J. Korean Math. Soc. **47** (2010), No. 4, pp. 705–718
DOI 10.4134/JKMS.2010.47.4.705

LIE BIALGEBRAS ARISING FROM POISSON BIALGEBRAS

Sei-Qwon Oh and Eun-Hee Cho

Abstract. It gives a method to obtain a natural Lie bialgebra from a Poisson bialgebra by an algebraic point of view. Let $\mathfrak{g}$ be a coboundary Lie bialgebra associated to a Poission Lie group G. As an application, we obtain a Lie bialgebra from a sub-Poisson bialgebra of the restricted dual of the universal enveloping algebra $U(\mathfrak{g})$.

Introduction

Assume throughout that G denotes a connected and simply connected Lie group with Lie algebra $\mathfrak{g}$, $\mathcal{O}(G)$ the coordinate ring of G and $U(\mathfrak{g})$ the universal enveloping algebra of $\mathfrak{g}$.

If G is a Poisson Lie group, then $\mathcal{O}(G)$ is a Poisson Hopf algebra and $\mathfrak{g}$ becomes a Lie bialgebra. Conversely, if $\mathfrak{g}$ has a Lie bialgebra structure, then G becomes a Poisson Lie group by [2, Chapter 1]. On the other hand, if $U(\mathfrak{g})$ has a co-Poisson Hopf structure with co-Poisson bracket δ, then $(\mathfrak{g}, \delta|_{\mathfrak{g}})$ becomes a Lie bialgebra. Conversely if $(\mathfrak{g}, \delta)$ is a Lie bialgebra, then the cobracket δ extends uniquely to a Poisson co-bracket on $U(\mathfrak{g})$, which makes $U(\mathfrak{g})$ into a co-Poisson Hopf algebra (see [2, Proposition 6.2.3]). Moreover, the coordinate ring $\mathcal{O}(G)$ is isomorphic as a Hopf algebra to the restricted dual $U(\mathfrak{g})^\circ$ of $U(\mathfrak{g})$ and it is sometimes more convenient to work on $U(\mathfrak{g})^\circ$ than to do on $\mathcal{O}(G)$. For instance, Hodges and his colleagues worked on restricted duals to obtain mathematical properties of a quantum group in [3] and [4]. Hence it makes sense mathematically to study a relationship between Lie bialgebras and restricted duals of their enveloping algebras.

Let $(A, \iota, m, \{\cdot,\cdot\}, \epsilon, \Delta)$ be a Poisson bialgebra and $\mathfrak{m} = \ker \epsilon$. In 1.5, we prove by an algebraic point of view that the pair $((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)$ is a natural Lie bialgebra obtained from $(A, \iota, m, \{\cdot,\cdot\}, \epsilon, \Delta)$.

Let $\mathfrak{g}$ be a coboundary Lie bialgebra. The restricted dual A of $U(\mathfrak{g})$ is the vector space spanned by all coordinate functions $c^M_{f,v}$, where M is a finite

Received January 31, 2008.
2000 *Mathematics Subject Classification.* 17B62, 17B63, 16W30.
Key words and phrases. Lie bialgebra, Poisson bialgebra.
This work was supported by the Korea Research Foundation Grant funded by the Korean Government, KRF-2007-314-C00007.

dimensional left $U(\mathfrak{g})$-module and $f \in M^*, v \in M$. Here we give an explicit Poisson bracket on A that is the Sklyanin Poisson bracket. Let B be a sub-Poisson bialgebra of the restricted dual A. Then, as an application of 1.5, we obtain a Lie bialgebra $((\mathfrak{m}_B/\mathfrak{m}_B^2)^*, \mathfrak{m}_B/\mathfrak{m}_B^2)$ arising from B, where $\mathfrak{m}_B$ is the kernel of the counit in B.

Assume throughout that $\mathbf{k}$ denotes a field of characteristic zero, all vector spaces considered here are over $\mathbf{k}$ and if A is a bialgebra with comultiplication Δ, then we use Sweedler's notation

$$\Delta(a) = \sum_{(a)} a' \otimes a'', \quad a \in A.$$

Recall that a Poisson algebra A is a $\mathbf{k}$-algebra with $\mathbf{k}$-bilinear map $\{\cdot,\cdot\}$, called a Poisson bracket, such that

(a) $(A, \{\cdot,\cdot\})$ is a Lie algebra over $\mathbf{k}$.
(b) $\{ab, c\} = a\{b, c\} + \{a, c\}b$ for all $a, b, c \in A$. (Leibniz rule)

1. Lie bialgebra arising from Poisson bialgebra

Definition 1.1. A Poisson algebra A with Poisson bracket $\{\cdot,\cdot\}$ is said to be a Poisson bialgebra if A is also a bialgebra $(A, \iota, m, \epsilon, \Delta)$ over $\mathbf{k}$ such that

$$\Delta(\{a, b\}) = \{\Delta(a), \Delta(b)\}_{A\otimes A} \tag{1}$$

for all $a, b \in A$, where the Poisson bracket $\{\cdot,\cdot\}_{A\otimes A}$ on $A \otimes A$ is defined by

$$\{a \otimes b, c \otimes d\}_{A\otimes A} = \{a, c\} \otimes bd + ac \otimes \{b, d\}$$

for all $a, b, c, d \in A$.

A Poisson bialgebra A is often denoted by $A = (A, \iota, m, \{\cdot,\cdot\}, \epsilon, \Delta)$. If a Poisson bialgebra A is a Hopf algebra, then A is called a Poisson Hopf algebra (see [2, 6.2.1] and [1, III.5.3]). Note, by (1), that

$$\Delta(\{a, b\}) = \sum a'b' \otimes \{a'', b''\} + \{a', b'\} \otimes a''b'' \tag{2}$$

for all elements a, b of a Poisson bialgebra A.

Lemma 1.2. *If $(A, \iota, m, \{\cdot,\cdot\}, \epsilon, \Delta)$ is a Poisson bialgebra, then $\epsilon(\{a, b\}) = 0$ for all $a, b \in A$.*

Proof. By (2), we have that

$$\begin{aligned}
\{a, b\} &= m \circ (\epsilon \otimes \mathrm{id}_A) \circ \Delta(\{a, b\}) \\
&= m \circ (\epsilon \otimes \mathrm{id}_A)(\textstyle\sum \{a', b'\} \otimes a''b'' + a'b' \otimes \{a'', b''\}) \\
&= \textstyle\sum \epsilon(\{a', b'\})a''b'' + \sum \epsilon(a'b')\{a'', b''\} \\
&= \textstyle\sum \epsilon(\{a', b'\})a''b'' + \{a, b\}
\end{aligned}$$

for $a, b \in A$ and thus we have $\sum \epsilon(\{a', b'\})a''b'' = 0$. Hence

$$0 = \epsilon(\textstyle\sum \epsilon(\{a', b'\})a''b'') = \sum \epsilon(\{a', b'\})\epsilon(a'')\epsilon(b'') = \epsilon(\{a, b\}),$$

as claimed. $\square$

Corollary 1.3. *In a Poisson bialgebra* $(A, \iota, m, \{\cdot,\cdot\}, \epsilon, \Delta)$, *set* $\ker\epsilon = \mathfrak{m}$. *Then* $\mathfrak{m}/\mathfrak{m}^2$ *is a Lie algebra with Lie bracket*

$$[a+\mathfrak{m}^2, b+\mathfrak{m}^2] = \{a,b\} + \mathfrak{m}^2, \quad a,b \in \mathfrak{m}. \tag{3}$$

Proof. The Lie bracket (3) is well-defined by Lemma 1.2. Clearly $(\mathfrak{m}/\mathfrak{m}^2, [\cdot,\cdot])$ is a Lie algebra. $\square$

1.4. Let $(A, \iota, m, \epsilon, \Delta)$ be a bialgebra and set

$$A^\circ = \{f \in A^* \mid f(I) = 0 \text{ for some ideal } I \text{ of } A \text{ such that } \dim(A/I) < \infty\}.$$

Then A°, called the restricted dual of A, becomes a bialgebra with bialgebra structure: For $f, g \in A^\circ$ and $a, b \in A$,

$$(fg)(a) = \sum f(a')g(a''), \quad \Delta(f)(a\otimes b) = f(ab).$$

Denote

$$P_\epsilon(A^\circ) = \{f \in A^\circ \mid f(ab) = \epsilon(a)f(b) + f(a)\epsilon(b), \quad \forall a,b \in A\}.$$

That is, $P_\epsilon(A^\circ) = \{f \in A^\circ \mid \Delta(f) = \epsilon\otimes f + f\otimes\epsilon\}$. It is well-known that $P_\epsilon(A^\circ)$ is a Lie algebra with Lie bracket

$$[f,g] = fg - gf$$

for all $f, g \in P_\epsilon(A^\circ)$.

Denote $\mathfrak{m} = \ker\epsilon$ and let $i : \mathfrak{m} \longrightarrow A$ be the canonical injection. Then i^* is a surjection of A^* onto $\mathfrak{m}^*$. Let $f \in \ker i^*$. Then $f(a - \epsilon(a)1) = 0$ for all $a \in A$ since $f(\mathfrak{m}) = 0$ and $a - \epsilon(a)1 \in \mathfrak{m}$. Thus $f = f(1)\epsilon$ for $f \in \ker i^*$. It follows that $\ker i^* = \mathbf{k}\epsilon$. Given $f, g \in \mathfrak{m}^*$, choose representatives $f' = i^{*-1}(f), g' = i^{*-1}(g)$. If $f'_1 = i^{*-1}(f), g'_1 = i^{*-1}(g)$, then $f'_1 = f' + \alpha\epsilon, g'_1 = g' + \beta\epsilon$ for some $\alpha, \beta \in \mathbf{k}$. Thus

$$\begin{aligned} f'_1g'_1 - g'_1f'_1 &= (f'+\alpha\epsilon)(g'+\beta\epsilon) - (g'+\beta\epsilon)(f'+\alpha\epsilon) \\ &= (f'g' + \beta f' + \alpha g' + \alpha\beta\epsilon) - (g'f' + \beta f' + \alpha g' + \alpha\beta\epsilon) \\ &= f'g' - g'f' \end{aligned}$$

since ϵ is the multiplicative identity in A^*. Hence $[f,g] = i^*(f'g' - g'f')$ is independent of representatives and defines a Lie bracket on $\mathfrak{m}^*$. Identifying $\{f \in \mathfrak{m}^* \mid f(\mathfrak{m}^2) = 0\}$ with $(\mathfrak{m}/\mathfrak{m}^2)^*$, $(\mathfrak{m}/\mathfrak{m}^2)^*$ is a Lie subalgebra of $\mathfrak{m}^*$ by [8, 2.1.2].

Lemma. *The linear map*

$$i^*|_{P_\epsilon(A^\circ)} : P_\epsilon(A^\circ) \longrightarrow (\mathfrak{m}/\mathfrak{m}^2)^*, \quad f \mapsto i^*|_{P_\epsilon(A^\circ)}(f) = f|_{\mathfrak{m}}$$

is a Lie isomorphism.

Proof. Note that $A = \mathbf{k}1_A \oplus \mathfrak{m}$ and if $f \in P_\epsilon(A^\circ)$, then $f(\mathfrak{m}^2) = 0$. Hence $i^*|_{P_\epsilon(A^\circ)}$ is well-defined. If $f \in \ker(i^*|_{P_\epsilon(A^\circ)})$, then

$$f(\alpha 1_A + a) = \alpha f(1_A) + f(a) = 0$$

for all $\alpha \in \mathbf{k}$ and $a \in \mathfrak{m}$. It follows that $i^*|_{P_\epsilon(A^\circ)}$ is injective. If $f \in \mathfrak{m}^*$ such that $f(\mathfrak{m}^2) = 0$, then f is extended to A, denoted by f', by setting

$$f'(\mathbf{k}1_A) = 0, \quad f'|_{\mathfrak{m}} = f.$$

Then, for any $\alpha, \beta \in \mathbf{k}$ and $a, b \in \mathfrak{m}$,

$$\begin{aligned} f'((\alpha 1_A + a)(\beta 1_A + b)) &= f(\alpha b) + f(\beta a) \\ &= \epsilon(\alpha 1_A + a) f'(\beta 1_A + b) + f'(\alpha 1_A + a)\epsilon(\beta 1_A + b). \end{aligned}$$

Hence $f' \in P_\epsilon(A^\circ)$ and thus $i^*|_{P_\epsilon(A^\circ)}$ is surjective. Now $i^*|_{P_\epsilon(A^\circ)}$ is a Lie isomorphism by the definition of Lie brackets. $\square$

1.5. Let us recall the definition for Lie bialgebra in [2, 1.3] and [9, 2.1.1]. A Lie bialgebra is a pair $(\mathfrak{g}, \psi)$, where $\mathfrak{g}$ is a Lie algebra and $\psi : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$, called cobracket, satisfying the following conditions:

(a) The dual map $\psi^* : \mathfrak{g}^* \wedge \mathfrak{g}^* \longrightarrow \mathfrak{g}^*$ makes $\mathfrak{g}^*$ a Lie algebra.

(b) The cobracket $\psi : \mathfrak{g} \longrightarrow \mathfrak{g} \wedge \mathfrak{g}$ is a 1-cocycle on $\mathfrak{g}$ with respect to the $\mathfrak{g}$-module structure on $\mathfrak{g} \wedge \mathfrak{g}$ given by the adjoint action. In other words, we have that for any $a, b \in \mathfrak{g}$,

$$\psi([a, b]) = a \cdot \psi(b) - b \cdot \psi(a),$$

where

$$a \cdot (b \otimes c) = [a \otimes 1 + 1 \otimes a, b \otimes c] = [a, b] \otimes c + b \otimes [a, c].$$

In a Lie bialgebra $(\mathfrak{g}, \psi)$, a Lie ideal $\mathfrak{b}$ of $\mathfrak{g}$ is said to be a Lie bialgebra ideal if $\psi(\mathfrak{b}) \subseteq \mathfrak{g} \otimes \mathfrak{b} + \mathfrak{b} \otimes \mathfrak{g}$. A Lie homomorphism $\varphi : (\mathfrak{g}, \psi) \longrightarrow (\mathfrak{g}', \psi')$ is said to be a Lie bialgebra homomorphism if $(\varphi \otimes \varphi) \circ \psi = \psi' \circ \varphi$. Note that if $\mathfrak{b}$ is a Lie bialgebra ideal of $(\mathfrak{g}, \psi)$, then $(\mathfrak{g}/\mathfrak{b}, \overline{\psi})$ is also a Lie bialgebra. A Lie bialgebra $(\mathfrak{g}, \psi)$ is frequently denoted by $(\mathfrak{g}, \mathfrak{g}^*)$.

Theorem. *Let $(A, \iota, m, \{\cdot, \cdot\}, \epsilon, \Delta)$ be a Poisson bialgebra and let $\mathfrak{m} = \ker \epsilon$. Then $((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)$ is a Lie bialgebra.*

Proof. We will show that the pair $((\mathfrak{m}/\mathfrak{m}^2)^*, \psi)$ is a Lie bialgebra, where $\psi : (\mathfrak{m}/\mathfrak{m}^2)^* \longrightarrow (\mathfrak{m}/\mathfrak{m}^2)^* \wedge (\mathfrak{m}/\mathfrak{m}^2)^*$ is defined by

$$\psi(f)(z_1 \otimes z_2) = f([z_1, z_2]) \tag{4}$$

for all $z_1, z_2 \in \mathfrak{m}/\mathfrak{m}^2$. It is enough to prove that ψ is a 1-cocycle on $(\mathfrak{m}/\mathfrak{m}^2)^*$. The natural $\mathbf{k}$-bilinear form $\langle \cdot, \cdot \rangle$ defined by

$$\langle \cdot, \cdot \rangle : (\mathfrak{m}/\mathfrak{m}^2)^* \times \mathfrak{m}/\mathfrak{m}^2 \longrightarrow \mathbf{k}, \quad \langle f, a + \mathfrak{m}^2 \rangle = f(a + \mathfrak{m}^2)$$

is a nondegenerate **k**-bilinear form. Identifying $(\mathfrak{m}/\mathfrak{m}^2)^*$ to $P_\epsilon(A^\circ)$ by 1.4, we have that, for $f, g \in (\mathfrak{m}/\mathfrak{m}^2)^*$ and $a, b \in \mathfrak{m}$,

$$\begin{aligned}
\langle \psi([f,g]), (a+\mathfrak{m}^2)\otimes(b+\mathfrak{m}^2)\rangle &= \langle [f,g], \{a,b\}+\mathfrak{m}^2\rangle \\
&= (fg)(\{a,b\}) - (gf)(\{a,b\}) \\
&= \sum f(a'b')g(\{a'',b''\}) + f(\{a',b'\})g(a''b'') \\
&\quad - g(a'b')f(\{a'',b''\}) - g(\{a',b'\})f(a''b'') \\
&= \sum (\epsilon(a')f(b') + f(a')\epsilon(b'))g(\{a'',b''\}) \\
&\quad + f(\{a',b'\})(\epsilon(a'')g(b'') + g(a'')\epsilon(b'')) \\
&\quad - (\epsilon(a')g(b') + g(a')\epsilon(b'))f(\{a'',b''\}) \\
&\quad - g(\{a',b'\})(\epsilon(a'')f(b'') + f(a'')\epsilon(b'')) \\
&= \sum f(b')g(\{a,b''\}) + f(a')g(\{a'',b\}) \\
&\quad + f(\{a',b\})g(a'') + f(\{a,b'\})g(b'') \\
&\quad - g(b')f(\{a,b''\}) - g(a')f(\{a'',b\}) \\
&\quad - g(\{a,b'\})f(b'') - g(\{a',b\})f(a'')
\end{aligned}$$

by (2). Let

$$\psi(f) = \sum f_1 \otimes f_2, \quad \psi(g) = \sum g_1 \otimes g_2.$$

Then, by (4), we have

$$\begin{aligned}
f(\{a,b\}) &= \langle \psi(f), (a+\mathfrak{m}^2)\otimes(b+\mathfrak{m}^2)\rangle = \sum f_1(a)f_2(b), \\
g(\{a,b\}) &= \langle \psi(g), (a+\mathfrak{m}^2)\otimes(b+\mathfrak{m}^2)\rangle = \sum g_1(a)g_2(b)
\end{aligned}$$

for all $a, b \in \mathfrak{m}$. Hence

$$\begin{aligned}
&\langle f\cdot\psi(g) - g\cdot\psi(f), (a+\mathfrak{m}^2)\otimes(b+\mathfrak{m}^2)\rangle \\
&= \langle \sum [f,g_1]\otimes g_2 + g_1\otimes[f,g_2] - [g,f_1]\otimes f_2 - f_1\otimes[g,f_2], \\
&\qquad (a+\mathfrak{m}^2)\otimes(b+\mathfrak{m}^2)\rangle \\
&= \sum f(a')g_1(a'')g_2(b) - g_1(a')f(a'')g_2(b) \\
&\quad + g_1(a)f(b')g_2(b'') - g_1(a)g_2(b')f(b'') \\
&\quad - g(a')f_1(a'')f_2(b) + f_1(a')g(a'')f_2(b) \\
&\quad - f_1(a)g(b')f_2(b'') + f_1(a)f_2(b')g(b'') \\
&= \sum f(a')g(\{a'',b\}) - g(\{a',b\})f(a'') \\
&\quad + f(b')g(\{a,b''\}) - g(\{a,b'\})f(b'') \\
&\quad - g(a')f(\{a'',b\}) + f(\{a',b\})g(a'') \\
&\quad - g(b')f(\{a,b''\}) + f(\{a,b'\})g(b'').
\end{aligned}$$

Thus we have $\psi([f,g]) = f\cdot\psi(g) - g\cdot\psi(f)$ for all $f,g\in(\mathfrak{m}/\mathfrak{m}^2)^*$ and so ψ is a 1-cocycle as claimed. □

Example 1.6. Let q be an indeterminate over $\mathbf{k}$. By [1, I.2.2], the coordinate ring of quantum $n\times n$-matrices, denoted by $\mathcal{O}_q(M_n(\mathbf{k}))$, is the $\mathbf{k}[q^{\pm1}]$-algebra generated by $x_{ij}, 1\le i,j\le n$, subject to the relations

$$x_{ij}x_{rs} = \begin{cases} qx_{rs}x_{ij} & i=r \text{ and } j<s,\\ qx_{rs}x_{ij} & i<r \text{ and } j=s,\\ x_{rs}x_{ij} & i<r \text{ and } j>s,\\ x_{rs}x_{ij} + (q-q^{-1})x_{is}x_{rj} & i<r \text{ and } j<s.\end{cases}$$

Thus

$$x_{ij}x_{rs} - x_{rs}x_{ij} = \begin{cases} (q-1)x_{rs}x_{ij} & i=r \text{ and } j<s,\\ (q-1)x_{rs}x_{ij} & i<r \text{ and } j=s,\\ 0 & i<r \text{ and } j>s,\\ q^{-1}(q-1)(q+1)x_{is}x_{rj} & i<r \text{ and } j<s.\end{cases}$$

Hence $\mathcal{O}_q(M_n(\mathbf{k}))/\langle q-1\rangle$ is the commutative $\mathbf{k}$-algebra $\mathbf{k}[\overline{x}_{ij} \mid i,j=1,\ldots,n]$. Moreover $\mathcal{O}_q(M_n(\mathbf{k}))/\langle q-1\rangle$ is a Poisson algebra with Poisson bracket

$$\{\overline{x}_{ij},\overline{x}_{rs}\} = \overline{(q-1)^{-1}(x_{ij}x_{rs} - x_{rs}x_{ij})}$$

by [1, III.5.4]. More precisely, we have that

$$\{\overline{x}_{ij},\overline{x}_{rs}\} = \begin{cases} \overline{x}_{rs}\overline{x}_{ij} & i=r \text{ and } j<s,\\ \overline{x}_{rs}\overline{x}_{ij} & i<r \text{ and } j=s,\\ 0 & i<r \text{ and } j>s,\\ 2\overline{x}_{is}\overline{x}_{rj} & i<r \text{ and } j<s.\end{cases}$$

The coordinate ring of $n\times n$-matrices is the commutative $\mathbf{k}$-algebra

$$\mathbf{k}[x_{ij} \mid i,j=1,\ldots,n],$$

denoted by $\mathcal{O}(M_n(\mathbf{k}))$, which is a bialgebra with the coalgebra structure

$$\epsilon(x_{ij}) = \delta_{ij}, \quad \Delta(x_{ij}) = \sum_{k=1}^{n} x_{ik}\otimes x_{kj}.$$

The algebra $\mathcal{O}(M_n(\mathbf{k}))$ is also a Poisson algebra with Poisson bracket

$$\{x_{ij},x_{rs}\} = \begin{cases} x_{ij}x_{rs} & i=r \text{ and } j<s,\\ x_{ij}x_{rs} & i<r \text{ and } j=s,\\ 0 & i<r \text{ and } j>s,\\ 2x_{is}x_{rj} & i<r \text{ and } j<s\end{cases} \tag{5}$$

by the above paragraph. Moreover $\mathcal{O}(M_n(\mathbf{k}))$ is a Poisson bialgebra since

$$\Delta(\{x_{ij},x_{rs}\}) = \{\Delta(x_{ij}),\Delta(x_{rs})\}$$

for all i,j,r,s, any Poisson bracket satisfies the Leibniz rule and Δ is an algebra homomorphism.

In $\mathfrak{m}/\mathfrak{m}^2$, set

$$e_{ij} = x_{ij} + \mathfrak{m}^2, \quad e_{kk} = (x_{kk} - 1) + \mathfrak{m}^2, \quad i \neq j,\ 1 \leq k \leq n.$$

Then $e_{ij},\ i,j = 1,\dots,n$, form a $\mathbf{k}$-basis of $\mathfrak{m}/\mathfrak{m}^2$ and satisfy

$$\begin{array}{ll}
[e_{ii}, e_{is}] = e_{is} & i < s, \\
[e_{ii}, e_{is}] = -e_{is} & i > s, \\
[e_{ij}, e_{is}] = 0 & i \neq j, i \neq s, \\
[e_{ii}, e_{ri}] = e_{ri} & i < r, \\
[e_{ii}, e_{ri}] = -e_{ri} & i > r, \\
[e_{ij}, e_{rj}] = 0 & i \neq j, r \neq j, \\
[e_{ij}, e_{rs}] = 0 & i < r, j < s, i \neq s, r \neq j, \\
[e_{ij}, e_{rs}] = 2e_{rj} & i < r, j < s, i = s, r \neq j, \\
[e_{ij}, e_{rs}] = 2e_{is} & i < r, j < s, i \neq s, r = j, \\
[e_{ij}, e_{rs}] = 0 & i < r, j > s, \\
[e_{ii}, e_{rr}] = 0 & i \neq r
\end{array}$$

by (5). The dual $(\mathfrak{m}/\mathfrak{m}^2)^*$ has the dual basis e_{ij}^* for e_{ij}, $i,j = 1,\dots,n$, satisfying

$$[e_{ij}^*, e_{rs}^*] = \delta_{jr} e_{is}^* - \delta_{si} e_{rj}^*$$

for all i,j,r,s. That is, $(\mathfrak{m}/\mathfrak{m}^2)^*$ is isomorphic to the general linear Lie algebra $\mathfrak{gl}_n(\mathbf{k})$. Moreover the pair $((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)$ is a Lie bialgebra by 1.5. Now the cobracket $\psi : (\mathfrak{m}/\mathfrak{m}^2)^* \longrightarrow (\mathfrak{m}/\mathfrak{m}^2)^* \wedge (\mathfrak{m}/\mathfrak{m}^2)^*$ is given by

$$\begin{array}{ll}
\psi(e_{ii}^*) = 0 & i = 1,\dots,n, \\
\psi(e_{ij}^*) = e_{ii}^* \wedge e_{ij}^* + e_{ij}^* \wedge e_{jj}^* + \sum\limits_{i<k<j} 2e_{ik}^* \wedge e_{kj}^* & i < j, \\
\psi(e_{ij}^*) = e_{ij}^* \wedge e_{ii}^* + e_{jj}^* \wedge e_{ij}^* + \sum\limits_{j<k<i} 2e_{kj}^* \wedge e_{ik}^* & i > j.
\end{array}$$

Example 1.7. Let $\mathfrak{b}$ denote the Lie ideal $\mathbf{k}(\sum_i e_{ii}^*)$ of $(\mathfrak{m}/\mathfrak{m}^2)^*$ in Example 1.6. Then $\mathfrak{b}$ is a Lie bialgebra ideal since $\psi(\mathfrak{b}) \subseteq (\mathfrak{m}/\mathfrak{m}^2)^* \otimes \mathfrak{b} + \mathfrak{b} \otimes (\mathfrak{m}/\mathfrak{m}^2)^*$ and thus $(\mathfrak{m}/\mathfrak{m}^2)^*/\mathfrak{b}$ is also a Lie bialgebra. In fact, it is checked immediately that the Lie bialgebra $(\mathfrak{m}/\mathfrak{m}^2)^*/\mathfrak{b}$ is isomorphic to the well-known Lie bialgebra $(\mathfrak{sl}_n(\mathbf{k}), \delta)$, where $\delta : \mathfrak{sl}_n(\mathbf{k}) \longrightarrow \mathfrak{sl}_n(\mathbf{k}) \wedge \mathfrak{sl}_n(\mathbf{k})$ is given by

$$(6) \qquad \delta(h_i) = 0, \quad \delta(E_{i,i+1}) = h_i \wedge E_{i,i+1}, \quad \delta(E_{i+1,i}) = h_i \wedge E_{i+1,i},$$

where E_{ij} is the $n \times n$-matrix with 0 for all positions except (i,j)-position and 1 for (i,j)-position and $h_i = E_{ii} - E_{i+1,i+1}$ for $i = 1,\dots,n-1$ (The cobracket δ is uniquely determined by (6) since δ is a 1-cocycle and $\mathfrak{sl}_n(\mathbf{k})$ is generated by $h_i, E_{i,i+1}, E_{i+1,i},\ i = 1,\dots,n-1$). The cobracket δ in (6) is the standard Lie bialgebra structure in $\mathfrak{sl}_n(\mathbf{k})$ (see [2, 1.3.8]).

2. Application

2.1. Let $A = (A, \iota, m, \epsilon, \Delta)$ be a bialgebra. Note that the dual A^* is an A-A bimodule:

$$(a\varphi b)(x) = \varphi(bxa), \quad \varphi \in A^*,\ a,\ b,\ x \in A. \tag{7}$$

For a left A-module M, the dual space M^* is a right A-module with structure

$$(fa)(x) = f(ax), \quad a \in A, f \in M^*, x \in M.$$

Let $\mathcal{C}$ be a class of finite dimensional left A-modules which is closed under finite direct sums and finite tensor products. For any $M \in \mathcal{C}, f \in M^*$ and $v \in M$, the coordinate function $c^M_{f,v} \in A^*$ is defined by

$$c^M_{f,v}(x) = f(xv), \quad x \in A.$$

Then $c^M_{f,v}$ is an element of the restricted dual A° of A since the annihilator I of M is an ideal of A such that the dimension of A/I is finite and $c^M_{f,v}(I) = 0$. It is well-known that the vector space $A(\mathcal{C})$ spanned by all coordinate functions $c^M_{f,v}, M \in \mathcal{C}, f \in M^*, v \in M$, is a sub-bialgebra of A° with structure

$$\begin{aligned} &c^M_{f,v} + c^N_{g,w} = c^{M\oplus N}_{(f,g),(v,w)}, && c^M_{f,v} c^N_{g,w} = c^{M\otimes N}_{f\otimes g, v\otimes w},\\ &\Delta(c^M_{f,v}) = \textstyle\sum_i c^M_{f,v_i} \otimes c^M_{f_i,v}, && \epsilon(c^M_{f,v}) = f(v), \end{aligned} \tag{8}$$

where $\{v_i\}$ and $\{f_i\}$ are dual bases for M and M^* (see [1, I.7]). Moreover if A is a Hopf algebra and $\mathcal{C}$ is closed under duals, then $A(\mathcal{C})$ is a Hopf algebra with antipode S defined by

$$S(c^M_{f,v}) = c^{M^*}_{v,f}, \quad M \in \mathcal{C}, f \in M^*, v \in M.$$

Observe that $A(\mathcal{C})$ has a left and right A-action induced by (7):

$$a \cdot c^M_{f,v} = c^M_{f,av}, \quad c^M_{f,v} \cdot a = c^M_{fa,v}, \quad a \in A. \tag{9}$$

2.2. Let $(\mathfrak{g}, \psi)$ be a Lie bialgebra and let Δ be the comultiplication of $U(\mathfrak{g})$. The cobracket ψ is extended uniquely to a Δ-derivation $\overline{\psi}$ from $U(\mathfrak{g})$ into $U(\mathfrak{g}) \otimes U(\mathfrak{g})$. That is,

$$\overline{\psi} : U(\mathfrak{g}) \longrightarrow U(\mathfrak{g}) \otimes U(\mathfrak{g})$$

is a $\mathbf{k}$-linear map such that $\overline{\psi}|_{\mathfrak{g}} = \psi$ and $\overline{\psi}(xy) = \overline{\psi}(x)\Delta(y) + \Delta(x)\overline{\psi}(y)$ for all $x, y \in U(\mathfrak{g})$.

Let $(\mathfrak{g}, \psi)$ be a coboundary Lie bialgebra such that the cobracket ψ determined by a classical r-matrix $r = \sum_i a_i \otimes b_i$. That is, r satisfies the modified classical Yang-Baxter equation and ψ is defined by

$$\psi(x) = x \cdot r = \sum_i [x, a_i] \otimes b_i + a_i \otimes [x, b_i] = [\Delta(x), r]_{U(\mathfrak{g})\otimes U(\mathfrak{g})}$$

for all $x \in \mathfrak{g}$ (refer to [2, 2.1] and [9, §4.1] for the definition of a coboundary Lie bialgebra). Then the extension map $\overline{\psi}$ of ψ to $U(\mathfrak{g})$ is given by $\overline{\psi}(x) = [\Delta(x), r]_{U(\mathfrak{g})\otimes U(\mathfrak{g})}$ for all $x \in U(\mathfrak{g})$.

Theorem. *Let $(\mathfrak{g}, \psi)$ be a coboundary Lie bialgebra such that the cobracket ψ is determined by a classical r-matrix r. Fix a class $\mathcal{C}$ of finite dimensional left $U(\mathfrak{g})$-modules which is closed under finite direct sums and finite tensor products. Denote by $A(\mathcal{C})$ the vector space spanned by all coordinate functions $c^M_{f,v}, M \in \mathcal{C}, f \in M^*, v \in M$. Then $A(\mathcal{C})$ is a Poisson bialgebra with Poisson bracket*

$$\{c^M_{f,v}, c^N_{g,w}\}(x) = \langle \overline{\psi}(x), c^M_{f,v} \otimes c^N_{g,w}\rangle \tag{10}$$

for all $x \in U(\mathfrak{g})$.

Remark. Observe that, in the above theorem, $A(\mathcal{C})$ is a sub-Poisson bialgebra of the restricted dual $U(\mathfrak{g})^\circ$ and we obtain a Lie bialgebra $((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)$ by applying 1.5 to $A(\mathcal{C})$, where $\mathfrak{m}$ is the kernel of the counit in $A(\mathcal{C})$.

Proof of Theorem. We have already known that $A(\mathcal{C})$ is a sub-bialgebra of the restricted dual $U(\mathfrak{g})^\circ$ with structure (8) by 2.1.

Denote $r = \sum_i a_i \otimes b_i$. Then

$$\overline{\psi}(x) = [\Delta(x), r]_{U(\mathfrak{g})\otimes U(\mathfrak{g})} = \sum_{(x)} \sum_i (x'a_i \otimes x''b_i - a_i x' \otimes b_i x'')$$

for all $x \in U(\mathfrak{g})$, thus

$$\begin{aligned}\{c^M_{f,v}, c^N_{g,w}\}(x) &= \sum_i \sum_{(x)} c^M_{f,v}(x'a_i)c^N_{g,w}(x''b_i) - \sum_i \sum_{(x)} c^M_{f,v}(a_i x')c^N_{g,w}(b_i x'') \\ &= \sum_i (c^M_{f,a_i v} c^N_{g,b_i w})(x) - \sum_i (c^M_{fa_i,v} c^N_{gb_i,w})(x).\end{aligned}$$

Hence

$$\{c^M_{f,v}, c^N_{g,w}\} = \sum_i (c^M_{f,a_i v} c^N_{g,b_i w}) - \sum_i (c^M_{fa_i,v} c^N_{gb_i,w}) \in A(\mathcal{C}), \tag{11}$$

that is, the Poisson bracket (10) is well-defined.

Let $\tau : U(\mathfrak{g})\otimes U(\mathfrak{g}) \longrightarrow U(\mathfrak{g})\otimes U(\mathfrak{g})$ be the flip. Since $U(\mathfrak{g})$ is cocommutative and $\tau(r) = -r$, we have that $\tau\overline{\psi}(x) = -\overline{\psi}(x)$ for all $x \in U(\mathfrak{g})$, thus we have immediately that $\{c^M_{f,v}, c^N_{g,w}\} = -\{c^N_{g,w}, c^M_{f,v}\}$ for all $c^M_{f,v}, c^N_{g,w} \in A(\mathcal{C})$ by (10).

For distinct numbers $s, t = 1, 2, 3$, denote by $r_{st} \in \mathfrak{g}\otimes\mathfrak{g}\otimes\mathfrak{g}$ the element with a_i for s-component, b_i for t-component and 1 for the other component. For instance, $r_{12} = \sum a_i \otimes b_i \otimes 1$ and $r_{31} = \sum b_i \otimes 1 \otimes a_i$. Note that $r_{st} = -r_{ts}$ for all distinct numbers $s, t = 1, 2, 3$, by the skew symmetry of r. Since $\Delta(a) =$

$a \otimes 1 + 1 \otimes a$ for all $a \in \mathfrak{g}$, we have

$$\begin{aligned}\{\{c^M_{f,v}, c^N_{g,w}\}, c^L_{h,u}\}(x) = &\langle \Delta^2(x)(r_{13}+r_{23})r_{12}, c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &- \langle r_{12}\Delta^2(x)(r_{13}+r_{23}), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &- \langle (r_{13}+r_{23})\Delta^2(x)r_{12}, c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &+ \langle r_{12}(r_{13}+r_{23})\Delta^2(x), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle\end{aligned}$$

for $x \in U(\mathfrak{g})$, where $\Delta^2 = (\Delta\otimes 1)\circ\Delta = (1\otimes\Delta)\circ\Delta$, by (10). Hence, by $r_{st} = -r_{ts}$ for all $s, t = 1, 2, 3$ and the coassociativity of Δ, we have that

$$\begin{aligned}&(\{\{c^M_{f,v}, c^N_{g,w}\}, c^L_{h,u}\} + \{\{c^N_{g,w}, c^L_{h,u}\}, c^M_{f,v}\} + \{\{c^L_{h,u}, c^M_{f,v}\}, c^N_{g,w}\})(x) \\ =& \langle \Delta^2(x)(r_{13}+r_{23})r_{12}, c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &- \langle r_{12}\Delta^2(x)(r_{13}+r_{23}), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &- \langle (r_{13}+r_{23})\Delta^2(x)r_{12}, c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &+ \langle r_{12}(r_{13}+r_{23})\Delta^2(x), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &+ \langle \Delta^2(x)(r_{21}+r_{31})r_{23}, c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &- \langle r_{23}\Delta^2(x)(r_{21}+r_{31}), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &- \langle (r_{21}+r_{31})\Delta^2(x)r_{23}, c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &+ \langle r_{23}(r_{21}+r_{31})\Delta^2(x), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &+ \langle \Delta^2(x)(r_{32}+r_{12})r_{31}, c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &- \langle r_{31}\Delta^2(x)(r_{32}+r_{12}), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &- \langle (r_{32}+r_{12})\Delta^2(x)r_{31}, c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ &+ \langle r_{31}(r_{32}+r_{12})\Delta^2(x), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u}\rangle \\ =& \langle ([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}])\Delta^2(x), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u})\rangle \\ &\quad - \langle \Delta^2(x)([r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}]), c^M_{f,v}\otimes c^N_{g,w}\otimes c^L_{h,u})\rangle \\ =& 0\end{aligned}$$

for any $c^M_{f,v}, c^N_{g,w}, c^L_{h,u} \in A(\mathcal{C})$ and $x \in U(\mathfrak{g})$ since

$$\begin{aligned}&[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] \\ =& (r_{12}r_{13} - r_{13}r_{12}) + (r_{12}r_{23} - r_{23}r_{12}) + (r_{13}r_{23} - r_{23}r_{13}) \\ =& \sum_{i,j}[a_i, a_j]\otimes b_i\otimes b_j + \sum_{i,j} a_i\otimes[b_i, a_j]\otimes b_j + \sum_{i,j} a_i\otimes a_j\otimes[b_i, b_j]\end{aligned}$$

is $\mathfrak{g}$-invariant. Hence the Poisson bracket (10) satisfies the Jacobi identity.

By (11), we have

$$\begin{aligned}\{c^M_{f,v}, c^N_{g,w}c^L_{h,u}\} &= \sum_i c^M_{f,a_iv}(c^N_{g,b_iw}c^L_{h,u} + c^N_{g,w}c^L_{h,b_iu})\\ &\quad - \sum_i c^M_{fa_i,v}(c^N_{gb_i,w}c^L_{h,u} + c^N_{g,w}c^L_{hb_i,u})\\ &= \{c^M_{f,v}, c^N_{g,w}\}c^L_{h,u} + c^N_{g,w}\{c^M_{f,v}, c^L_{h,u}\}.\end{aligned}$$

It follows that the Poisson bracket (10) satisfies the Leibniz rule.

Let us prove that $\Delta(\{c^M_{f,v}, c^N_{g,w}\}) = \{\Delta(c^M_{f,v}), \Delta(c^N_{g,w})\}$ for all elements $c^M_{f,v}$, $c^N_{g,w} \in A(\mathcal{C})$. Note that $\Delta(c^M_{f,v}) = \sum_j c^M_{f,v_j} \otimes c^M_{f_j,v}$, $\Delta(c^N_{g,w}) = \sum_k c^N_{g,w_k} \otimes c^N_{g_k,w}$, where $\{v_j\}, \{f_j\}$ are dual bases for M and M^* and $\{w_k\}, \{g_k\}$ are dual bases for N and N^*. Now, for any $x, y \in U(\mathfrak{g})$,

$$\begin{aligned}\Delta(\{c^M_{f,v}, c^N_{g,w}\})(x\otimes y) &= \langle \overline{\psi}(xy), c^M_{f,v} \otimes c^N_{g,w}\rangle\\ &= \langle \overline{\psi}(x)\Delta(y), c^M_{f,v} \otimes c^N_{g,w}\rangle + \langle \Delta(x)\overline{\psi}(y), c^M_{f,v} \otimes c^N_{g,w}\rangle\\ &= \sum_{j,k}\langle \overline{\psi}(x), c^M_{f,v_j} \otimes c^N_{g,w_k}\rangle\langle \Delta(y), c^M_{f_j,v} \otimes c^N_{g_k,w}\rangle\\ &\qquad + \sum_{j,k}\langle \Delta(x), c^M_{f,v_j} \otimes c^N_{g,w_k}\rangle\langle \overline{\psi}(y), c^M_{f_j,v} \otimes c^N_{g_k,w}\rangle\\ &= \sum_{j,k}(\{c^M_{f,v_j}, c^N_{g,w_k}\} \otimes c^M_{f_j,v}c^N_{g_k,w})(x\otimes y)\\ &\qquad + \sum_{j,k}(c^M_{f,v_j}c^N_{g,w_k} \otimes \{c^M_{f_j,v}, c^N_{g_k,w}\})(x\otimes y)\\ &= \{\Delta(c^M_{f,v}), \Delta(c^N_{g,w})\}(x\otimes y).\end{aligned}$$

Hence we have $\Delta(\{c^M_{f,v}, c^N_{g,w}\}) = \{\Delta(c^M_{f,v}), \Delta(c^N_{g,w})\}$ for all elements $c^M_{f,v}, c^N_{g,w} \in A(\mathcal{C})$. This completes the proof. □

Proposition 2.3. *Let $(\mathfrak{g}, \psi)$ be a coboundary Lie bialgebra such that $\mathfrak{g}$ is connected and simply connected and let $\mathcal{C}$ be the set of all finite dimensional left $U(\mathfrak{g})$-modules. Then $A(\mathcal{C})$ is the restricted dual $U(\mathfrak{g})^\circ$. Moreover the given Lie bialgebra $(\mathfrak{g}, \psi)$ is isomorphic to $((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)$, where $\mathfrak{m}$ is the kernel of the counit ϵ of $A(\mathcal{C})$.*

Proof. Note that the set of all finite dimensional left $U(\mathfrak{g})$-modules is closed under finite direct sums and finite tensor products. Since every element of the restricted dual $U(\mathfrak{g})^\circ$ is represented by a coordinate function $c^M_{f,v}$ for some finite dimensional left $U(\mathfrak{g})$-module M, we have immediately that $A(\mathcal{C})$ is the restricted dual $U(\mathfrak{g})^\circ$. Moreover $((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)$ is a Lie bialgebra by 2.2 and 1.5, and $\mathfrak{g} = (\mathfrak{m}/\mathfrak{m}^2)^*$ by [7, 7.11]. Thus $\mathfrak{g}^*$ is equal to $\mathfrak{m}/\mathfrak{m}^2$ as a Lie algebra by (3) and (10). It follows that the Lie bialgebra $((\mathfrak{m}/\mathfrak{m}^2)^*, \mathfrak{m}/\mathfrak{m}^2)$ is equal to $(\mathfrak{g}, \psi) = (\mathfrak{g}, \mathfrak{g}^*)$. □

Example 2.4. In the symplectic Lie algebra $\mathfrak{sp}_4$, set

$$\begin{array}{llll} h_1 = E_{11} - E_{22} - E_{33} + E_{44}, & h_2 = E_{22} - E_{44}, & & \\ e_1 = E_{12} - E_{43}, & e_2 = E_{24}, & e_3 = E_{14} + E_{23}, & e_4 = E_{13}, \\ f_1 = E_{21} - E_{34}, & f_2 = E_{42}, & f_3 = E_{41} + E_{32}, & f_4 = E_{31} \end{array}$$

(see [5, 8.3] for $\mathfrak{sp}_4$). Let H be the subspace of $\mathfrak{sp}_4$ spanned by h_1, h_2 and let $\alpha_1, \alpha_2 \in H^*$ be defined by

$$\begin{array}{ll} \alpha_1(h_1) = 2, & \alpha_2(h_1) = -2, \\ \alpha_1(h_2) = -1, & \alpha_2(h_2) = 2. \end{array}$$

Then $e_1, e_2, e_3, e_4, f_1, f_2, f_3, f_4$ are weight vectors with weights

$$\begin{array}{llll} \mathrm{wt}(e_1) = \alpha_1, & \mathrm{wt}(e_2) = \alpha_2, & \mathrm{wt}(e_3) = \alpha_1 + \alpha_2, & \mathrm{wt}(e_4) = 2\alpha_1 + \alpha_2, \\ \mathrm{wt}(f_1) = -\alpha_1, & \mathrm{wt}(f_2) = -\alpha_2, & \mathrm{wt}(f_3) = -(\alpha_1 + \alpha_2), & \mathrm{wt}(f_4) = -(2\alpha_1 + \alpha_2). \end{array}$$

Hence α_1, α_2 are positive simple roots. It is well-known that

$$r = e_1 \wedge f_1 + 2e_2 \wedge f_2 + e_3 \wedge f_3 + 2e_4 \wedge f_4 \in \mathfrak{sp}_4 \wedge \mathfrak{sp}_4$$

satisfies the modified classical Yang-Baxter equation and gives the standard Lie bialgebra structure ψ in $\mathfrak{sp}_4$ such that

$$\begin{array}{ll} \psi(h_1) = 0, & \psi(h_2) = 0, \\ \psi(e_1) = e_1 \wedge h_1, & \psi(e_2) = 2e_2 \wedge h_2, \\ \psi(e_3) = e_3 \wedge h_1 + 2e_3 \wedge h_2 - 4e_1 \wedge e_2, & \psi(e_4) = 2e_4 \wedge h_1 + 2e_4 \wedge h_2 - 2e_1 \wedge e_3, \\ \psi(f_1) = f_1 \wedge h_1, & \psi(f_2) = 2f_2 \wedge h_2, \\ \psi(f_3) = f_3 \wedge h_1 + 2f_3 \wedge h_2 - 4f_1 \wedge f_2, & \psi(f_4) = 2f_4 \wedge h_1 + 2f_4 \wedge h_2 - 2f_1 \wedge f_3 \end{array}$$

(see [9, Exercise 4.1.11]).

The weight lattice $\mathbf{P}$ in $\mathfrak{sp}_4$ is a free abelian group with basis consisting of the fundamental dominant integral weights λ_1, λ_2, where $\lambda_i(h_j) = \delta_{ij}$ for $i, j = 1, 2$. Hence

$$\alpha_1 = 2\lambda_1 - \lambda_2, \quad \alpha_2 = -2\lambda_1 + 2\lambda_2.$$

The natural $\mathfrak{sp}_4$-module $V = \mathbf{k}^4$ is an irreducible highest weight module with highest weight λ_1. In fact, set

$$v_1 = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, v_2 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, v_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}, v_4 = \begin{pmatrix} 0 \\ 0 \\ -1 \\ 0 \end{pmatrix} \in V.$$

Then v_1 is a highest weight vector with highest weight λ_1 and

$$v_1 \in V_{\lambda_1}, v_2 = f_1 v_1 \in V_{-\lambda_1+\lambda_2}, v_3 = f_2 v_2 \in V_{\lambda_1-\lambda_2}, v_4 = f_1 v_3 \in V_{-\lambda_1}.$$
$$(v_1 \xrightarrow{f_1} v_2 \xrightarrow{f_2} v_3 \xrightarrow{f_1} v_4)$$

Here we simply write $c_{f,v}$ for $c^V_{f,v}$, $v \in V, f \in V^*$. Observe that

$v_1^* = (V^*)_{-\lambda_1}, v_2^* = v_1^* e_1 \in (V^*)_{\lambda_1-\lambda_2}, v_3^* = v_2^* e_2 \in (V^*)_{-\lambda_1+\lambda_2}, v_4^* = v_3^* e_1 \in (V^*)_{\lambda_1}$

$$(v_1^* \xrightarrow{e_1} v_2^* \xrightarrow{e_2} v_3^* \xrightarrow{e_1} v_4^*)$$

and

$$\mathcal{C} = \{\mathbf{k}, V^n, V^{\otimes n} \mid n = 1, 2, \ldots\}$$

is a class of $U(\mathfrak{sp}_4)$-modules closed under finite direct sums and finite tensor products. Thus $A(\mathcal{C})$ is a sub-Poisson bialgebra of the Poisson bialgebra $U(\mathfrak{sp}_4)^\circ$ by 2.2. Set

$$\begin{array}{llll} h_1^* = \overline{c_{v_1^*,v_1} - 1}, & h_2^* = \overline{c_{v_2^*,v_2} + c_{v_1^*,v_1} - 2}, & & \\ x_1^* = \overline{c_{v_1^*,v_2}}, & x_2^* = \overline{c_{v_1^*,v_3}}, & y_1^* = \overline{c_{v_2^*,v_1}}, & y_2^* = \overline{c_{v_3^*,v_1}} \end{array}$$

in $\mathfrak{m}/\mathfrak{m}^2$, where $\mathfrak{m} = \ker \epsilon$. Let S be the antipode of $U(\mathfrak{sp}_4)^\circ$. Since $V^* \cong V$ as a $U(\mathfrak{sp}_4)$-module and $m \circ (\mathrm{id}_{U(\mathfrak{sp}_4)^*} \otimes S) \circ \Delta = \epsilon 1$, we have

$$\begin{array}{llll} \overline{c_{v_3^*,v_4}} = -\overline{c_{v_1^*,v_2}}, & \overline{c_{v_2^*,v_4}} = -\overline{c_{v_1^*,v_3}}, & \overline{c_{v_2^*,v_3}} = 0, & \overline{c_{v_1^*,v_4}} = 0, \\ \overline{c_{v_4^*,v_3}} = -\overline{c_{v_2^*,v_1}}, & \overline{c_{v_3^*,v_1}} = -\overline{c_{v_4^*,v_2}}, & \overline{c_{v_3^*,v_2}} = 0, & \overline{c_{v_4^*,v_1}} = 0. \end{array}$$

It follows that $\mathfrak{m}/\mathfrak{m}^2$ is a 6-dimensional Lie algebra with structure

$$\begin{array}{lll} [h_1^*, h_2^*] = 0, & [h_1^*, x_1^*] = -x_1^*, & [h_1^*, x_2^*] = -x_2^*, \\ [h_1^*, y_1^*] = -y_1^*, & [h_1^*, y_2^*] = -y_2^*, & [h_2^*, x_1^*] = 0, \\ [h_2^*, x_2^*] = -2x_2^*, & [h_2^*, y_1^*] = 0, & [h_2^*, y_2^*] = -2y_2^*, \\ [x_1^*, x_2^*] = 0, & [y_1^*, y_2^*] = 0, & [x_i^*, y_j^*] = 0 \quad (i, j = 1, 2), \end{array}$$

where $[h_1^*, h_2^*] = \overline{\{c_{v_1^*,v_1} - 1, c_{v_2^*,v_2} + c_{v_1^*,v_1} - 2\}} \in \mathfrak{m}/\mathfrak{m}^2$, etc. and the dual Lie algebra $(\mathfrak{m}/\mathfrak{m}^2)^*$ is a six dimensional Lie algebra with the following structure

$$\begin{array}{lll} [h_1, h_2] = 0, & [h_1, x_1] = 2x_1, & [h_1, x_2] = 0, \\ [h_1, y_1] = -2y_1, & [h_1, y_2] = 0, & [h_2, x_1] = -x_1, \\ [h_2, x_2] = x_2, & [h_2, y_1] = y_1, & [h_2, y_2] = -y_2, \\ [x_1, x_2] = 0, & [y_1, y_2] = 0, & [x_1, y_1] = h_1, \\ [x_1, y_2] = 0, & [x_2, y_1] = 0, & [x_2, y_2] = h_1 + 2h_2. \end{array}$$

Hence the Lie algebra $(\mathfrak{m}/\mathfrak{m}^2)^*$ is a Lie bialgebra with cobracket ψ satisfying

$$\begin{array}{ll} \psi(h_1) = 0, & \psi(h_2) = 0, \\ \psi(x_1) = x_1 \wedge h_1, & \psi(x_2) = x_2 \wedge h_1 + 2x_2 \wedge h_2, \\ \psi(y_1) = y_1 \wedge h_1, & \psi(y_2) = y_2 \wedge h_1 + 2y_2 \wedge h_2. \end{array}$$

References

[1] K. A. Brown and K. R. Goodearl, *Lectures on Algebraic Quantum Groups*, Advanced Courses in Mathematics. CRM Barcelona. Birkhauser Verlag, Basel, 2002.

[2] V. Chari and A. Pressley, *A Guide to Quantum Groups*, Cambridge University Press, Cambridge, 1994.

[3] T. J. Hodges and T. Levasseur, *Primitive ideals of* C_q[SL(3)], Comm. Math. Phys. **156** (1993), no. 3, 581–605.

[4] T. J. Hodges, T. Levasseur, and M. Toro, *Algebraic structure of multiparameter quantum groups*, Adv. Math. **126** (1997), no. 1, 52–92.

[5] J. Hong and S.-J. Kang, *Introduction to Quantum Groups and Crystal Bases*, Graduate Studies in Mathematics, 42. American Mathematical Society, Providence, RI, 2002.

[6] J. E. Humphreys, *Linear Algebraic Groups*, Graduate Texts in Mathematics, vol. 21, Springer-Verlag Inc., thirding printing, 1987.

[7] J. C. Jantzen, *Lectures on Quantum Groups*, Graduate Studies in Mathematics, vol. 6, American Mathematical Society, Providence, 1996.

[8] A. Joseph, *Quantum Groups and Their Primitive Ideals*, A series of modern surveys in mathematics, vol. 3. Folge·Band 29, Springer-Verlag, Berlin, 1995.

[9] L. I. Korogodski and Y. S. Soibelman, *Algebras of Functions on Quantum Groups. Part I.*, Mathematical Surveys and Monographs, 56. American Mathematical Society, Providence, RI, 1998.

[10] L. A. Lambe and D. E. Radford, *Introduction to the Quantum Yang-Baxter Equation and Quantum Groups: an algebraic approach*, Mathematics and its Applications, 423. Kluwer Academic Publishers, Dordrecht, 1997.

Sei-Qwon Oh
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
E-mail address: sqoh@cnu.ac.kr

Eun-Hee Cho
Department of Mathematics
Chungnam National University
Daejeon 305-764, Korea
E-mail address: ehcho@cnu.ac.kr