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GORENSTEIN-INJECTORS, GORENSTEIN-FLATORS

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Abstract. Over a ring R, let P_R be a finitely generated projective right R-module. Then we define the G-injector (G-projector) if P_R preservers Gorenstein injective modules (Gorenstein projective modules), the G-flator if P_R preservers Gorenstein flat modules. G-injector (G-flator) and G-injector are characterized focus primarily on the cases where R is a Gorenstein ring, and under this condition we also study the relations between the injector (projector, flator) and the G-injector (G-projector, G-flator). Over any ring we also give the characteristics of G-injector (G-flator) by the Gorenstein injective (Gorenstein flat) dimensions of modules.

Introduction

Unless otherwise stated, throughout this paper R will denote an associative ring with identity, P_R will denote a finitely generated projective right R-module. ${}_RP^*$ will denote its R-dual ${}_RP^* = \mathrm{Hom}_R(P_R, R)$ and S will denote its R-endomorphism ring $S = \mathrm{End}_R(P_R)$.

Let ${}_R\mathcal{M}$ and ${}_S\mathcal{M}$ be categories of all left R-modules and of all left S-modules respectively. By an (R, S) adjoint triple is meant a triple $(\mathscr{G}, \mathscr{F}, \mathscr{H})$ of additive functors

$$\mathscr{F} :_R \mathcal{M} \to_S \mathcal{M} \ \text{ and } \ \mathscr{G}, \ \ \mathscr{H} :_S \mathcal{M} \to_R \mathcal{M}$$

such that there are natural isomorphisms

$$\mathrm{Hom}_R(\mathscr{G}(N), M) \cong \mathrm{Hom}_S(N, \mathscr{F}(M)),$$
$$\mathrm{Hom}_S(\mathscr{F}(M), N) \cong \mathrm{Hom}_R(M, \mathscr{H}(N))$$

for all $M \in_R \mathcal{M}$ and all $N \in_S \mathcal{M}$. When ${}_SP_R$ is a bimodule with P_R finitely generated projective then by [11] $\mathscr{F}, \mathscr{G}, \mathscr{H}$ are naturally equivalent to the functors

$$\mathscr{F}_P = P_R \otimes_R () :_R \mathcal{M} \longrightarrow_S \mathcal{M},$$
$$\mathscr{G} = \mathrm{Hom}_R(P_R, R) \otimes_S () :_S \mathcal{M} \longrightarrow_R \mathcal{M},$$
$$\mathscr{H}_P = \mathrm{Hom}_S(P_R,) :_S \mathcal{M} \longrightarrow_R \mathcal{M}.$$

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Anderson [1] defined the injector, the projector and studied the characteristics of them. Moreover, he gave some conditions under which P_R preserves the injective modules (projective modules). In [2], Miller defined the flator and generated Anderson's results. He also gave the relations among flators, injectors, and projectors. In 1995, Enochs and Jenda [6] defined the Gorenstein injective modules, Gorenstein projective modules. And defined Gorenstein flat modules in [8]. As we all know these notions generalized the usual injective, projective and flat modules.

In this paper we shall define the G-injector (the G-projector), the perfect G-injector, the G-flator and the perfect G-flator which generalize Anderson's injector (projector), perfect injector, and Miller's flator, perfect flator respectively.

When P_R is a finitely generated projective generator, we know that the ring R is Morita equivalent to $S = \mathrm{End}(P_R)$ (see [2]). It is easy to know that P_R preserves Gorenstein injective modules (Gorenstein projective modules), and Gorenstein injective envelopes (Gorenstein projective covers). When P_R is not a generator, R and S are not Morita equivalent; still, for many projective modules P, a considerable amount of information is often available about S. Here, we focus primarily on the cases where R is a Gorenstein ring. We consider the conditions that P_R preserves G-injective (G-projective) modules and G-flat modules. We shall show that if P_R is a G-injector (G-flator), then $P_R \otimes -$ preservers modules with finite Gorenstein injective (Gorenstein flat) dimensions, that is, if N is a left R-module of finite Gorenstein injective (Gorenstein flat) dimension, then ${}_SP_R \otimes N$ is a left S-module of finite Gorenstein injective (Gorenstein flat) dimension. We also give the relation between the G-injector (G-flator) and the Gorenstein injective dimensions of modules.

1. Preliminaries

In this section we shall recall some definitions which we use later.

Definition 1.1 (Gorenstein injective module)**.** A left R-module M is said to be Gorenstein injective if and only if there is an exact sequence

$$\cdots \to_R E_1 \to_R E_0 \to_R E^0 \to_R E^1 \to \cdots$$

of injective left R-modules such that $M = \mathrm{Ker}({}_RE^0 \to_R E^1)$ and such that for any injective left R-module E, $\mathrm{Hom}(E, -)$ leaves the complex above exact. Dually, we can define the Gorenstein projective module.

A left R-module N is said to be Gorenstein flat (or G-flat) if there exists an $(\mathcal{I}nj \otimes -)$-exact exact sequence

$$\cdots \to_R F_1 \to_R F_0 \to_R F^0 \to_R F^1 \to \cdots$$

of flat R-modules such that $N = \mathrm{Ker}({}_RF^0 \to_R F^1)$ (see [7]). The above exact sequence is called a complete flat resolution.

Definition 1.2 (Strongly Gorenstein injective module)**.** If all injective (resp., projective) modules and homomorphisms of the complete injective (resp., projective) resolution in Definition 1.1 are the same, then M is called strongly Gorenstein projective (resp., injective).

If all flat modules and homomorphisms of the complete flat resolution in Definition 1.1 are the same, then N is called strongly Gorenstein flat (see [3]).

Definition 1.3 (Injector and flator)**.** We call ${}_SP_R$ an injector (projector, flator) in case $F_P = P_R \otimes -$ preserves injective (projective, flat) modules where $S = \mathrm{End}(P_R)$, that is, $F_P(M)$ is S-injective (projective, flat) whenever M is R-injective (projective, flat) (see [1] and [10]).

Definition 1.4 (Gorenstein injective preenvelope)**.** A Gorenstein injective preenvelope of an R-module M, we mean a morphism $\varphi :_R M \to_R G$ where G is a Gorenstein injective module such that for any morphism $f :_R M \to_R G'$ with G' is Gorenstein injective, there is a $g :_R F \to_R F'$ such that $g \circ \varphi = f$. Dually, we can define the Gorenstein flat (Gorenstein projective) precover (see [7]).

Definition 1.5 (Gorenstein injective dimension)**.** For an R-module M we said its Gorenstein injective dimension is equal to or less than n if it has a Gorenstein injective resolution whose length is equal to or less than n, we denote it by $\mathrm{Gid}_R\, M \leq n$ (see [9]).

Definition 1.6 (Copure injective)**.** An R-module M is said to be copure injective (copure flat) if $\mathrm{Ext}^1_R(E, M) = 0$ ($\mathrm{Tor}^R_1(E, M) = 0$) for any injective R-module E. Also, M is said to be strongly copure injective (strongly copure flat) if $\mathrm{Ext}^i_R(E, M) = 0$ ($\mathrm{Tor}^R_i(E, M) = 0$) for any injective R-module E and any $i > 0$ (see [5]).

Definition 1.7 ($T(p)$)**.** In general $\varphi :_R P^*_S \otimes_S P_R$ is not an isomorphism, the image of φ is a two sided ideal $T = T(P)$ of R called the trace ideal of P, thus $T = \mathrm{Im}\varphi = \Sigma \mathrm{Im} f \quad (f \in P^*)$ (see [1]).

Remark. By $\mathcal{I}(R)$ we denote the class of injective left R-modules, and by $\widetilde{\mathcal{GI}}(R)$, $\widetilde{\mathcal{GF}}(R)$ we denote the classes of all R-modules with finite Gorenstein injective, flat dimensions respectively. By $\mathcal{I}(S)$ we denote the class of injective left S-modules, and by $\widetilde{\mathcal{GI}}(S)$, $\widetilde{\mathcal{GF}}(S)$ we denote the classes of all S-modules with finite Gorenstein injective, flat dimensions respectively.

2. Gorenstein-injectors

Let P_R be a finitely generated projective module, and $S = \mathrm{End}(P_R)$. We call P_R a G-injector if P_R preservers Gorenstein injective modules, that is, for any Gorenstein injective left R-module ${}_RM$, ${}_SP_R \otimes_R M$ is a Gorenstein injective left S-module.

Proposition 2.1. *Let ${}_SP_R$ be an injector. If ${}_RP^*_S \otimes_S -$ preserves injective modules, then P_R is a G-injector.*

Proof. Assume that M is a Gorenstein injective left R-module, then there exists a $\mathrm{Hom}(\mathcal{I}(R), -)$-exact exact sequence

$$\cdots \to E_2 \to E_1 \to E_0 \to E^0 \to E^1 \to E^2 \to \cdots$$

of injective left R-modules such that

$$M = \mathrm{Ker}(E^0 \to E^1).$$

Applying $P_R \otimes -$ to the above exact sequence, we get an exact sequence:

$$(1) \qquad \cdots \to P_R \otimes_R E_1 \to P_R \otimes_R E_0 \to P_R \otimes_R E^0 \to P_R \otimes_R E^1 \to \cdots .$$

Since P_R is an injector each $P_R \otimes_R E_i$ and each $P_R \otimes_R E^i$ are injective. Clearly, $P_R \otimes_R M = \mathrm{Ker}(P_R \otimes_R E^0 \to P_R \otimes_R E^1)$. Note that

$$\mathrm{Hom}_S(E', P_R \otimes E) \cong \mathrm{Hom}_R({}_RP_S^* \otimes E', E)$$

for any left S-module E' and any left R-module E, and ${}_RP_S^* \otimes -$ preserves injective modules; then (1) is a $\mathrm{Hom}(\mathcal{I}(S), -)$-exact exact sequence of injective left S-modules. Thus P_R is a G-injector. □

Lemma 2.2. *Let ${}_SP_R$ be an (R, S)-bimodule with ${}_SP$ projective and P_R finitely generated projective. If R is Gorenstein, then $S = \mathrm{End}(P_R)$ is Gorenstein.*

Proof. Since R is a Gorenstein ring, S is Notherian by [1], [7, Theorem 9.1.17] gives that P_R has finite injective dimension. Now we assume $id(P_R) = n < \infty$, That is, there exists an injective resolution of P_R:

$$0 \to_S P_R \to E_R^0 \to E_R^1 \to E_R^2 \to \cdots \to E_R^n \to 0.$$

Applying $\mathrm{Hom}_R({}_SP_R, -)$ to this resolution, we get

$$0 \to \mathrm{Hom}_S({}_SP_R, {}_SP_R) \to \mathrm{Hom}_S({}_SP_R, E_R^0) \to \cdots \to \mathrm{Hom}_S({}_SP_R, E_R^n) \to 0$$

as each $\mathrm{Hom}_S({}_SP_R, E_R^i)$ is injective as an S-module. So S has finite self-injective dimension. Thus S is Gorenstein. □

Corollary 2.3. *Let ${}_SP_R$ be an (R, S)-bimodule with ${}_SP$ projective and P_R finitely generated projective. If R is n-Gorenstein, then $S = \mathrm{End}(P_R)$ is m-Gorenstein ($m \leq n$).*

Proposition 2.4. *Let ${}_SP_R$ be an (R, S)-bimodule with ${}_SP$ projective and P_R finitely generated projective, and let R be a Gorenstein ring. If ${}_SP_R$ is a G-injector, then P_R is an injector.*

Proof. Let E_R be an injective left R-module. Then ${}_SP_R \otimes_R E$ is a Gorenstein injective left S-module. Since R is Gorenstein and ${}_RE$ is injective by [7, Theorem 9.1.17], we know that ${}_RE$ has finite projective dimension. Assume that $pd_R(E) = n$. So there exists a projective resolution of E

$$(2) \qquad 0 \to_R P_n \to \cdots \to_R P_2 \to_R P_1 \to_R E \to 0.$$

Since ${}_SP_R$ is left S-projective, ${}_SP_R \otimes_R P_i$ is projective by [1, Theorem 3.1]. Applying ${}_SP_R \otimes -$ to (2), we get a projective resolution of ${}_RE$

$$(3) \quad 0 \to_S P_R \otimes_R P_n \to \cdots \to_S P_R \otimes_R P_2 \to_S P_R \otimes_R P_1 \to_S P_R \otimes_R E \to 0.$$

Then ${}_SP_R \otimes_R E$ has finite projective dimension, and so the injective dimension of ${}_SP_R \otimes_R E$ is finite by [7, Theorem 9.1.17]. Thus ${}_SP_R \otimes_R E$ is injective by [7, Proposition 10.1.2]. □

Lemma 2.5. *Let R be Gorenstein, and let M be a Gorenstein injective R-module. Then there exists an exact sequence $0 \to K \to E \to M \to 0$ such that E is injective and K is Gorenstein injective.*

Proof. See [13, Lemma 5.4.3]. □

Theorem 2.6. *Let R be an n-Gorenstein ring, let ${}_SP_R$ be an (R, S)-bimodule with ${}_SP$ projective and P_R finitely generated projective. If P_R is an injector, then P_R is a G-injector.*

Proof. Suppose that M is a Gorenstein injective left R-module. By Lemma 2.5, we can construct an exact sequence

$$(4) \qquad \cdots \to E_{i+1} \to E_i \to \cdots \to E_1 \to E_0 \to M \to 0$$

in which each E_i is injective and each $K_i = \text{Coker}(E_{i+1} \to E_i)$ $(i \geq 1)$ is Gorenstein injective. Consider the short exact sequences $0 \to_R K_1 \to_R E_0 \to_R M \to 0$ and $0 \to_R K_{i+1} \to_R E_i \to_R K_i \to 0$ for all i.

Applying ${}_SP_R \otimes -$ to (4), we have the short exact sequences

$$0 \to_S P_R \otimes_R K_1 \to_S P_R \otimes_R E_0 \to_S P_R \otimes_R M \to 0$$

and

$$0 \to_S P_R \otimes_R K_{i+1} \to_S P_R \otimes_R E_i \to_S P_R \otimes_R K_i \to 0$$

for all i. Since $P_R \otimes_R E_i$ is injective, we have the following exact sequence:

$$\cdots \to_S P_R \otimes E_{i+1} \to_S P_R \otimes E_i \to \cdots \to_S P_R \otimes E_1 \to_S P_R \otimes E_0 \to_S P_R \otimes_R M \to 0.$$

By Corollary 2.3, S is m-Gorenstein. [7, Theorem 10.1.13] gives that each ${}_SP_R \otimes K_i$ is Gorenstein injective for $i \geq m-1$. By [7, Theorem 10.1.4] and the short exact sequence

$$0 \to_S P_R \otimes K_{m-1} \to_S P_R \otimes E_{m-2} \to_S P_R \otimes K_{m-2} \to 0,$$

we conclude that ${}_SP_R \otimes K_{m-2}$ is Gorenstein injective. By repeating this argument for the other short exact sequences we have ${}_SP_R \otimes M$ is a Gorenstein injective left S-module. □

Proposition 2.7. *Let ${}_SP_R$ be a finitely generated projective left R-module. If $T(P)$ is right R-flat and ${}_RP_S^* \otimes -$ preserves injective modules, then P_R is a G-injector.*

Proof. Since $T(P)$ is right R-flat, P_R is an injector by [1, Theorem 2.2]. So P_R is a G-injector by Proposition 2.1. □

A left R-module M is Gorenstein injective if and only if $\mathrm{Ext}^i_R(\mathcal{I}(R), M) = 0$ and M admits a proper left injective resolution by [9]. But if $\mathrm{Ext}^i_R(\mathcal{I}(R), M) = 0$, then M is strongly copure injective. Using the relations between Gorenstein injective modules and copure injective modules, we get the following results.

Proposition 2.8. *Let P_R be an injector, and let $P_R \otimes_R M$ be copure injective for any Gorenstein injective left R-module M. Then P_R is a G-injector.*

Proof. Assume that M is a Gorenstein injective left R-module. So there exists a $\mathrm{Hom}(\mathcal{I}(R), -)$-exact exact sequence

$$\cdots \to_R E_1 \to_R E_0 \to_R E^0 \to_R E^1 \to \cdots$$

of injective R-modules such that $M = \mathrm{Ker}(E^0 \to E^1)$. Applying ${}_SP_R \otimes -$ to this exact sequence, we get the following exact sequence of copure injective S-modules

$$\cdots \to_S P_R \otimes_R E_1 \to_S P_R \otimes_R E_0 \to_S P_R \otimes_R E^0 \to_S P_R \otimes_R E^1 \to \cdots ,$$

where ${}_SP_R \otimes_R M = \mathrm{Ker}({}_SP_R \otimes_R E^0 \to_S P_R \otimes_R E^1)$. By using the hypothesis for any injective S-module ${}_SE$, the functor $\mathrm{Hom}_S({}_SE, -)$ makes the above exact sequence exact. Thus ${}_SP_R \otimes_R M$ is Gorenstein injective. □

Proposition 2.9. *Let ${}_SP_R$ be an injector and ${}_SP_R \otimes_R M$ be copure injective for all Gorenstein injective R-modules M. If a left R-module $N \in \widetilde{\mathcal{GI}}(R)$, then the left S-module ${}_SP_R \otimes_R N \in \widetilde{\mathcal{GI}}(S)$.*

Proof. We proceed by induction on $\mathrm{Gid}_R(N) = n$. If $n = 0$, then N is Gorenstein injective. Hence the result follows by Proposition 2.8. Now suppose inductively that the result has been proved for all values smaller than n, and so we prove it for n. As $\mathrm{Gid}_R(N) = n$, there is an exact sequence $0 \to N \to M \to C \to 0$ of R-modules such that M is Gorenstein injective and $\mathrm{Gid}_R(C) \leq n-1$ by [9, Proposition 2.18]. Applying ${}_SP_R \otimes_R -$ to the above exact sequence, we get the following exact sequence of S-modules.

$$0 \to_S P_R \otimes_R N \to_S P_R \otimes_R M \to_S P_R \otimes_R C \to 0.$$

Then we can use the induction hypothesis for C and conclude that ${}_SP_R \otimes_R C$ has finite Gorenstein injective dimension. By Proposition 2.8 the left S-module ${}_SP_R \otimes M$ is Gorenstein injective. Then the proceeding exact sequence implies that ${}_SP_R \otimes N$ has finite Gorenstein injective dimension. □

Corollary 2.10. *Let ${}_SP_R$ be a finitely generated projective left R-module, and let ${}_SP_R$ be a G-injector. If a left R-module $N \in \widetilde{\mathcal{GI}}(R)$, then the ${}_SP_R \otimes_R N \in \widetilde{\mathcal{GI}}(S)$.*

Next we shall study the relation between the G-injectors and the Gorenstein injective dimensions of the left R-modules.

Theorem 2.11. *Let ${}_SP_R$ be a finitely generated projective right R-module, and $S = \mathrm{End}(P_R)$. Then ${}_SP_R$ is a G-injector* $\Leftrightarrow$ $\mathrm{Gid}_S({}_SP_R \otimes_R M) \le \mathrm{Gid}_R({}_RM)$ *for any left R-module M.*

Proof. ($\Longrightarrow$) Let M be a left R-module. Then M has a Gorenstein injective resolution:

$$0 \to M \to_R E^0 \to_R E^1 \to_R E^2 \to \cdots .$$

Applying ${}_SP_R \otimes -$ to this resolution, we get the following exact sequence:

$$0 \to_S P_R \otimes_R M \to_S P_R \otimes_R E^0 \to_S P_R \otimes_R E^1 \to \cdots .$$

Where each ${}_SP_R \otimes_R E^i \quad (i \ge 0)$ is a Gorenstein injective left S-module as P_R is a G-injector. So $\mathrm{Gid}_S({}_SP_R \otimes_R M) \le \mathrm{Gid}_R({}_RM)$.

($\Longleftarrow$) Suppose that $\mathrm{Gid}_S({}_SP_R \otimes_R M) \le \mathrm{Gid}_R({}_RM)$. Let M be a Gorenstein injective left R-module. Then $\mathrm{Gid}_R({}_RM) = 0$, so $\mathrm{Gid}_S({}_SP_R \otimes_R M) = 0$, and thus ${}_SP_R \otimes_R M$ is a Gorenstein injective module. Therefore, ${}_SP_R$ is a Gorenstein injector. □

From the above theorem a natural problem is: under what conditions does the $\mathrm{Gid}_S({}_SP_R \otimes M) = \mathrm{Gid}_R({}_RM)$ hold? Next we shall discuss this problem. For this reason, we shall give the following new definition.

Definition 2.12 (Bi-G-injector)**.** We call P_R a Bi-G-injector if ${}_SP_R$ is a G-injector and for any left R-module ${}_RM$, ${}_SP_R \otimes_R M$ is Gorenstein injective then ${}_RM$ is Gorenstein injective. When P_R is a finitely generated projective generator, P_R is a Bi-G-injector.

Theorem 2.13. *Let ${}_SP_R$ be a finitely generated projective left R-module. Then ${}_SP_R$ is a Bi-G-injector* $\Leftrightarrow$ $\mathrm{Gid}_S({}_SP_R \otimes_R M) = \mathrm{Gid}_R({}_RM)$ *for any left R-module M.*

Proof. ($\Longleftarrow$) It is trivial.

($\Longrightarrow$) Suppose that P_R is a Bi-injector. Let M be a left R-module. Then $\mathrm{Gid}_S({}_SP_R \otimes_R M) \le \mathrm{Gid}_R({}_RM)$ by Theorem 2.11.

Next we shall show $\mathrm{Gid}_S({}_SP_R \otimes_R M) \ge \mathrm{Gid}_R({}_RM)$.

If $\mathrm{Gid}({}_SP_R \otimes_R M) = \infty$. Then it is trivial.

If $\mathrm{Gid}({}_SP_R \otimes_R M) = n < \infty$. Then there exists a Gorenstein injective resolution of M:

$$0 \to_R M \to_R E^0 \to_R E^1 \to_R E^2 \to \cdots \to_R E^n,$$

where each $E^i (i \ge 0)$ is Gorenstein injective. Note that ${}_SP_R$ is a Bi-G-injector and finitely generated projective right R-module, then

$$0 \to_S P_R \otimes_R M \to_S P_R \otimes_R E^0 \to_S P_R \otimes_R E^1 \to \cdots \to P_R \otimes_R E^n$$

is a Gorenstein injective resolution of ${}_SP_R \otimes_R M$. Since $\mathrm{Gid}({}_SP_R \otimes_R M) = n < \infty$, then $\mathrm{Im}({}_SP_R \otimes_R E^{n-2} \to_S P_R \otimes_R E^{n-1})$ is Gorenstein injective, but ${}_SP_R \otimes_R \mathrm{Im}(E^{n-3} \to E^{n-2}) \cong \mathrm{Im}({}_SP_R \otimes_R E^{n-2} \to_S P_R \otimes_R E^{n-1})$ and

${}_SP_R$ is a Bi-G-injector, hence $\mathrm{Im}(E^{n-3} \to E^{n-2})$ is Gorenstein injective. So $\mathrm{Gid}({}_RM) \leq n$. □

In [1] F. W. Anderson defined the perfect injector. For a finitely generated R-module P_R, ${}_SP_R$ is called a perfect injector if $P_R \otimes_R -$ preserves injective envelopes. We call P_R is a perfect G-injector if $P_R \otimes_R -$ preserves Gorenstein injective preenvelopes. Of course, every perfect G-injector is a G-injector.

Theorem 2.14. *Let R be a Gorenstein ring, and let ${}_SP_R$ be an (R,S)-bimodule with P_R finitely generated projective and ${}_SP$ projective, $S = \mathrm{End}(P_R)$. Then P_R is an injector $\Leftrightarrow$ P_R is a perfect G-injector.*

Proof. ($\Longleftarrow$) By Proposition 2.4.

($\Longrightarrow$) Suppose that P_R is an injector. Then P_R is a G-injector by Theorem 2.6. First, we shall prove $\mathrm{Hom}_R({}_SP_R, -)$ preserves Gorenstein injective left S-modules. Let ${}_SM$ be a Gorenstein injective left S-module, then there is a complete resolution of injective modules

(4) $\mathbb{M}_1 = \cdots \to_S E_2 \to_S E_1 \to_S E_0 \to_S E^0 \to_S E^1 \to_S E^1 \to_S E^2 \to \cdots,$

where ${}_SM = \mathrm{Ker}({}_SE^0 \to_S E^1)$. Applying $\mathrm{Hom}_S({}_SP_R, -)$ to (4) we get the following exact sequence:

(5)

$$\mathbb{M}_2 = \cdots \to \mathrm{Hom}({}_SP_R, {}_SE_0) \to \mathrm{Hom}({}_SP_R, {}_SE^0) \to \mathrm{Hom}({}_SP_R, {}_SE^1) \to \cdots.$$

Since ${}_SP_R$ is R-projective and ${}_SE_i$ is injective, by [12, Theorem 3.44] each $\mathrm{Hom}_s({}_SP_R, {}_SE_i)$ is an injective R-module. Note that

$$\mathrm{Hom}_R({}_RE', \mathrm{Hom}({}_SP_R, {}_SE_i)) \cong \mathrm{Hom}_S({}_SP_R \otimes_R E', {}_SE_i)$$

for any left R-module E' and any left S-module. For each injective R-module E', ${}_SP_R \otimes_R E'$ is an injective S-module as ${}_SP_R$ preserves injective modules. Then the sequence $\mathrm{Hom}_S({}_SP_R \otimes_R E', \mathbb{M}_1)$ is exact, this gives that the sequence $\mathrm{Hom}_R({}_RE', \mathbb{M}_2)$ is also exact. So $\mathrm{Hom}_S({}_SP_R, -)$ preserves Gorenstein injective modules.

Next, we shall show that ${}_SP_R$ preserves Gorenstein injective preenvelopes. Let ${}_RM \to_R G$ be a Gorenstein injective preenvelope of ${}_RM$. Since ${}_SP_R$ is a G-injector then ${}_SP_R \otimes_R G$ is Gorenstein injective. For any Gorenstein injective left S-module ${}_SG'$ we get the commutative diagram:

(6)

$$\begin{array}{ccc}
\mathrm{Hom}({}_SP_R \otimes_R G, {}_SG') & \longrightarrow & \mathrm{Hom}({}_SP_R \otimes_R M, {}_SG') \\
\| & & \| \\
\mathrm{Hom}({}_RG, \mathrm{Hom}({}_SP_R, {}_SG')) & \longrightarrow & \mathrm{Hom}({}_RM, \mathrm{Hom}({}_SP_R, {}_SG')).
\end{array}$$

$\mathrm{Hom}({}_SP_R, -)$ preserves Gorenstein injective modules. So

$$\mathrm{Hom}({}_RG, \mathrm{Hom}({}_SP_R, {}_SG')) \to \mathrm{Hom}({}_RM, \mathrm{Hom}({}_SP_R, {}_SG')) \to 0$$

is exact.

By the commutative diagram (6) we know that

$$\mathrm{Hom}_S({}_SP_R \otimes G, G') \to \mathrm{Hom}_S({}_SP_R \otimes M, G')$$

is surjective. So ${}_SP_R \otimes_R M \to_S P_R \otimes_R G$ is a Gorenstein injective preenvelope. □

Remark 2.15. Dually, one can also define the Gorenstein projector. We call P_R a G-projector if P_R preserves Gorenstein projective modules, that is, for any Gorenstein projective left R-module ${}_RM$, ${}_SP_R \otimes_R M$ is a Gorenstein projective left S-module. All the results, concerning Gorenstein injective, have a Gorenstein projective counterpart.

3. G-flator

Let P_R be a finitely generated projective right R-module, and $S = \mathrm{End}(P_R)$. We call P_R a G-flator if P_R preservers Gorenstein flat modules, that is, for any Gorenstein flat module ${}_RM$, ${}_SP_R \otimes_R M$ is a Gorenstein flat left S-module.

Proposition 3.1. *Let ${}_SP_R$ be finitely generated projective and $-_S \otimes_S P_R$ preserve injective right S-modules, $S = \mathrm{End}(P_R)$. If P_R is a flator, then P_R is a G-flator.*

Proof. Let F be a Gorenstein flat module. By the definition of the Gorenstein flat module, there exists an $I_R \otimes -$ (I_R is an arbitrary injective right R-module) complete flat resolution

$$\mathbb{F} = \cdots \to_R F_1 \to_R F_0 \to_R F^0 \to_R F^1 \to \cdots \tag{7}$$

such that $F = \mathrm{Ker}({}_RF^0 \to_R F^1)$. Applying $P_R \otimes_R -$ to (7), we get a resolution of flat modules

$$\cdots \to_S P_R \otimes_R F_1 \to_S P_R \otimes_R F_0 \to_S P_R \otimes_R F^0 \to_S P_R \otimes_R F^1 \to \cdots . \tag{8}$$

By the hypothesis, for each injective S-module I_S, $I_S \otimes_S P_R$ is an injective right R-module. So $(I_S \otimes_S P_R) \otimes_R \mathbb{F}$ is exact by the definition of the Gorenstein flat modules. But $I_S \otimes_S (P_R \otimes F_i) \cong (I_S \otimes_S P_R) \otimes_R F_i$; hence (8) is a $(I_S \otimes -)$-exact exact sequence for any injective S-module. So ${}_SP \otimes_R F$ is Gorenstein flat. Thus ${}_SP_R$ is a Gorenstein flator. □

From the last proposition we know that under some conditions a flator is a G-flator. Next we shall consider the problem under what conditions a G-flator is a flator.

Proposition 3.2. *Let R be a Gorenstein ring and let ${}_SP_R$ be an (R, S)-bimodule with ${}_SP$ projective and P_R finitely generated projective. If P_R is a G-flator, then P_R is a flator.*

Proof. Suppose that ${}_RF$ is a flat left R-module. Then it is a Gorenstein flat module. If ${}_SP_R$ is a G-flator, then ${}_SP_R \otimes_R F$ is a Gorenstein flat left S-module. Since ${}_SP_R$ is left S-projective, it has finite injective dimension by [7]. Assume that $id({}_SP) \leq n$. By the proof of Lemma 2.2, we know that ${}_SP_R \otimes_R F$ has the finite injective dimension. Then the flat dimension of ${}_SP_R \otimes_R F$ is also finite. By [7, Corollary 10.3.4], the flat dimension of ${}_SP_R \otimes_R F$ is 0. So ${}_SP_R \otimes_R F$ is flat. Thus P_R is a flator. □

Bennis defined the strongly Gorenstein flat modules in [3]. Next, we shall use the strongly Gorenstein modules to character the Gorenstein flator.

Lemma 3.3. *Let R be a right coherent ring. Then a module is Gorenstein flat if and only if it is a direct summand of strongly Gorenstein flat modules.*

Proof. The direct implication is immediately from [3, Theorem 3.5]. For the converse implication, it sufficient to prove that a direct summand of the strongly Gorenstein flat modules is a Gorenstein flat module. By [9, Theorem 3.7] we know that the Gorenstein flat modules closed under direct summand. However, a strongly Gorenstein flat module is a Gorenstein flat module. So we get the result. □

Definition 3.4 (Gorenstein flat dimension)**.** As done in [8] (and similar to the Gorenstein projective case), we define the Gorenstein flat dimension, Gfd_RM, of a module M by declaring that $Gfd_RM \leq n$ if and only if M has a resolution of Gorenstein flat modules of length n.

Lemma 3.5. *Let R be a Gorenstein ring. Then an R-module M is strongly Gorenstein flat if and only if there exists a short exact sequence $0 \to_R M \to_R F \to_R M \to 0$, where F is a flat R-module.*

Proof. (⇒) It is straightforward.

(⇐) Since R is a Gorenstein ring, it is a Notherian ring. By [7, Theorem 12.3.1], $Gfd_RM \leq \infty$ for all R-modules M. Assume $Gfd_RM \leq n$. Then $\mathrm{Tor}_R^{n+1}(M, X) = 0$ for all X with finite injective dimension. Furthermore, by the dual case of [4, Proposition 3.15] (it is also right under noncommutative condition), we know that M is strongly Gorenstein flat. □

Lemma 3.6. *Let R be a Gorenstein ring and let ${}_SP_R$ be a flator with ${}_SP$ is a projective left S-module. Then ${}_SP_R$ is a G-flator.*

Proof. First, we shall prove that ${}_SP_R \otimes_R -$ preserves strongly Gorenstein flat left R-modules. Suppose that M is a strongly Gorenstein flat left R-module, by Lemma 3.5 there exists a short exact sequence $0 \to_R M \to_R F \to_R M \to 0$ of left R-modules, where F is a flat left R-module. Applying ${}_SP_R \otimes_R -$ to the above exact sequence, we have the following exact sequence

$$0 \to_S P_R \otimes_R M \to_S P_R \otimes_R F \to_S P_R \otimes_R M \to 0,$$

where ${}_SP_R \otimes_R F$ is a flat left S-module. By Lemmas 2.2 and 3.5 we know that ${}_SP_R \otimes_R M$ is strongly Gorenstein flat.

Now, we shall show ${}_SP_R \otimes_R -$ preserves Gorenstein flat left R-modules. Suppose that M is Gorenstein flat. Then by Lemma 3.3, there exists a left

R-module Q and a strongly Gorenstein flat left R-module G such that ${}_RG \cong_R M \bigoplus_R Q$. Applying ${}_SP_R \otimes_R -$ to the above formula we get ${}_SP_R \otimes_R G \cong_S P_R \otimes_R M \bigoplus_S P_R \otimes_R Q$. Note that ${}_SP_R \otimes_R G$ is a strongly Gorenstein flat left S-module, so ${}_SP_R \otimes_R M$ is a Gorenstein flat left S-module by Lemma 3.3. □

Theorem 3.7. *Let R be a Gorenstein ring, and let ${}_SP_R$ be an (R, S)-bimodule with ${}_SP$ projective and P_R finitely generated projective. Then P_R is a flator if and only if P_R is a G-flator.*

Proof. (⇐) Following Proposition 3.2.
(⇒) By the proof of Lemma 3.6. □

Corollary 3.8. *Let ${}_SP_R$ be a finitely generated projective left R-module. If ${}_SP_R \otimes_R -$ preserves strongly Gorenstein flat left R-modules, then ${}_SP_R$ is a G-flator.*

Following [9], we know that there are some relations between Gorenstein flat modules and strongly copure flat modules.

Proposition 3.9. *Let P_R be a flator and ${}_SP_R \otimes_R F$ be a copure flat left S-module for any Gorenstein flat R-module F. Then P_R is a G-flator.*

Proof. Assume that F is a Gorenstein flat left R-module. So there exists an $(\mathcal{I}(R) \otimes -)$-exact exact sequence

$$\cdots \to_R F_1 \to_R F_0 \to_R F^0 \to_R F^1 \to \cdots$$

of flat R-modules such that $F = \mathrm{Ker}(F^0 \to F^1)$. Applying ${}_SP_R \otimes -$ to this exact sequence, we get the following exact sequence of copure flat S-modules

$$\cdots \to_S P_R \otimes_R F_1 \to_S P_R \otimes_R F_0 \to_S P_R \otimes_R F^0 \to_S P_R \otimes_R F^1 \to \cdots,$$

where ${}_SP_R \otimes_R F = \mathrm{Ker}({}_SP_R \otimes_R F^0 \to_S P_R \otimes_R F^1)$. By using the hypothesis for any injective right S-module E_S, the functor $E_S \otimes -$ makes the above exact sequence exact. Thus ${}_SP_R \otimes_R M$ is Gorenstein flat. □

Proposition 3.10. *Let P_R be a flator and ${}_SP_R \otimes_R F$ be a copure flat for any Gorenstein flat R-module F. If ${}_RN \in \widetilde{\mathcal{GF}(R)}$, then ${}_SP_R \otimes_R N \in \widetilde{\mathcal{GF}(S)}$.*

Proof. We proceed by induction on $Gfd_R(N) = n$. If $n = 0$, then N is Gorenstein flat. Hence the result follows by Proposition 3.9. Now suppose inductively that the result has been proved for all values smaller than n, and so we prove it for n. As $Gfd_R(N) = n$, there is an exact sequence $0 \to C \to M \to N \to 0$ of R-modules such that M is Gorenstein flat and $\mathrm{Gid}_R(C) \leq n - 1$ by [9, Proposition 2.18]. Applying ${}_SP_R \otimes_R -$ to the above exact sequence, we get the following exact sequence of S-modules.

$$0 \to_S P_R \otimes_R C \to_S P_R \otimes_R M \to_S P_R \otimes_R N \to 0.$$

Then we can use the induction hypothesis for C and conclude that ${}_SP_R \otimes_R C$ has finite Gorenstein flat dimension. By Proposition 3.9 the left S-module

${}_SP_R \otimes M$ is Gorenstein flat. Then the proceeding exact sequence implies that ${}_SP_R \otimes N$ has finite Gorenstein flat dimension. □

Corollary 3.11. *Let ${}_SP_R$ be a finitely generated projective left R-module, and ${}_SP_R$ be a G-flator. If a left R-module $N \in \widetilde{\mathcal{GF}}(R)$, then the left S-module ${}_SP_R \otimes_R N \in \widetilde{\mathcal{GF}}(S)$.*

Next, we shall study the relations between the G-flator and the Gorenstein flat dimension of the left R-modules.

Theorem 3.12. *Let P_R be a finitely generated projective R-module. Then P_R is a G-flator if and only if $Gfd(P_R \otimes_R M) \leq Gfd({}_RM)$.*

Proof. ($\Longrightarrow$) Let M be a left R-module. Then M has a Gorenstein flat resolution:

$$\cdots \to_R F_2 \to_R F_1 \to_R F_0 \to M \to 0.$$

Applying ${}_SP_R \otimes -$ to this resolution, we get the following exact sequence:

$$\cdots \to_S P_R \otimes_R F_2 \to_S P_R \otimes_R F_1 \to_S P_R \otimes_R F_0 \to_S P_R \otimes M \to 0.$$

Where each ${}_SP_R \otimes_R F_i \quad (i \geq 0)$ is a Gorenstein flat left S-module as P_R is a G-flator. So $Gfd_S({}_SP_R \otimes_R M) \leq Gfd_R({}_RM)$.

($\Longleftarrow$) Suppose that $Gfd_S({}_SP_R \otimes_R M) \leq Gfd_R({}_RM)$. Let M be a Gorenstein flat left R-module. Then $Gifd_R({}_RM) = 0$, so $Gfd_S({}_SP_R \otimes_R M) = 0$, and thus ${}_SP_R \otimes_R M$ is a Gorenstein flat module. Therefore, ${}_SP_R$ is a Gorenstein flator. □

Definition 3.13. We call P_R a perfect G-flator if ${}_SP_R$ preserves Gorenstein flat precovers, that is, if $G \to M$ is a Gorenstein flat precover of M, then ${}_SP_R \otimes_R G \to P_R \otimes_R M$ is also a Gorenstein flat precover of $P_R \otimes_R M$.

Theorem 3.14. *Let R be a Gorenstein ring, and let ${}_SP_R$ be an injector with ${}_SP$ projective and $- \otimes_R P_S^*$ preserve injective modules. Then P_R is a flator if and only if P_R is a perfect G-flator.*

Proof. Similar to the proof of Theorem 2.14, it is easy to see that if P_R is a flator, then it is a G-flator. First, we shall prove ${}_RP_S^* \otimes_S -$ preserves Gorenstein flat modules. For any Gorenstein flat module ${}_SM$, by the definition of Gorenstein flat modules, there exists a $J_S \otimes_S$-exact complete flat resolution of flat modules (where J_S is an arbitrary injective right S-module)

$$\mathbb{F} = \cdots \to_S F_1 \to_S F_0 \to_S F^0 \to_S F^1 \to \cdots \tag{9}$$

such that ${}_SM \cong \mathrm{Ker}({}_SF^0 \to_S F^1)$. Applying ${}_RP_S^* \otimes_S -$ to (9), we get a sequence of left R-modules

$$\cdots \to_R P_S^* \otimes_S F_1 \to_R P_S^* \otimes_S F_0 \to_R P_S^* \otimes_S F^0 \to_R P_S^* \otimes_S F^1 \to \cdots. \tag{10}$$

Since P_R is an injector, by [1, Theorem 2.1], ${}_RP_S^*$ is right S-flat and left R-projective. So (9) is an exact sequence of flat left R-modules. Moreover,

$${}_RP_S^* \otimes_S M \cong \mathrm{Ker}({}_RP_S^* \otimes_S F_0 \to_R P_S^* \otimes_S F^0).$$

For each injective R-module I_R, by the supposition $I_R \otimes_R P^*$ is an injective right S-module. So $(I_R \otimes_R P^*) \otimes \mathbb{F}$ is exact. Note that

$$I_R \otimes_R (P_S^* \otimes_S F_i) \cong (I_R \otimes_R P_S^*) \otimes_S F_i, \quad i \geq 0$$

for any right R-module I_R, then (10) is an $(I_R \otimes -)$-exact exact sequence. Hence ${}_RP_S^*$ preserves Gorenstein flat modules.

Suppose that ${}_RG \to_R M$ is a Gorenstein flat precover of left R-module M. Since P_R is a Gorenstein flator, ${}_SP_R \otimes_R G$ is Gorenstein flat. Using the same method in the proof of Theorem 2.14 and the definition of the Gorenstein flat precover, we can get the result. □

Proposition 3.15. *Let R be a quasi-Frobenius ring, and let ${}_RP_S$ be an (R, S)-bimodule with ${}_SP$ projective and P_R finitely generated projective. Then the following are equivalent.*

(1) *P_R is a Gorenstein injector.*
(2) *P_R is a Gorenstein projector.*
(3) *P_R is a Gorenstein flator.*

Proof. We only prove (1) ⇒ (2), (2) ⇒ (3) and (3) ⇒ (1) are similarly.

Let M be a Gorenstein injective right R-module, M is also Gorenstein projective by [7, P_{257} Exercise 5]. Since P_R is a Gorenstein injector, ${}_SP_R \otimes M$ is Gorenstein injective. $S = \mathrm{End}(P_R)$ is also a quasi-Frobenius ring by Lemma 2.2, so ${}_SP_R \otimes M$ is Gorenstein projective by [7, P_{257} Exercise 5]. □

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