi-Armendariz (by $f(x)$).

Next we consider the general case of $n \geq 4$. Consider $g(x) = A + Bx \in R[x]$ with

$$A = E_{12} + E_{23} + \cdots + E_{(n-2)(n-1)} + E_{(n-1)n} \text{ and } B = E_{(n-1)1} + (-E_{n2}).$$

We first show $g(x)^n = 0$. Use $\phi_{(s,t)}$ to denote the sum of all products of s-number of A's and t-number of B's. Then we have $(A+Bx)^n = \sum_{i=0}^{n} \phi_{(n-i,i)} x^i$. Note that

$$(*) \qquad \phi_{(n,0)} = A^n = 0 \text{ and } BA^{\ell}B = 0 \text{ for } \ell = 0, 1, \ldots, n-4.$$

Setting $f_k = A^{n-k-1}BA^k$ to compute $\phi_{(n-1,1)}$, we have

$$f_0 = -E_{12},$$
$$f_k = E_{k(k+1)} + (-E_{(k+1)(k+2)}) \ \text{(for } k = 1, 2, \ldots, n-2) \text{ and}$$
$$f_{n-1} = E_{(n-1)n}.$$

So $\phi_{(n-1,1)} = \sum_{i=0}^{n-1} f_i = 0$. Next by $(*)$, $\phi_{(n-2,2)} = BA^{n-3}BA + BA^{n-2}B + ABA^{n-3}B = (-E_{n2}) + (E_{(n-1)1} + E_{n2}) + (-E_{(n-1)1}) = 0$. In case of $\phi_{(n-k,k)}$

($k \geq 3$), every term contains B^2 or BA^hB ($h \leq n-4$), so that $\phi_{(n-k,k)}=0$ for all $k \geq 3$ by ($*$). Therefore $g(x)^n = (A+Bx)^n = \sum_{i=0}^{n} \phi_{(n-i,i)} x^i = 0$ but $A^{n-1}B = -E_{12} \neq 0$, entailing that R is not n-semi-Armendariz.

2. Properties and more examples

In this section we examine the interesting properties of the class of n-semi-Armendariz rings, finding various kinds of examples of n-semi-Armendariz rings. Use X to denote a nonempty set of commuting indeterminates over rings and let $R[X]$ be the polynomial ring with X over a ring R.

Theorem 2.1. (1) *A ring R is n-semi-Armendariz if and only if so is $R[X]$.*

(2) *A ring R is semi-Armendariz if and only if so is $R[X]$.*

(3) *If a ring R is semi-Armendariz, then $N(R[X]) \subseteq N(R)[X]$.*

Proof. (1) It suffices by Lemma 1.1(2) to prove that $R[x]$ is n-semi-Armendariz if so is R. Suppose that R is n-semi-Armendariz for a positive integer $n \geq 2$ and let $f(T) = f_0 + f_1T + \cdots + f_mT^m \in R[x][T]$ with $f(T)^n = 0$, where $f_i = \sum_{j=0}^{k_i} a_{ij}x^j$ in $R[x]$ for $i = 0, 1, \ldots, m$. We apply the proof of [1, Theorem 2], letting $k = k_0 + k_1 + \cdots + k_m$. Then $f(x^k) = f_0 + f_1x^k + \cdots + f_mx^{km} \in R[x]$ and the set of coefficients of the f_i's equals the set of coefficients of $f(x^k)$. Since $f(T)^n = 0$ and x commutes with elements of R, $f(x^k)^n = 0$ in $R[x]$. Since R is n-semi-Armendariz, $a_{i_1}a_{i_2}\cdots a_{i_n} = 0$ for any choice of a_{i_j}'s with $j = 1, \ldots, n$. Thus $f_{s_1}f_{s_2}\cdots f_{s_n} = 0$ for any choice of f_{s_ℓ}'s in $\{f_0, f_1, \ldots, f_m\}$.

Next letting $g \in R[X]$ with $g^n = 0$, there is a finite subset X_0 of X such that $g \in R[X_0]$; hence it suffices to consider the case of X being finite. Then the induction enables us to decide that $R[X_0]$ is also n-semi-Armendariz, with the help of the result above. Thus $R[X]$ is n-semi-Armendariz.

(2) is obtained from (1).

(3) Let R be a semi-Armendariz ring. Letting $f \in N(R[X])$, there is a finite subset X_0 of X such that $f \in N(R[X_0])$, say $X_0 = \{x_1, \ldots, x_k\}$. $N(R[x_1]) \subseteq N(R)[x_1]$ by Lemma 1.1(7). $R[x_1]$ and $R[x_1, x_2]$ are semi-Armendariz by (2) and so Lemma 1.1(7) gives

$$N(R[x_1,x_2]) = N(R[x_1][x_2]) \subseteq N(R[x_1])[x_2] \subseteq N(R)[x_1][x_2] = N(R)[x_1,x_2].$$

Inductively we can get $N(R[X_0]) \subseteq N(R)[X_0]$, entailing $N(R[X]) \subseteq N(R)[X]$. □

There can be a natural conjecture that R is an n-semi-Armendariz ring if R/I and I are n-semi-Armendariz for a nonzero proper ideal I of R, where I is considered as an n-semi-Armendariz ring without identity. However there is a counterexample as in the following. Let R be an algebra over a commutative ring S. The *Dorroh extension* of R by S, written by $R \oplus_D S$, is the ring $R \oplus S$ with operations $(r_1, s_1) + (r_2, s_2) = (r_1 + r_2, s_1 + s_2)$ and $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for $r_i \in R$ and $s_i \in S$.

Example 2.2. Let D be the Dorroh extension $\left(\begin{smallmatrix} 0 & \mathbb{Z}_m \\ 0 & 0 \end{smallmatrix}\right) \oplus_D \mathbb{Z}_m$, where $\mathbb{Z}_m$ is the ring of integers modulo m and the exponents of distinct primes in the primary decomposition of m are all 1. Then $\mathbb{Z}_m$ is reduced. Consider $R = U_n(D)$ for $n \geq 2$. Then, by Theorem 1.2, R is not n-semi-Armendariz since D is not reduced. Set

$$I = U_n\left(\begin{pmatrix} 0 & \mathbb{Z}_m \\ 0 & 0 \end{pmatrix} \oplus_D 0\right).$$

Then I is an ideal of R such that $\frac{R}{I} \cong U_n(\mathbb{Z}_m)$. So R/I is n-semi-Armendariz by Theorem 1.2 and I is n-semi-Armendariz from $I^n = 0$.

But we have an affirmative answer to the preceding conjecture, taking a stronger condition "I is reduced" instead of the one "I is n-semi-Armendariz".

Theorem 2.3. *For a ring R and a positive integer $n \geq 2$ suppose that R/I is an n-semi-Armendariz ring for some ideal I of R. If I is reduced, then R is n-semi-Armendariz.*

Proof. From the condition that I is reduced, we first have $bIa \subseteq I$, $(bIa)^2 = 0$, and $bIa = 0$ whenever $ab = 0$ for $a, b \in R$. Applying this result we also get $aI_1aI_2a \cdots aI_{n-1}a = 0$ when $a^n = 0$ for $a \in R$ and some positive integer $n \geq 2$, where $I_k = I$ for all $k = 1, 2, \ldots n-1$. For, $a^n = 0$ implies $a^{n-1}Ia = 0$, and then we have $aIaIa^{n-2} = 0$, and so on.

Let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ such that $f(x)^n = 0$ for a positive integer $n \geq 2$. R/I is n-semi-Armendariz by hypothesis and so we have

$$a_{i_1}a_{i_2} \cdots a_{i_n} \in I \tag{1}$$

for all a_{i_j}'s in $\{a_0, a_1, \ldots, a_m\}$, where $j = 1, \ldots, n$. If $m = 0$ we are done, and so assume $m \geq 1$. We proceed by induction on m.

From $f(x)^n = 0$, we have $a_0^n = 0$ and so

$$a_0I_1a_0I_2a_0 \cdots a_0I_{n-1}a_0 = 0. \tag{2}$$

Let $a_{s_1}a_{s_2} \cdots a_{s_n}$ be a product, in a term in $f(x)^n$, containing a_0; say $a_{s_t} = a_0$. Then

$$(a_{s_1}a_{s_2} \cdots a_{s_n})^{2n-1} \in (a_{s_1} \cdots a_{s_{t-1}})(a_0I_1a_0I_2a_0 \cdots a_0I_{n-1}a_0)(a_{s_{t+1}} \cdots a_{s_n})$$

by (1) and so $(a_{s_1}a_{s_2} \cdots a_{s_n})^{2n-1} = 0$ by (2). But I is reduced and so $a_{s_1}a_{s_2} \cdots a_{s_n} = 0$. Consequently we get $(\sum_{i=1}^{m} a_i x^i)^n = 0$ and moreover $(\sum_{i=0}^{m-1} a_{i+1} x^i)^n = 0$. Now, by the induction hypothesis, $a_{i_1}a_{i_2} \cdots a_{i_n} = 0$ for all a_{i_j}'s in $\{a_1, \ldots, a_m\}$, where $j = 1, \ldots, n$. Consequently we obtain $a_{i_1}a_{i_2} \cdots a_{i_n} = 0$ for all a_{i_j}'s in $\{a_0, a_1, \ldots, a_m\}$, where $j = 1, \ldots, n$. Therefore R is n-semi-Armendariz. □

However the n-semi-Armendarizness need not be preserved by factor rings. In Example 1.7, $\mathbb{Z}_w[x, y]$ is reduced (hence v-semi-Armendariz) but

$$\mathbb{Z}_w[x, y]/(Rx^v + Rx^2y^2 + Ry^v)$$

is not v-semi-Armendariz.

Armendariz rings are abelian by the proof of [1, Theorem 6], but semi-Armendariz rings need not be abelian as can be seen by the 2 by 2 upper triangular matrix ring over a reduced ring. Since reduced rings are semiprime and Armendariz, one may ask whether abelian semiprime rings are semi-Armendariz. In a similar point of view, one may also conjecture that commutative rings are semi-Armendariz. However the following answers them negatively.

Example 2.4. (1) Let S be a reduced ring and n be a positive integer. Next consider $D_{2^n}(S)$ over S and let $R_n = D_{2^n}(S)$. Each R_n is a 2^n-semi-Armendariz ring by Theorem 1.2 and Lemma 1.1(2). Define a map $\sigma : R_n \to R_{n+1}$ by $A \mapsto \left(\begin{smallmatrix} A & 0 \\ 0 & A \end{smallmatrix}\right)$. Then R_n can be considered as a subring of R_{n+1} via σ (i.e., $A = \sigma(A)$ for $A \in R_n$). Notice that $D = \{R_n, \sigma_{nm}\}$, with $\sigma_{nm} = \sigma^{m-n}$ whenever $n \leq m$, is a direct system over $I = \{1, 2, \ldots\}$. Set $R = \varinjlim R_n$ be the direct limit of D, where n goes to infinity. With the help of Example 1.6, there exists a positive integer N such that R_n is not N-semi-Armendariz for some n; hence R is not semi-Armendariz. But every nonzero idempotent in R_n is such that the diagonal is an idempotent in S and elsewhere is zero by [5, Lemma 2]. Thus R is abelian. Next letting S be a domain, we get that R is prime by [7, Proposition 1.3].

(2) Let F be the Galois field of order 2^n, where n is any positive integer and $F[x, y]$ be the polynomial ring with commuting indeterminates x, y over F. Next consider $R = \frac{F[x,y]}{(x^2,y^2)}$ with (x^2, y^2) the ideal of S generated by x^2 and y^2. Then R is commutative and $(x + yT)^2 = 0$, where T is an indeterminate over R. But $xy \neq 0$ implies that R is not semi-Armendariz.

For given an abelian ring R the following may be a useful method to check whether R is n-semi-Armendariz, if it is available.

Proposition 2.5. *For an abelian ring R the following conditions are equivalent*:

(1) *R is n-semi-Armendariz*;

(2) *eR and $(1 - e)R$ are n-semi-Armendariz for every idempotent e of R*;

(3) *eR and $(1 - e)R$ are n-semi-Armendariz for some idempotent e of R.*

Proof. (1)⇒(2) is obtained by Lemma 1.1(2) since eR and $(1-e)R$ are subrings of R. (2)⇒(3) is obvious.

(3)⇒(1): Suppose that eR and $(1 - e)R$ are n-semi-Armendariz for some idempotent e of R, and consider $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ with $f(x)^n = 0$ for a positive integer $n \geq 2$. Next let $f_1(x) = ef(x) \in eR[x]$ and $f_2(x) = (1-e)f(x) \in (1-e)R[x]$. Since R is abelian, $f_1(x)^n = ef(x)^n = 0$ and $f_2(x)^n = (1 - e)f(x)^n = 0$. By the condition (3), we obtain that $ea_{s_1}a_{s_2} \cdots a_{s_n} = ea_{s_1}ea_{s_2}e \cdots ea_{s_n}e = 0$ and $(1 - e)a_{s_1}a_{s_2} \cdots a_{s_n} = (1 - e)a_{s_1}(1 - e)a_{s_2}(1 - e) \cdots (1-e)a_{s_n}(1-e) = 0$ for all a_{s_j}'s in $\{a_0, a_1, \ldots, a_m\}$, where $j = 1, 2, \ldots, n$; hence every $a_{s_1}a_{s_2} \cdots a_{s_n} = 0$, concluding that R is n-semi-Armendariz. □

An element a in a ring R is called *regular* if a is neither left nor right zero-divisor. $[-]$ means the Gauss function.

Proposition 2.6. *Let R be a ring and J be an ideal of R such that every element in $R\backslash J$ is regular and $J^n = 0$. Then R is ℓ-semi-Armendariz for $\ell \geq n$.*

Proof. We use freely the condition that every element in $R\backslash J$ is regular. Let $f(x) = \sum_{i=0}^{m} a_i x^i \in R[x]$ and suppose $f(x)^n = 0$. Then $a_0^n = 0 = a_m^n$, so that $a_0, a_m \in J$.

In the coefficient $\cdots + a_1^n + \cdots = 0$ of x^n, every term (except a_1^n) contains a_0 and thus is contained in J, entailing $a_1^n \in J$. We then get $a_1 \in J$. We proceed by induction on $k = 0, 1, \ldots, [\frac{m}{2}]$. In the coefficient $\cdots + a_k^n + \cdots = 0$ of x^{kn}, every term (except a_k^n) contains a_h with $h < k$ and so is contained in J, entailing $a_k^n \in J$. Thus $a_k \in J$. The computation, based on $a_m \in J$, from a_m to $a_{[\frac{m}{2}]+1}$ is similar. Therefore $a_i \in J$ for $i = 0, \ldots, m$. Now since $J^n = 0$, we have that $a_{i_1} a_{i_2} \cdots a_{i_n} = 0$ for any choice of a_{i_j}'s in $\{a_0, \ldots, a_m\}$, where $j = 1, \ldots, n$, concluding that R is n-semi-Armendariz.

Next since $J^\ell = 0$ for every $\ell \geq n$, we obtain that R is ℓ-semi-Armendariz by the same computation as above. □

Any local ring R with $J(R)^n = 0$ is ℓ-semi-Armendariz for $\ell \geq n$ by Proposition 2.6, where $J(R)$ is the Jacobson radical of R.

Corollary 2.7. *Let p be a prime and $\mathbb{Z}_p$ be the ring of integers modulo p. Consider*

$$R = \begin{pmatrix} 0 & \mathbb{Z}_p \\ 0 & 0 \end{pmatrix} \oplus_D \mathbb{Z}_p.$$

Then $D_n(R)$ is m-semi-Armendariz for $m \geq n+1$.

Proof. Let $I = \left(\begin{smallmatrix} 0 & \mathbb{Z}_p \\ 0 & 0 \end{smallmatrix}\right) \oplus_D 0$ and $J = \{(a_{ij}) \in D_n(R) \mid a_{ii} \in I\}$. Since $D_n(R)/J$ is isomorphic to $\mathbb{Z}_p$ and $J^{n+1} = 0$, $D_n(R)$ is local. Thus every element in $D_n(R)\backslash J$ is regular and then $D_n(R)$ is m-semi-Armendariz (for $m \geq n+1$) by Proposition 2.6. □

In the following we have a similar result to [1, Theorem 5].

Proposition 2.8. *Let h, k, m be integers ≥ 2 such that h divides k and k divides m. Suppose that R is a ring of characteristic h. Then R is reduced if and only if $R[x]/(x^m)$ is semi-Armendariz, where $(x^m) = R[x]x^m$.*

Proof. If R is reduced, then $R[x]/(x^m)$ is Armendariz by [1, Theorem 5]. Conversely let $R[x]/(x^m)$ be semi-Armendariz and assume on the contrary that there exists $0 \neq r \in R$ with $r^2 = 0$. We use $\bar{x}$ for $x + (x^m)$. If $h = k = m = 2$ or $h = k = m = 3$, then $(r + \bar{x})^k = 0$ and $r\bar{x}^{k-1} \neq 0$; hence $R[x]/(x^m)$ is not k-semi-Armendariz, a contradiction. Suppose $m \geq 4$ and $k \leq m-1$. Say $m = \ell k$. Since R is of characteristic h,

$$(r + \bar{x}^\ell + \bar{x}^{m-1})^k = r^k + \cdots + kr(\bar{x}^\ell + \bar{x}^{m-1})^{k-1} + (\bar{x}^\ell + \bar{x}^{m-1})^k = \bar{x}^{\ell k} = \bar{x}^m = 0.$$

But $r\bar{x}^{\ell(k-1)} \neq 0$ and so $R[x]/(x^m)$ is not k-semi-Armendariz, a contradiction. □

3. Commutative n-semi-Armendariz rings

We observe in this section the structure of commutative n-semi-Armendariz rings, applying the arguments in [1]. Let R be a commutative ring and $f \in R[x]$. Let A_f be the content of f, i.e., the ideal of R generated by the coefficients of f. It is obvious that $A_{fg} \subseteq A_f A_g$ for $f, g \in R[x]$. Note that R is Armendariz if and only if for $f, g \in R[x]$ with $A_{fg} = 0$ we have $A_f A_g = 0$. The following is obtained from the definition.

Lemma 3.1. *A commutative ring R is n-semi-Armendariz if and only if for $f \in R[x]$ with $A_{f^n} = 0$ we have $(A_f)^n = 0$.*

It is shown by [1, Theorem 8 and Corollary 9] that $A_{fg} = A_f A_g$ for all $f, g \in R[X]$ if and only if for $f_1, \ldots, f_n \in R[X]$ we get $A_{f_1 \cdots f_n} = A_{f_1} \cdots A_{f_n}$ if and only if every homomorphic image of R is Armendariz. By [14, Theorem 2.2] every homomorphic image of a PID is Armendariz. In the following we see similar results for n-semi-Armendariz rings.

Theorem 3.2. (1) *Let R be a commutative ring. Then $A_{f^n} = (A_f)^n$ for each $f \in R[x]$ if and only if every homomorphic image of R is n-semi-Armendariz.*

(2) *Let R be a ring and S be a multiplicative monoid in R consisting of central regular elements. Then R is n-semi-Armendariz if and only if so is $S^{-1}R$.*

Proof. (1) Assume that $A_{f^n} = (A_f)^n$ for each $f \in R[x]$. Then every factor ring $\bar{R}$ of R also satisfies this condition, so that $\bar{R}$ is n-semi-Armendariz by Lemma 3.1. Conversely assume that every homomorphic image of R is n-semi-Armendariz. Consider $S = \frac{R}{A_{f^n}}[x]$ for $f \in R[x]$. Then $(\bar{f})^n = 0$ in S, and since R/A_{f^n} is n-semi-Armendariz we have $(A_f)^n / A_{f^n} = 0$, entailing $(A_f)^n = A_{f^n}$.

(2) It suffices to prove by Lemma 1.1(2) that $S^{-1}R$ is n-semi-Armendariz when so is R. Suppose that R is n-semi-Armendariz. Consider

$$f(x) = \sum_{i=0}^{m} \alpha_i x^i \in S^{-1}R[x]$$

with $f(x)^n = 0$. We can assume that $\alpha_i = a_i u^{-1}$ with $a_i \in R$ for all i and regular $u \in S$. Setting $f_1(x) = \sum_{i=0}^{m} a_i x^i$, we have $0 = f(x)^n = (f_1(x)u^{-1})^n = f_1(x)^n u^{-n}$, entailing $f_1(x)^n = 0$. Since R is n-semi-Armendariz, $a_{i_1} \cdots a_{i_n} = 0$ for any choice of a_{i_j}'s with $j = 1, \ldots, n$; hence $\alpha_{i_1} \cdots \alpha_{i_n} = a_{i_1} \cdots a_{i_n} u^{-n} = 0$ for any choice of α_{i_j}'s with $j = 1, \ldots, n$. Therefore $S^{-1}R$ is n-semi-Armendariz. □

About Theorem 3.2(1), there exist commutative reduced rings whose homomorphic images need not be n-semi-Armendariz as can be seen by Example 1.7.

For a commutative ring R let $T(R)$ be the total quotient ring of R and S be an overring of R (i.e., $R \subseteq S \subseteq T(R)$). The following is shown by Theorem 3.2(2) and Lemma 1.1(2).

Corollary 3.3. (1) *Let R be a commutative ring and S be an overring of R. Then R is n-semi-Armendariz if and only if S is n-semi-Armendariz if and only if $T(R)$ is n-semi-Armendariz.*

(2) *Let R be a commutative ring and P be a prime ideal of R such that $R \backslash P$ contains no zero-divisors. Then R is n-semi-Armendariz if and only if so is R_P.*

Let R be a commutative ring such that 0 is P-primary and $P^2 = 0$. Then R is Armendariz by [1, Proposition 13]. We get a similar result for n-semi-Armendariz rings in the following.

Proposition 3.4. *Let R be a commutative ring and Q be an ideal of R such that Q is P-primary and $P^n \subseteq Q$. Then R/Q and $R[X]/Q[X]$ are ℓ-semi-Armendariz for $\ell \geq n$.*

Proof. Let $f = \sum_{i=0}^{m} a_i x^i \in \frac{R}{Q}[x]$ with $f^n = 0$. Since Q is P-primary, $Q[x]$ is $P[x]$-primary in $R[x]$ and so $R[x]/P[x]$ is a commutative domain, entailing $f \in P[x]$. Then $a_i \in P$ for all i, and so R/Q is n-semi-Armendariz since $P^n \subseteq Q$. Also since $P^\ell \subseteq Q$ for every $\ell \geq n$, we obtain that R/Q is ℓ-semi-Armendariz by the same computation as above. Next by Theorem 2.1(1) $R[X]/Q[X](\cong \frac{R}{Q}[X])$ is ℓ-semi-Armendariz. □

Example 3.5. Use (r) to denote the ideal of a ring R generated by r.

(1) Let $\mathbb{Z}$ be the ring of integers and p be a prime. Since (p^n) $(n \geq 2)$ is primary for (p) and $(p)^n = (p^n)$, $\mathbb{Z}/(p^n)$ is ℓ-semi-Armendariz for $\ell \geq n$ by Proposition 3.4. This result is also shown by [14, Theorem 2.2].

(2) Let $S = \mathbb{Z}[x, y]$ be the polynomial ring with commuting indeterminates x, y over $\mathbb{Z}$. Since $Q = (x^n, y^n, x^i y^j)$ $(n \geq 2$ and $i + j = n)$ is primary for (x, y) and $(x, y)^n = Q$, S/Q is ℓ-semi-Armendariz for $\ell \geq n$ by Proposition 3.4.

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