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ON THE TOPOLOGY OF DIFFEOMORPHISMS OF SYMPLECTIC 4-MANIFOLDS

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Abstract. For a closed symplectic 4-manifold X, let $\mathrm{Diff}_0(X)$ be the group of diffeomorphisms of X smoothly isotopic to the identity, and let $\mathrm{Symp}(X)$ be the subgroup of $\mathrm{Diff}_0(X)$ consisting of symplectic automorphisms. In this paper we show that for any finitely given collection of positive integers $\{n_1, n_2, \dots, n_k\}$ and any non-negative integer m, there exists a closed symplectic (or Kähler) 4-manifold X with $b_2^+(X) > m$ such that the homologies H_i of the quotient space $\mathrm{Diff}_0(X)/\mathrm{Symp}(X)$ over the rational coefficients are non-trivial for all odd degrees $i = 2n_1 - 1, \dots, 2n_k - 1$.

The basic idea of this paper is to use the local invariants for symplectic 4-manifolds with contact boundary, which are extended from the invariants of Kronheimer for closed symplectic 4-manifolds, as well as the symplectic compactifications of Stein surfaces of Lisca and Matić.

1. Introduction

The purpose of this paper is to investigate the existence of closed symplectic smooth 4-manifolds having non-trivial homotopy groups of certain diffeomorphisms of symplectic 4-manifolds, which was first initiated by P. B. Kronheimer in [9]. To be precise, let (X, ω_0) be a closed symplectic 4-manifold. Let $\mathrm{Diff}_0(X)$ be the group of diffeomorphisms of X smoothly isotopic to the identity, and let $\mathrm{Symp}(X)$ be the subgroup of $\mathrm{Diff}_0(X)$ consisting of symplectic automorphisms. Let Λ_0 be the space of 2-forms on X that are symplectic and cohomologous to ω_0. Now consider the map

$$\Psi : \mathrm{Diff}_0(X) \to \Lambda_0, \quad f \mapsto (f^{-1})^*\omega_0.$$

Then Ψ induces an injection from the quotient space $\mathrm{Diff}_0(X)/\mathrm{Symp}(X)$ to the space Λ_0. Since $\mathrm{Diff}_0(X)$ is connected, it follows from a well-known theorem of Moser in [13] that the image of Ψ is the connected component of Λ_0 containing ω_0.

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Using families of the Seiberg-Witten solutions, Kronheimer in [9] defined an interesting invariant which is an obstruction to some higher homotopy groups of the quotient space $\mathrm{Diff}_0(X)/\mathrm{Symp}(X)$ or equivalently Λ_0. One of the key ingredients in [9] is a result of C. Taubes about the constraints on symplectic forms [23, 24]. By the work of Taubes, Kronheimer showed the following interesting theorem.

Theorem 1.1. *For each positive integer n, there exists a closed symplectic 4-manifold (Y_n, ω_n) with $b_2^+(Y_n) > 2n+1$ such that both homotopy group π_{2n-1} and homology group H_{2n-1} of the quotient space $\mathrm{Diff}_0(Y_n)/\mathrm{Symp}(Y_n)$ over the rational coefficients are non-trivial.*

His explicit examples are algebraic surfaces of general type. Kronheimer's result in [9] was motivated by a result of P. Seidel [19]. By using symplectic Floer homology, Seidel showed that there are diffeomorphisms in $\mathrm{Diff}_0(X)$ on an open symplectic 4-manifold which cannot be symplectically isotopic to the identity.

In view of Kronheimer's results on the homotopy groups of diffeomorphisms in symplectic 4-manifolds, the following general question seems to be worthy of further investigation:

Question 1.2. *Is there a closed symplectic* (*or Kähler*) *4-manifolds (X, ω_0) having the non-trivial homologies*

$$H_i(\mathrm{Diff}_0(X)/\mathrm{Symp}(X); \mathbb{Q})$$

for all positive integers i?

Another interesting motivation for this question can also be found in a series of Ruberman's applications of the Seiberg-Witten theory to an obstruction to smooth isotopy in dimension 4 and the topology of the space of all metrics with positive scalar curvature. In particular, Ruberman, among other things, showed in [16] that π_0 of the diffeomorphism group of certain 4-manifolds is infinitely generated (see [15], [16], and [17] for more applications). It is also worth mentioning some related works of McMullen and Taubes, Smith, and Vidussi. They independently showed that the moduli spaces of symplectic forms modulo diffeomorphisms on certain simply connected 4-manifolds are disconnected (see [12], [20], and [25]).

Let M be a smooth closed 3-manifold. A *contact structure* on M is a distribution ξ of tangent 2-planes locally defined by a 1-form θ such that $\theta \wedge d\theta$ is nowhere vanishing. Clearly $\theta \wedge d\theta$ defines an orientation on Y and this orientation does not depend of the choice of the sign of θ. When M has a fixed orientation, we say that ξ is *positive* (*negative*, respectively) if the orientation on M induced by ξ coincides with (is opposite to, respectively) the given orientation. From now on, we assume that every contact structure of this paper is positive. A *4-manifold with contact boundary* is a pair (X, ξ) consisting of a connected oriented smooth 4-manifold with boundary and a positive contact structure ξ on the boundary ∂X. A symplectic structure ω on the oriented 4-manifold X with a contact structure ξ on ∂X is *compatible* if the symplectic 2-form ω satisfies $\omega|_\xi > 0$ at every point of the boundary (see [1] for more details on contact structures).

In [10], Kronheimer and Mrowka introduced monopole invariants for smooth 4-manifolds with contact boundary, and extended the results of Taubes in [22] and [23] to this non-compact settings. Moreover, using the monopole invariants for 4-manifolds with contact boundary in [10], Kronheimer also obtained local versions of the results in Theorem 1.1. More precisely, let B^4 be the unit ball in $\mathbb{C}^2$, X' be the quotient $B^4/\mathbb{Z}_{n+1}$, and let ξ be the contact structure on the lens space at the boundary obtained from the embedding in $\mathbb{C}^2/\mathbb{Z}_{n+1}$. Let X_{n+1} be the resolution obtained from X' by resolving the singular point with a sphere C of self-intersection $-(n+1)$. Then the boundary of X_{n+1} is a lens space $L(n+1, 1)$. It is well-known that any lens space $L(p, q)$ is obtained by $-\frac{p}{q}$-surgery on the unknot, with $-\frac{p}{q} < -1$ except for S^3 and $S^1 \times S^2$. By Proposition 5.3 in [7], X_{n+1} admits a Stein structure with $L(n+1, 1)$ as its oriented boundary. Thus if we denote by J^* the dual of an almost complex structure J, there exists a smooth strictly pluri-subharmonic function $\phi : X_{n+1} \to \mathbb{R}$ such that the 2-form $\omega_\phi = dJ^*d\phi$ is non-degenerate and closed (so X_{n+1} admits a Kähler form ω_ϕ) such that its boundary admits a contact structure ξ obtained by a 1-form $-J^*d\phi$. In his paper [9], Kronheimer stated the following theorem whose proof was omitted.

Theorem 1.3. *Let Λ be the space of symplectic 2-forms that are cohomologous to ω_ϕ. Then there is a family ω_u parameterized by $u \in S^{2n-1}$ which represents a non-trivial class in homology of the space Λ. In particular, the family cannot be extended to a family parameterized by the ball. Indeed, if ω_ν ($\nu \in B^{2n}$) is any family of symplectic forms compatible with a contact structure ξ and extending the given family on S^{2n-1}, then there exists at least one $\nu \in B^{2n}$ for which the pairing of ω_ν with the sphere C is positive.*

For the sake of completeness, we provide a proof of this theorem in Sections 2 and 3, relatively in detail.

On the other hand, using the symplectic compactifications of Stein surfaces and the Seiberg-Witten theory, Lisca and Matić showed, among other things, in [11] that given any positive integer n, there exists homology 3-spheres with at least n homotopic, but non-isomorphic tight contact structures. Their proof of the result seems to give many implications to answer Question 1.2. Indeed, combining Kronheimer's local results with the plumbing construction using disk bundles over a sphere, in this paper we give a positive partial answer to Question 1.2 as follows.

Theorem 1.4. *For any given collection of positive integers $\{n_1, n_2, \ldots, n_k\}$ and any non-negative integer m, there exists a closed symplectic 4-manifold X (or closed Kähler minimal surface of general type) with $b_+(X) > m$ having the* *non-trivial homologies*

$$H_i(\mathrm{Diff}_0(X)/\mathrm{Symp}(X);\mathbb{Q})$$

for all $i = 2n_1 - 1, \ldots, 2n_k - 1$.

As a simple corollary, we have:

Corollary 1.5. *For any positive odd integer* k*, there exists a closed symplectic 4-manifold* X (*or closed Kähler minimal surface of general type*) *having the non-trivial homologies*

$$H_i(\mathrm{Diff}_0(X)/\mathrm{Symp}(X);\mathbb{Q})$$

for all odd degrees i *between* 0 *and* k *inclusive.*

We organize this paper as follows. In Section 2, we set up and review detail constructions for the local results by Kronheimer which was stated without proof. In Section 3, we prove Theorem 1.3. In Section 4, we construct examples of closed symplectic 4-manifolds to give a partial answer to Question 1.2. It seems that we are able to extend the result of Theorem 1.4 to even degrees using the global invariants of even degrees which can be constructed analogously. We hope we return this issue elsewhere.

2. Local invariants

In this section, we set up and review the facts necessary to detect the local versions of Kronheimer's results stated in Theorem 1.3, in detail.

2.1. 4-manifolds with contact boundary

Let (X, ξ) be a compact oriented smooth 4-manifold with a contact structure ξ on the boundary ∂X which is compatible with the boundary orientation, and let X^+ be the smooth manifold obtained from X by attaching the open cylinder $[1, \infty) \times \partial X$. According to [10], since ∂X is a contact 3-manifold, we can give a symplectic structure ω_0 and its compatible Riemannian metric g_0 to $[1, \infty) \times \partial X$ for which ω_0 has length $\sqrt{2}$ and is self-dual. These two in turn give X^+ a metric and a symplectic structure outside a compact set. Now extend the metric g_0 to all of X^+, also called g_0. In [10], the triple (X^+, ω_0, g_0) is called an *AFAK* (asymptotically flat almost Kähler) manifold (see Subsection 2(iii) in [10] for more detailed constructions). By Lemma 2.1 in [10], ω_0 provides a canonical *spin*^{c} structure $\mathbf{s}_0 = (W^+, W^-, \rho)$, a spinor Φ_0 of unit length, and a unique spin connection A_0 on $X^+\backslash X$ satisfying $D^+_{A_0}\Phi = 0$. As in [10], we write $Spin^c(X, \xi)$ for the set of isomorphisms of *spin*^{c} structures $\mathbf{s}$ on X^+, equipped with an isomorphism $\mathbf{s} \to \mathbf{s}_0$ on $X^+\backslash X$. For the sake of convenience, we use the same notations A_0 and Φ_0 for arbitrary extensions of them to all of X^+.

From now on, assume that we have provided suitable function spaces on X^+ to define a moduli space of pairs which solves the monopole equations and which are asymptotic to (A_0, Φ_0) on the ends of X^+.

Let $\eta \in L^2_{l-1}(i\mathbf{su}(W^+))$ for $l > 4$. The Seiberg-Witten equations, perturbed by η, are the following pair of equations for a spin connection A and a section Φ of W^+

$$\begin{aligned} \rho(F^+_{\widehat{A}}) - \{\Phi \otimes \Phi^*\} &= \rho(F^+_{\widehat{A}_0}) - \{\Phi_0 \otimes \Phi_0^*\} + \eta, \\ D^+_A \Phi &= 0, \end{aligned} \tag{2.1}$$

where $\widehat{A}$ means the induced connection on $\det(W^+)$ and $\{\Phi \otimes \Phi^*\}$ denotes the traceless part of the endomorphism $\Phi \otimes \Phi^*$.

Let $R(X^+)$ denote the space of all Riemannian metrics g on X^+ of class C^l, and let $N(X) = e^{-\varepsilon_0 \tilde{t}} C^r(i\mathbf{su}(W^+))$ with norm $||\eta||_{N(X)} = ||e^{\varepsilon_0 \tilde{t}}\eta||_{C^r}$ for some fixed $r \geq l$, where $\varepsilon_0 > 0$ and $\tilde{t}$ is an extension of the function t on $[1, \infty) \times \partial X$ to all of X^+. As in the case of closed 4-manifolds, we let $\mathcal{P}$ be the subset of $R(X^+) \times N(X^+)$ consisting of pairs (g, η) such that η is self-dual with respect to g. Let L^2_l and L^2_{l,A_0} $(l > 4)$ be the Sobolev spaces of imaginary 1-forms and sections of W^+. We define

$$\mathcal{C} = \{(A, \Phi) \mid (A - A_0) \in L^2_l \text{ and } (\Phi - \Phi_0) \in L^2_{l,A_0}\}$$

and

$$\mathcal{G} = \{u : X^+ \to \mathbb{C} \mid |u| = 1 \text{ and } 1 - u \in L^2_{l+1}\}.$$

Then $\mathcal{G}$ is a Hilbert Lie group acting freely on $\mathcal{C}$. Thus, unlike the Seiberg-Witten equations on closed 4-manifolds, there is no Banach sub-manifold such as $\mathcal{P}_{\text{red}}$ for which the corresponding Seiberg-Witten equations have a reducible solution. This is the reason why no restriction on b_2^+ is necessary to define monopole invariants for 4-manifolds with contact boundary.

We now have a family of the Seiberg-Witten equations (2.1) parameterized by $\mathcal{P}$, and write $\mathcal{M}$ for the parameterized space of solutions modulo the gauge group $\mathcal{G}$ as follows.

$$\mathcal{M} = \{([A, \Phi], (g, \eta)) \mid (2.1) \text{ hold}\}.$$

The transversality and compactness results of Theorem 2.4 in [10] say that $\mathcal{M}$ is a Banach manifold of $\mathcal{C}/\mathcal{G} \times \mathcal{P}$ and that the projection

$$\pi : \mathcal{M} \to \mathcal{P}$$

is a smooth and proper Fredholm map of index

$$\operatorname{ind} \pi = \langle e(W^+; \Phi_0), [X, \partial X] \rangle$$

and has orientable index bundle. An orientation of the index bundle can be specified by a choice of homology orientation of (X, ξ) described in [10]. From now on, we assume that such an orientation is chosen.

In order to define local invariants derived from the Seiberg-Witten equations, we suppose that the index of π is negative, and thus we write $\operatorname{ind} \pi = -d$ $(d > 0)$. Let us denote by Δ the image of π.

Proposition 2.1. *Under the assumptions stated in this subsection, there exists a well-defined homomorphism*

$$Q_{d-1} : H_{d-1}(\mathcal{P}\backslash\Delta; \mathbb{Z}) \to \mathbb{Z}.$$

Proof. The proof is very similar to it in [9]. For the sake of convenience, we briefly explain it. Since $\mathcal{P}$ is contractible, any closed chain S in $\mathcal{P}\backslash\Delta$ of dimension $d-1$ is the boundary of a singular chain T^d in $\mathcal{P}$ of dimension d. Now arrange that T^d is transverse to π, and we define $Q_{d-1}(S)$ to be an integer $\langle [\mathcal{M}]_{-d}, \pi^{-1}(T^d)\rangle$ obtained by counting the points of $\mathcal{M}$ over T^d. It is straightforward to show that this definition is independent of the choice of T^d. This completes the proof. □

2.2. Special cases

In this subsection, we apply the above constructions to special cases of symplectic 4-manifolds with contact boundary (X, ξ) which are compatible with ξ.

Let (X, ξ) be a symplectic 4-manifold with contact boundary, and let ω be a symplectic form on X, compatible with ξ. As before, let X^+ be the 4-manifold obtained by attaching the cone $[1, \infty) \times \partial X$ to X, together with a metric g_0 and symplectic form ω_0 defined outside a compact set of X^+. Then the manifold X^+ has a symplectic form ω on the compact submanifold X, and a symplectic form ω_0 on the complement of X in X^+. Note that the compatibility condition between ω and ξ does not guarantee that ω and ω_0 match on ∂X. However, by Lemma 4.1 in [10] using the argument of patching two symplectic forms, there exists a symplectic form, also denoted by ω, on all of X^+ which is an extension of the symplectic form ω on $X\backslash U$, where U is a collar neighborhood of ∂X of X, and is asymptotic to ω_0 on the end of X^+.

Let (A_0, Φ_0) be the canonical spin connection and spinor for the *spin*^{c} structure $\mathbf{s}_0$ defined on all of X^+. Let $E \to X^+$ be a line bundle with a trivialization outside a compact set and the first Chern class $e \in H^2_c(X^+, \mathbb{Z})$ so that the spinor bundles W^+ and W_0^+ for $\mathbf{s}$ and $\mathbf{s}_0$, respectively, are related by $W^+ = W_0^+ \otimes E$. In order to apply the construction in the previous subsection, we assume that the index $-d$ of the *spin*^{c} structure $\mathbf{s}$ is negative (so $d > 0$). Since d is non-zero, $\mathbf{s}$ and $\mathbf{s}_0$ are distinct and so e is non-zero.

We will need the following dimension formula later.

Lemma 2.2. *The formula $d(\mathbf{s})$ is given by*

$$d(\mathbf{s}) = -\langle e^2 + c_1(W_0^+; \Phi_0) \cup e, [X, \partial X]\rangle.$$

Proof. By the relationship $ch(W^+; \Phi_0) = ch(W_0^+; \Phi_0)ch(E)$, it is easy to see

$$c_1(W^+; \Phi_0) = 2c_1(E) + c_1(W_0^+; \Phi_0),$$

$$\frac{1}{2}c_1(W^+; \Phi_0)^2 - c_2(W^+; \Phi_0) = c_1(E)^2 + c_1(W_0^+; \Phi_0) \cup c_1(E) + \frac{1}{2}c_1(W_0^+; \Phi_0)^2 - c_2(W_0^+; \Phi_0).$$

Thus we get

$$\begin{aligned} c_2(W^+;\Phi_0) = & -c_1(E)^2 - c_1(W_0^+;\Phi_0)\cup c_1(E) - \frac{1}{2}c_1(W_0^+;\Phi_0)^2 + c_2(W_0^+;\Phi_0) \\ & + \frac{1}{2}(2c_1(E) + c_1(W_0^+;\Phi_0))^2 \\ = & \ c_1(E)^2 + c_1(W_0^+;\Phi_0)\cup c_1(E) + c_2(W_0^+;\Phi_0). \end{aligned}$$

Since $d(\mathbf{s}) = -\langle c_2(W^+;\Phi_0), [X,\partial X]\rangle$ and $\langle c_2(W_0^+;\Phi_0), [X,\partial X]\rangle = 0$, we have the formula, as required. □

2.3. Local invariants

In this subsection, we finish the constructions of local invariants which are obstructions to the non-triviality of the homologies of the space Λ_0 or equivalently $\mathrm{Diff}_0(X)/\mathrm{Symp}(X)$.

Let $\Lambda = \Lambda(e, \mathbf{s}_0)$ be the space of 2-forms ω in $\Omega^2(X^+)$ satisfying the following three conditions: (1) ω is symplectic, (2) $\langle[\omega]\cup e, [X,\partial X]\rangle \le 0$, and (3) $\mathbf{s}_\omega \cong \mathbf{s}_0$.

Note that the space Λ is smaller than Λ_0 defined in Section 1. The purpose of this subsection is to construct homomorphisms τ_* from the homology $H_*(\Lambda;\mathbb{Z})$ to the homology $H_*(\mathcal{P}\backslash\Delta;\mathbb{Z})$. To do so, we again use the principle established in [22] and [10] that the basic classes of a 4-manifold which were defined using the Seiberg-Witten equations constrain the cohomology class of a symplectic form. By composing τ_* with Q, we then can have homomorphisms, also called Q_{d-1}, from the homology $H_{d-1}(\Lambda;\mathbb{Z})$ to $\mathbb{Z}$, which is the purpose of this section.

More precisely, for each compact subset $K \subset \Lambda$, we give a homotopy class of maps

$$\tau_K : K \to \mathcal{P}\backslash\Delta$$

such that if $K \subset K' \subset \Delta$ are compact subsets, then $\tau_{K'}|_K$ is homotopic to τ_K. Thus we will have an element $\tau \in \varinjlim_K [K, \mathcal{P}\backslash\Delta]$.

To define τ_K, we use Theorem 4.2 in [10]. As in the proof of Theorem 4.2, any element (A,Ψ) of $\mathcal{C}(X^+,\mathbf{s})$ can be written in terms of a triple (a,α,β), where a is a connection in E, $\alpha \in \Omega^{0,0}(E)$, and $\beta \in \Omega^{0,2}(E)$. As in [10], we also consider the following perturbed Seiberg-Witten equations by introducing a parameter $r \ge 1$ with $\eta = 0$ as follows.

$$\begin{aligned} \bar\partial_a\alpha + \bar\partial_a^*\beta = 0, \\ 2iF_a^\omega - \frac{r}{4}(1 - |\alpha|^2 + |\beta|^2) = 0, \\ 2F_a^{0,2} - \frac{r}{2}\bar\alpha\beta = 0, \end{aligned} \tag{2.2}$$

where $F_a^\omega = \frac{1}{2}\langle F_a, \omega\rangle$. Following [22], Kronheimer and Mrowka proved the following lemma:

Lemma 2.3. *Under the hypothesis $e \neq 0$ and $\langle[\omega]\cup e, [X,\partial X]\rangle \le 0$, there exists a constant $r_0 = r_0(\omega, g_\omega, e)$ such that for all $r \ge r_0$ the equations* (2.2) *have no solutions for the metric g_ω on X^+ with perturbing term $\eta = 0$.*

As a consequence, we can say that $(g_\omega, 0) \in \mathcal{P}\backslash\Delta$, when $r \geq r_0$. As in closed symplectic 4-manifolds, this r_0 depends on the geometry of X^+ and its almost complex structure. Thus this implies that we can choose a single sufficiently large r_0 so that the lemma above holds for all $\omega \in K$. This in turn gives a map τ_K from the compact set K to $\mathcal{P}\backslash\Delta$, as required. Clearly the homotopy class of the map does not depend on the choice of g_ω, and so on. Now the family of the maps on compact subsets induces a well-defined map on homologies

$$\tau_* : H_i(\Lambda; \mathbb{Z}) \to H_i(\mathcal{P}\backslash\Delta; \mathbb{Z}).$$

3. Local results: Proof of Kronheimer's Theorem 1.3

The purpose of this section is to give a proof of Theorem 1.3 by Kronheimer, for the sake of reader's convenience. As before, let X_{n+1} be the resolution obtained from X' by resolving the singular point with a sphere C of self-intersection $-(n+1)$. In addition, as in [9] we assume the following holds:

- We are given an analytic family $\mathbb{X}$ of Kähler manifolds X_u with the same contact boundary as X'.
- X_u are the fibers of a map $p : \mathbb{X} \to U$, where U is an open ball about $0 \in U \subset \mathbb{C}^n$.
- All fibers X_u are smooth, except for $X_0 = X'$ which has a single quotient singularity at x_0.
- All X_u $(u \neq 0)$ are embedded in $\mathbb{C}^N$, with the Kähler form inherited from the Fubini-Study metric on $\mathbb{C}^N$.
- We have a smooth family of manifolds $\tilde{\mathbb{X}}$ and a commutative diagram

$$\begin{array}{ccc} \tilde{\mathbb{X}} & \xrightarrow{\sigma} & \mathbb{X} \\ {\scriptstyle \tilde{p}}\big\downarrow & & \big\downarrow{\scriptstyle p} \\ U & = & U \end{array}$$

  such that $\tilde{X}_0 = X_{n+1}$ is a minimal resolution of X_0.
- The family $\tilde{\mathbb{X}}$ has a C^∞ trivialization $\tilde{\mathbb{X}} \to \tilde{X}_0 \times U$.

Note that using the trivialization we can regard the Fubini-Study forms on X_u as giving a family of exact symplectic forms ω_u on the fixed manifold $X_{n+1} = \tilde{X}_0$. Clearly all the forms ω_u in the family as exact symplectic forms are cohomologous. Furthermore, let e denote the Poincaré dual of the homology class represented by the exceptional 2-sphere C in $\tilde{X}_0$. All the pairings $\langle [\omega_u] \cup e, [X, \partial X]\rangle$ are zero. Therefore, we have a $(2n-1)$-sphere $S = S^{2n-1}$ in the space Λ. By Lemma 2.2, $C^2 = -(n+1)$, and the adjunction equality, the index $-d$ for the Seiberg-Witten equations with the *spin*^{c} structure $\mathbf{s} = \mathbf{s}_0 + e$ is given by

$$d = -\langle e^2 + c_1(W_0^+; \Phi_0) \cup e, [X, \partial X]\rangle = 2n.$$

Thus it makes sense to evaluate the homomorphism Q_{2n-1} on S.

To complete the proof of Theorem 1.3, we need to show that S^{2n-1} is an essential homology class in $H_{2n-1}(\Lambda;\mathbb{Z})$. To do so, it suffices to show that $Q_{2n-1}(S^{2n-1}) = \pm 1$ as follows. Let μ_u be any smooth family of Kähler forms on the fibers $\tilde{X}_u$ of $\tilde{p}$. Let $\sigma^*\omega_u$ be the pull-back of the forms ω_u under σ. Since the form $\sigma^*\omega_0$ is degenerate along the complex exceptional curve C, we define a new family of Kähler forms

$$\tilde{\omega}_u = \sigma^*\omega_u + \psi(u)\mu_u,$$

where $\psi : U \to \mathbb{R}$ is a non-negative C^∞ bump function supported near 0 and equal to zero on the small sphere S^{2n-1}. Since $\tilde{\omega}_u$ coincides with $\sigma^*\omega_u = \omega_u$ for u in the small sphere S^{2n-1}, we can regard the family $\tilde{\omega}_u$, parameterized by a ball B^{2n} which bounds S^{2n-1}, as an extension of the family ω_u of symplectic forms on the fixed manifold $\tilde{X}_0$. Let us denote $\tilde{g}_u$ for the Kähler metric with respect to the Kähler form $\tilde{\omega}_u$. Then we have a map

$$T^{2n} : B^{2n} \to \mathcal{P}, \quad u \mapsto (\tilde{g}_u, 0).$$

Now we need the following proposition which is analogous to Proposition 4.1 in [9].

Proposition 3.1. *When r is sufficiently large, the solutions of the perturbed Seiberg-Witten equations* (2.2) *on the Kähler manifold X^+ correspond to algebraic curves in X^+ homologous to C, whose fundamental class is Poincaré dual to e.*

Proof. To show this, first act on the first equation of (2.2) with $\bar{\partial}_a$ and use the last equation of (2.2) to get

$$\bar{\partial}_a\bar{\partial}_a^*\beta + \frac{r}{4}|\alpha|^2\beta = 0.$$

Thus we get $|\bar{\partial}_a^*\beta|^2 = -\frac{r}{4}|\alpha|^2|\beta|^2$, and so either $\alpha = 0$ or $\beta = 0$. On the other hand, it follows from the proof of Theorem 4.2 in [10] that for sufficiently large r we have the following inequality

$$\int_{X^+} \frac{r}{2} iF_a^\omega \geq \int_{X^+} \Big(\frac{1}{4}|\nabla_a\alpha|^2 + \frac{1}{2}|\tilde{\nabla}_a\beta|^2 + \frac{r^2}{32}(1-|\alpha|^2-|\beta|^2)^2 + \frac{r^2}{16}|\beta|^2\Big).$$

As in [10], a gauge transformation can be chosen so that the left hand side of the inequality is the pairing $r\pi\langle[\omega]\cup e, [X,\partial X]\rangle$. Thus if $\alpha = 0$, then we get

$$\begin{aligned} r\pi\langle[\omega]\cup e, [X,\partial X]\rangle &\geq \int_{X^+} \Big(\frac{1}{2}|\tilde{\nabla}_a\beta|^2 + \frac{r^2}{32}(1+|\beta|^4)\Big) \\ &\geq \int_{X^+} \frac{r^2}{32}, \end{aligned}$$

which implies that $\pi\langle[\omega]\cup e, [X,\partial X]\rangle$ is infinite. This is a contradiction. Therefore $\beta = 0$, and the zero set of α is a curve C whose fundamental class is

Poincaré dual to e. Conversely, it is a well-known procedure that from the algebraic curve C we can obtain the holomorphic bundle $(E, \bar{\partial}_a)$ and the section α up to isomorphism (see e.g. [3]). This completes the proof. □

Now we return to the proof of Theorem 1.3. Note that there is only one such curve in our manifold $\tilde{X}_0$ by construction and no such curve for $u \neq 0$. Thus for sufficiently large r the image of T^{2n} meets $\Delta = \pi(\mathcal{M})$ only at $T^{2n}(0)$, and there is only one solution in $\mathcal{M}$ over $T^{2n}(0)$.

As a final step, we need to show that the map π is transverse to T^{2n}, as in Proposition 3.1 in [9]. However, in Section 4 of [9], Kronheimer provided a detailed argument showing that for the examples in his paper the map π is transverse to T^{2n} at $T^{2n}(0)$ by comparing the deformation theory of the solutions (a, α, β) of the Seiberg-Witten equations with the deformation theory of the curve C given by the zero set of α (see [14] for related discussions). Certainly his argument goes through in our case, too. Thus this completes the proof of Theorem 1.3.

4. Global results: Proof of Theorem 1.4

The main purpose of this section is to give a partial answer to Question 1.2 positively.

In order to get global results on closed symplectic (or Kähler) manifolds from the local result of Theorem 1.3, we need to use the symplectic compactifications of Stein surfaces of P. Lisca and G. Matić in [11].

Theorem 4.1 (Theorem 3.2 or Corollary 3.3 in [11])**.** *Let X be a Stein surface, and let $\phi : X \to \mathbb{R}$ be a smooth strictly pluri-subharmonic function such that the boundary of X is the regular level set of ϕ. Then there exist a holomorphic embedding j of X as a domain inside a closed Kähler minimal surface S of general surface such that the pull-back of the Kähler form of S to X equals $\omega_\phi = dJ^* d\phi$.*

Moreover, we can refine the statement in Theorem 4.1 as follows (see [21] and [2] for a similar argument).

Lemma 4.2. *Let X be a Stein surface. For any non-negative integer m, there exists a holomorphic embedding j of X as a domain inside a closed Kähler minimal surface S of general type with $b_2^+(S\backslash j(X)) > m$ (so $b_2^+(S) > m$).*

Proof. Suppose that $b_2^+(S\backslash j(X)) = m$ in Theorem 4.1. Then we first extend X into X' by attaching a 2-handle with framing $tb(K)-1$ along a Legendrian knot satisfying $tb(K) > 1$ contained in a standard 3-ball D^3 in ∂X (see [7] for the definitions of Legendrian knots and $tb(K)$). By the choice of such a knot with the framing, the well-known Eliashberg's theorem in [5] and [6] and its Gompf's refinement in [7] imply that X' is also a Stein surface. Now we embedd X' into a closed Kähler minimal surface S' of a general type. Since we have attached a 2-handle with positive framing, we have a second homology class with positive self-intersection number in $S'\backslash X$. Thus clearly we have $b_2^+(S'\backslash j(X)) > m$, so that $b_2^+(S') > m$. This completes the proof. □

We also need the following symplectic gluing result of symplectic forms, provided that the two corresponding contact structures on the glued region are isomorphic.

Lemma 4.3 (Lemma 4.1 in [11]). *Let X_1 and X_2 be two Stein surfaces with boundary ∂X_i $(i = 1, 2)$, and suppose that ∂X_i are diffeomorphic to the connected 3-manifold M. Let $\phi : X_1 \to \mathbb{R}$ be a smooth strictly pluri-subharmonic function having ∂X_1 as a level set, and let X_1 have the symplectic structure $\omega_1 = dJ^* d\phi$. Suppose that the contact structures ξ_1 and ξ_2 induced on M are isomorphic. Then there exist a J-compatible symplectic form ω_2 on the interior of X_2 and a symplectic embedding of the interior of a collar $U_1 \subset X_1$ around ∂X_1 as a subcollar of the interior of a collar $U_2 \subset X_2$ around ∂X_2.*

4.1. Simple cases

We first show the following simple theorem which is a slight generalization of Theorem 1.1 of Kronheimer, in the sense that we do not need any restriction on $b_2^+(X)$.

Theorem 4.4. *For any positive integer n, there exists a closed symplectic (Kähler) 4-manifold X having the non-trivial homology*

$$H_{2n-1}(\mathrm{Diff}_0(X)/\mathrm{Symp}(X); \mathbb{Q}).$$

Proof. Once again, let B^4 be the unit ball in $\mathbb{C}^2$, X' be the quotient $B^4/\mathbb{Z}_{n+1}$, and let X_{n+1} be the resolution obtained from X' by resolving the singular point with a sphere C of self-intersection $-(n+1)$. Then the boundary is a lens space $L(n+1, 1)$ and admits the contact structure ξ obtained from the embedding in $\mathbb{C}^2/\mathbb{Z}_{n+1}$. Moreover, X_{n+1} becomes a Stein manifold with contact boundary. By Theorem 1.3, we also have a family ω_u parameterized by $u \in S^{2n-1}$ which represents a non-trivial class in homology of the space consisting of symplectic 2-forms that are cohomologous to ω_ϕ. According to Theorem 4.1, we can consider the Kähler embedding $j : X_{n+1} \to S$, where S is a minimal surface of general type.

Our aim now is to extend the family ω_u of symplectic forms on X_{n+1} to a family of symplectic forms on all of S which represents a non-trivial class in homology of the space Λ. To do so, we first choose a symplectic form ω_{u_0} from the family ω_u for $u \in S^{2n-1}$. Then, by Lemma 4.3, we can extend ω_{u_0} to a symplectic form $\tilde{\omega}_{u_0}$ defined on all of S. Now it remains to extend the rest of the family ω_u defined on X_{n+1} to all of S. Since ω_u and ω_{u_0} are isotopic, they are connected by a path of cohomologous symplectic forms between ω_u and ω_{u_0}. Let Ω_s^u be such a smooth path over X_{n+1} of cohomologous symplectic forms such that $\Omega_0^u = \omega_{u_0}$ and $\Omega_1^u = \omega_u$. Assuming the collar U of ∂X_{n+1} is diffeomorphic to $\partial X_{n+1} \times [0, 1)$, we can define a new symplectic form $\tilde{\omega}_u$ as follows:

$$\tilde{\omega}_u = \begin{cases} \omega_u, & \text{on } X_{n+1}\backslash U \\ \Omega_t^u(y,t), & (y,t) \in U \cong \partial X_{n+1} \times [0,1) \\ \tilde{\omega}_{u_0}, & \text{on } S\backslash X_{n+1}. \end{cases}$$

Then all new symplectic forms $\tilde{\omega}_u$ are continuous and further cohomologous, since two forms $\tilde{\omega}_{u_0}$ and $\tilde{\omega}_u$ are connected by a path of symplectic forms given by

$$\tilde{\Omega}_s^u = \begin{cases} \Omega_{1-s}^u, & \text{on } X_{n+1}\backslash U \\ \Omega_{(1-s)t}^u, & \text{on } U \\ \tilde{\omega}_{u_0}, & \text{on } S\backslash X_{n+1}. \end{cases}$$

We next claim that the new family $\tilde{\omega}_u$ for $u \in S^{2n-1}$ is an essential element in the space of Λ. Indeed, suppose that there exists a singular chain T^{2n} in the space Λ of dimension $2n$ which bounds the family $\tilde{\omega}_u$. Then the restriction of T^{2n} to the submanifold X_{n+1} of S induces a singular chain of dimension $2n$, also denoted T^{2n}, in the space consisting of symplectic 2-forms that are cohomologous to ω_ϕ, which bounds the family ω_u for $u \in S^{2n-1}$. But this is a contradiction. This completes the proof. □

4.2. General cases

The purpose of this subsection is to prove Theorem 1.4 which is a partial answer to Question 1.2.

For any rational number $-\frac{p}{q} \in \mathbb{Q}$, we have a continued fraction expansion of the form

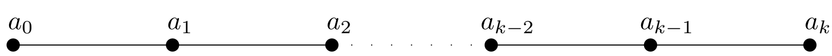
$$-\frac{p}{q} = a_0 - \cfrac{1}{a_1 - \cfrac{1}{a_2 - \cdots - \cfrac{1}{a_k}}}, \tag{4.1}$$

where $a_j \in \mathbb{Z}$ for $0 \leq j \leq k$. We will abbreviate (4.1) by writing

$$-\frac{p}{q} = [a_0, a_1, \ldots, a_k].$$

Note that lens spaces are a special case of Seifert fibered spaces. Our orientation convention will be that a lens space $L(p,q)$ is obtained by $-\frac{p}{q}$-Dehn surgery on an unknot in S^3 (see [8], [18], or [4]). Using the continued fraction expansion $-\dfrac{p}{q} = [a_0, a_1, \ldots, a_k]$, we get $L(p,q)$ as the boundary of the 4-manifold obtained by plumbing together $k+1$ disk bundles over S^2 with the Euler number a_j, according to the following linear chain Γ:

Γ :



where each dot represents a sphere S_j ($j = 0, 1, \ldots, k+1$) which has self-intersection number a_j and intersects only S_{j-1} and S_{j+1}.

Theorem 4.5. *For any given collection of positive integers* $\{n_1, n_2, \ldots, n_k\}$ *and any non-negative integer* m*, there exists a closed symplectic 4-manifold* X (*or closed Kähler minimal surface of general type*) *with* $b_2^+(X) > m$ *having the non-trivial homologies*

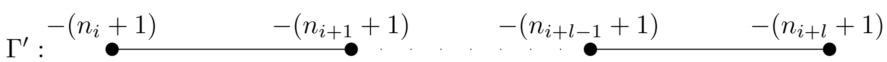
$$H_i(\mathrm{Diff}_0(X)/\mathrm{Symp}(X); \mathbb{Q})$$

for all $i = 2n_1 - 1, \ldots, 2n_k - 1$.

Proof. To construct such a closed symplectic 4-manifold, we use the 4-manifold obtained by plumbing together the disk bundles of cotangent bundles over a sphere. As noted earlier, Seifert fibered 3-manifolds are the boundaries of the plumbed 4-manifolds $P(\Gamma)$ associated to the plumbing graph Γ, and a lens space $L(p, q)$ with

$$-\frac{p}{q} = [-(n_1+1), -(n_2+1), \ldots, -(n_k+1)]$$

is a special case of Seifert fibered 3-manifolds, associated to the linear graph Γ with weights $-(n_1+1), -(n_2+1), \ldots, -(n_k+1)$. In addition, given a 4-manifold $P(\Gamma)$ obtained by plumbing on the linear graph Γ, let Γ' be a sub-chain in Γ as shown below:



Then the plumbing on Γ' gives rise to a sub-manifold $P(\Gamma')$ of $P(\Gamma)$ with the lens space $L(p', q')$ as a boundary, where

$$-\frac{p'}{q'} = [-(n_{i+1}+1), -(n_{i+2}+1), \ldots, -(n_{i+l}+1)].$$

Here $P(\Gamma')$ is taken so that $P(\Gamma')$ is strictly contained in the interior of $P(\Gamma)$ (e.g., see [18] for the construction). Thus the closure of $P(\Gamma)\backslash P(\Gamma')$ is a smooth compact 4-manifold with oriented boundary $-L(p', q') \cup L(p, q)$ in our case. In particular, if we take Γ' to be $\overset{-(n_i+1)}{\bullet}$, then $P(\Gamma)$ contains the sub-manifold $P(\Gamma')$ with the lens space $L(n_i+1, 1)$ as its boundary, which will be used in the proof, later.

Moreover, it is a theorem of Y. Eliashberg or a construction of Gompf in [7] that by attaching 2-handles along Legendrian knots K with framing $tb(K) - 1$ we can assume that the plumbed 4-manifold is a Stein manifold with the lens space as a contact boundary. So, we assume that the plumbed 4-manifold $P(\Gamma)$ associated to the linear graph Γ is a Stein manifold with contact boundary. In order to compactify the Stein manifold $P(\Gamma)$, we use Theorem 4.1 of Lisca and Matić and Lemma 4.2 concerning the symplectic compactifications of Stein surfaces to get such a compactification X with $b_2^+(X) > m$.

Let $\phi : P(\Gamma) \to \mathbb{R}$ be a smooth strictly pluri-subharmonic function such that the sub-level set of ϕ for a regular value is $\partial P(\Gamma)$, and such that the pull-back of the Kähler form of X to $P(\Gamma)$ equals $\omega_\phi = dJ^* d\phi$. Using Lemma 4.2, we next extend the exact Kähler form ω_ϕ to all of X. We may also assume that for $i = 1, \ldots, k$, X contains X_{n_i+1} as a Stein sub-manifold whose boundary is a lens space $L(n_i + 1, 1)$. Thus over each Stein sub-manifold X_{n_i+1} we have an essential element cohomologous to ω_ϕ, parameterized by a $(2n_i - 1)$-sphere, in the homology $H_{2n_i-1}(\mathrm{Diff}_0(X_{n_i+1})/\mathrm{Symp}(X_{n_i+1}), \mathbb{Q})$.

Finally, we apply again Lemma 4.2 and the argument in the proof of Theorem 4.1 to the essential element in order to obtain a family of symplectic forms defined on all of X, parameterized by a $(2n_i - 1)$-sphere. Then an easy argument proving Theorem 4.5 implies that the extended family of symplectic forms induces an essential element in the homology $H_{2n_i-1}(\mathrm{Diff}_0(X)/\mathrm{Symp}(X); \mathbb{Q})$. Since we can apply the same argument to any X_{n_i+1}, we have completed the proof for the odd degrees, as stated. □

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