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ANALYSIS OF A REVERSED TRAPEZOIDAL FIN USING A 2-D ANALYTIC METHOD

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ABSTRACT. A reversed trapezoidal fin is analyzed using a two-dimensional analytical method. Heat loss from the reversed trapezoidal fin is presented as a function of the fin shape factor, fin base thickness and the fin base height. The relationship between the fin tip length and the convection characteristic number as well as that between the fin tip length and the fin base height for equal amounts of heat loss are analyzed. Also the relationship between the fin base thickness and the fin shape factor for equal amount of heat loss is presented. One of the results shows that the heat loss decreases linearly with the increase of the fin shape factor.

1. INTRODUCTION

Extended surfaces or fins are well known to be a simple and effective means of increasing heat dissipation in many engineering and industrial applications such as the cooling of combustion engines, electronic equipments, many kinds of heat exchangers, and so on. As a result, a great deal of attention has been directed to the fin problems and many studies for the various shapes of fins have been presented. The most commonly studied fins are longitudinal rectangular, triangular, trapezoidal fins and the annular or circular fins. For example, Sen and Trinh [1] studied the rate of heat loss from a rectangular fin governed by the power law-type temperature dependence. Kang and Look [2] analyzed the trapezoidal fin with various lateral surface slopes while Yeh [3] investigated the optimum dimensions of rectangular fins and cylindrical pin fins. Kang and Look [4] presented the analysis of thermally asymmetric triangular fin using a two-dimensional analytical method. Sikka and Iqbal [5] made an analysis of the heat transfer characteristics of a circular fin dissipating heat from its surface by convection and radiation. In all these studies, fin base temperature is given as a constant for the fin base boundary condition.

The effect of fin base thickness variation is considered for the common shape fin analysis in some studies. For example, Abrate and Newnham [6] studied heat conduction in an array of triangular fins with an attached wall using the finite element method. Kang [7] presented an optimum procedure for a pin fin based on the increasing rate of heat loss. Recently, Kang [8] optimized a pin fin with variable fin base thickness for fixed fin volumes.

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Also, studies on the more unique shape of the fin have been reported. For this kind of paper, Bejan and Almogbel [9] reported the geometric (constructal) optimization of T-shaped fin assemblies, where the objective is to maximize the global conductance of the assembly, subject to total volume and fin-material constraints while Hashizume et al. [10] analyzed fin efficiency of serrated fins and derived the theoretical fin efficiency in the form of a function of modified Bessel functions. Kundu and Das [11] analyzed and optimized elliptical disk fins using a semi-analytical technique.

In this study, a reversed trapezoidal with various fin lateral surface slopes is analyzed using a two dimensional analytic method. For this analysis, both the fin base thickness and base height can be varied. Therefore, the thermal resistance from the inside wall to the fin base can be changed due to the variation of fin base thickness and the fin height. Under these conditions, heat loss from the reversed trapezoidal fin is presented as a function of the fin shape factor, fin base thickness and the fin base height. The relationship between the fin tip length and the convection characteristic number as well as that between the fin tip length and the fin base height for equal amounts of heat loss are analyzed. Also the relationship between the fin base thickness and the fin shape factor for equal amount of heat loss is presented.

2. A 2-D Analytical Method

The schematic diagram of a reversed trapezoidal fin is shown in Fig. 1. For this schematic diagram, dimensionless two-dimensional governing differential equation under steady state is

$$\frac{\partial^2\theta}{\partial X^2} + \frac{\partial^2\theta}{\partial Y^2} = 0. \tag{2.1}$$

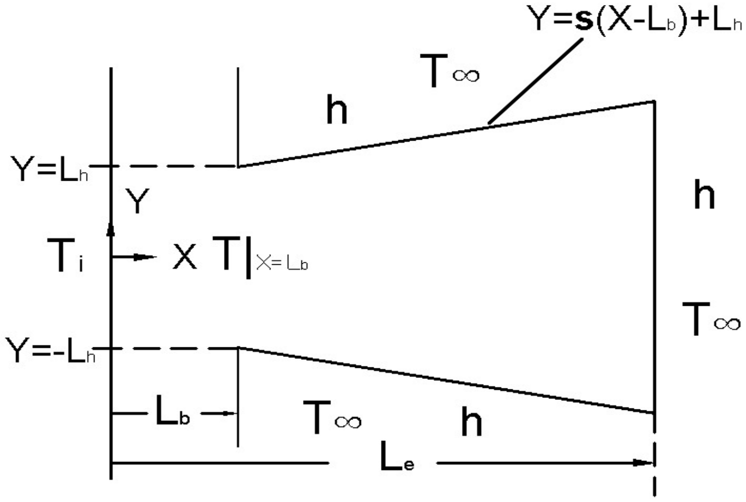

FIGURE 1. Geometry of a reversed trapezoidal fin

Three boundary conditions and one energy balance condition are required to solve the governing differential equation and these conditions are given as Eqs. (2.2)∼(2.5).

$$\left.\frac{\partial\theta}{\partial X}\right|_{X=L_b} + \frac{1-\theta|_{X=L_b}}{L_b} = 0 \tag{2.2}$$

$$\left.\frac{\partial\theta}{\partial Y}\right|_{Y=0} = 0 \tag{2.3}$$

$$\left.\frac{\partial\theta}{\partial X}\right|_{X=L_e} + M\cdot\theta|_{X=L_e} = 0 \tag{2.4}$$

$$\begin{aligned} -\int_0^{L_h} \left.\frac{\partial\theta}{\partial X}\right|_{X=L_b} dY \;&=\; M\cdot\int_{L_h}^{s(L_e-L_b)+L_h} \theta\sqrt{(1/s)^2+1}\,dY \\ &\quad - \int_0^{s(L_e-L_b)+L_h} \left.\frac{\partial\theta}{\partial X}\right|_{X=L_e} dY \end{aligned} \tag{2.5}$$

Fin base boundary condition is represented by Eq. (2.2) and it means that heat conduction from the inner wall to the fin base and heat conduction through the fin base are the same. There is no heat transfer through the center surface, as the shape of the fin and ambient conditions are symmetric, and this boundary condition is written by Eq. (2.3). Equation (2.4) is a fin tip boundary condition and it means physically that heat conduction through the tip is equal to the heat transfer by convection from the tip surface. Finally, energy balance equation (2.5) indicates that the energy conducted into the reversed trapezoidal fin at the upper half base must escape from the fin by convection via the upper flat sloped lateral surface and by conduction to the upper half fin tip.

For a rectangular fin case (i.e. ξ=1 or s=0), the governing Eq. (2.1) and Eqs. (2.2)-(2.4) for three boundary conditions are also used except that Eq. (2.5) for the energy balance condition is replaced into Eq. (2.6) for the fin top boundary condition.

$$\left.\frac{\partial\theta}{\partial Y}\right|_{Y=L_h} + M\cdot\theta|_{Y=L_h} = 0 \tag{2.6}$$

When Eq. (2.1) with three boundary conditions (2.2), (2.3) and (2.4) are solved, the temperature distribution θ(X, Y) within the reversed trapezoidal fin can be obtained using the separation of variables procedure. The result is

$$\theta(X,Y) = \sum_{n=1}^{\infty} \frac{g_1(\lambda_n)\cdot f(X)\cdot\cos(\lambda_n Y)}{g_2(\lambda_n)+g_3(\lambda_n)} \tag{2.7}$$

where,

$$f(X) = \cosh(\lambda_n X) + g_4(\lambda_n)\cdot\sinh(\lambda_n X)$$

$$g_1(\lambda_n) = \frac{4\sin(\lambda_n L_h)}{2\lambda_n L_h + \sin(2\lambda_n L_h)}$$

$$g_2(\lambda_n) = \cosh(\lambda_n L_b) - L_b\cdot\lambda_n\cdot\sinh(\lambda_n L_b)$$

$$g_3(\lambda_n) = g_4(\lambda_n) \cdot \{\sinh(\lambda_n L_b) - L_b \cdot \lambda_n \cdot \cosh(\lambda_n L_b)\}$$

$$g_4(\lambda_n) = -\frac{\lambda_n \cdot \tanh(\lambda_n L_e) + M}{\lambda_n + M \cdot \tanh(\lambda_n L_e)}.$$

The eigenvalues λ_n can be obtained using Eq. (2.8), which is a arranged form of Eq. (2.5).

$$0 = g_5(\lambda_n) - g_6(\lambda_n) + g_7(\lambda_n)[\{g_8(\lambda_n) + g_9(\lambda_n)\} \cdot \{g_{10}(\lambda_n) + g_{11}(\lambda_n) - g_{12}(\lambda_n) - g_{13}(\lambda_n)\} + \{g_{14}(\lambda_n) + g_{15}(\lambda_n)\} \cdot \{g_{16}(\lambda_n) + g_{17}(\lambda_n) - g_{18}(\lambda_n) - g_{19}(\lambda_n)\}] \quad (2.8)$$

where,

$$g_5(\lambda_n) = \{\sinh(\lambda_n L_b) + g_4(\lambda_n) \cdot \cosh(\lambda_n L_b)\} \sin(\lambda_n L_h)$$

$$g_6(\lambda_n) = \{\sinh(\lambda_n L_e) + g_4(\lambda_n) \cdot \cosh(\lambda_n L_e)\} \sin\{(\lambda_n (2 - \xi) L_h\}$$

$$g_7(\lambda_n) = M/(\lambda_n \cdot \sqrt{1 + s^2})$$

$$g_8(\lambda_n) = \cosh\{\lambda_n (L_b - \frac{L_h}{s})\}$$

$$g_9(\lambda_n) = g_4(\lambda_n) \cdot \sinh\{\lambda_n (L_b - \frac{L_h}{s})\}$$

$$g_{10}(\lambda_n) = \sinh\{\frac{\lambda_n}{s}(2 - \xi) L_h\} \cdot \cos\{\lambda_n (2 - \xi) L_h\}$$

$$g_{11}(\lambda_n) = s \cdot \cosh\{\frac{\lambda_n}{s}(2 - \xi) L_h\} \cdot \sin\{\lambda_n (2 - \xi) L_h\}$$

$$g_{13}(\lambda_n) = s \cdot \sin(\lambda_n L_h) \cosh(\frac{\lambda_n L_h}{s})$$

$$g_{14}(\lambda_n) = \sinh\{\lambda_n (L_b - \frac{L_h}{s})\}$$

$$g_{15}(\lambda_n) = g_4(\lambda_n) \cosh\{\lambda_n (L_b - \frac{L_h}{s})\}$$

$$g_{16}(\lambda_n) = \cosh\{\frac{\lambda_n}{s}(2 - \xi) L_h\} \cdot \cos\{\lambda_n (2 - \xi) L_h\}$$

$$g_{17}(\lambda_n) = s \cdot \sinh\{\frac{\lambda_n}{s}(2 - \xi) L_h\} \cdot \sin\{\lambda_n (2 - \xi) L_h\}$$

$$g_{18}(\lambda_n) = \cos(\lambda_n L_h) \cosh(\frac{\lambda_n L_h}{s})$$

$$g_{19}(\lambda_n) = s \cdot \sin(\lambda_n L_h) \sinh(\frac{\lambda_n L_h}{s}).$$

The first eigenvalue λ_1 is obtained by using an incremental search method from Eq. (2.8) and then the rest eigenvalues λ_n (n=2, 3, 4, $\cdots$) are calculated from Eq. (2.10). That is, the direct application of the orthogonality principle in the separation of variables method produces Eq. (2.9).

$$\int_0^{L_h} \cos(\lambda_1 Y) \cos(\lambda_n Y) \, dY = \frac{\sin\{(\lambda_1 - \lambda_n) L_h\}}{2(\lambda_1 - \lambda_n)} + \frac{\sin\{(\lambda_1 + \lambda_n) L_h\}}{2(\lambda_1 + \lambda_n)} = 0 \quad (2.9)$$

Algebraic manipulations of Eq. (2.9) produces Eq. (2.10) from which the eigenvalues λ_n (n=2, 3, 4, $\cdots$) may be more easily calculated by using the Newton-Raphson method.

$$\lambda_n = (2\lambda_1 + \lambda_n) - 2(\lambda_1 + \lambda_n)\frac{\tan(\lambda_n L_h)}{\tan(\lambda_1 L_h) + \tan(\lambda_n L_h)} \tag{2.10}$$

The eigenvalues λ_n (n=1, 2, 3, 4, $\cdots$) can be calculated using Eq. (2.11) that is derived from Eq. (2.6) for a rectangular fin case.

$$\lambda_n \cdot \tan(\lambda_n) = M \tag{2.11}$$

The heat loss conducted into the fin through the fin base for both the reversed trapezoidal and rectangular fins is calculated by

$$q = -2\int_0^{l_h} k\frac{\partial T}{\partial x}\bigg|_{x=l_b} l_w dy.$$

Then, the dimensionless heat loss from the reversed trapezoidal fin is denoted by

$$Q = \frac{q}{k\varphi_i l_w} = -2\sum_{n=1}^{\infty}\frac{g_1(\lambda_n)\cdot g_5(\lambda_n)}{g_2(\lambda_n) + g_3(\lambda_n)}.$$

3. Results and Discussions

For two different fin base height and three different fin tip length cases, the temperature profile along the fin center line is represented in Fig. 2. The normalized position of X, NPX, is given as $(X - L_b)/(L_e - L_b)$ so that NPX=0 represents the position at the fin base and NPX=1

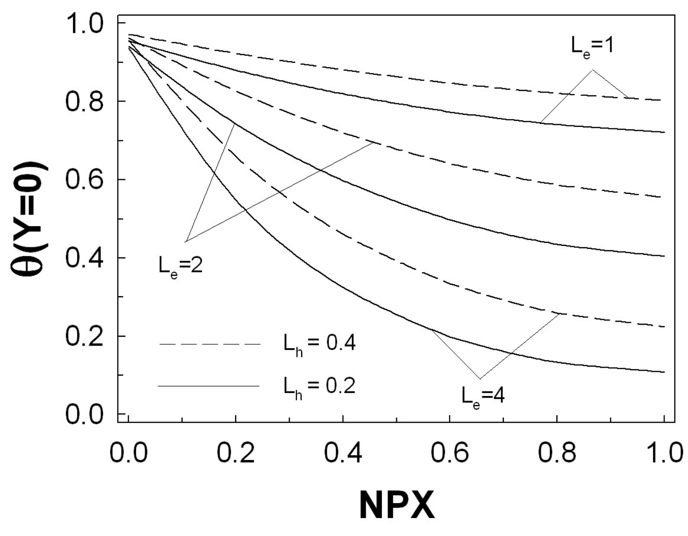

FIGURE 2. Dimensionless temperature profile along the normalized position of X (M=0.1, L_b=.1, ξ=0.5)

θ(NPY)/θ(NPY=0) (%)			
NPY	$\xi = 0$	$\xi = 0.5$	$\xi = 1$
0.2	99.14	99.54	99.81
0.4	96.57	98.16	99.23
0.6	92.34	95.88	98.26
0.8	86.52	92.71	96.92
1.0	79.20	88.69	95.20

TABLE 1. Temperature ratio with the variations of NPY at $X = L_e$ ($M = 0.2, L_b = 0.1, L_e = 2.1, L_h = 0.5$)

at the fin tip. As the fin base height increases from 0.2 to 0.4, the fin temperature along the fin center line increases for all three values of L_e. It can be inferred that, at the center of the fin tip, the difference between the temperature for L_h=0.2 and that for L_h=0.4 increases first and then decreases as the fin tip length increases from 1 to 4.

Table 1 lists the ratio of temperature with the variation of normalized poison of Y [i.e. NPY =Y/$\{$(2-ξ)$L_h\}$] at the fin tip. The shape of the fin becomes rectangular for ξ=1 and it becomes the reversed trapezoidal fin for which the fin tip height is twice the fin base height in the case of ξ=0. This table illustrates that the ratio of temperature decreases more remarkably with the increase of NPY as the fin shape factor decreases from 1 to 0. It also can be noted that the ratio decreases as ξ decreases for the fixed value of NPY.

Figure 3 represents the dimensionless heat loss as a function of the fin shape factor for different values of dimensionless fin tip length. As expected, the heat loss decreases as the fin

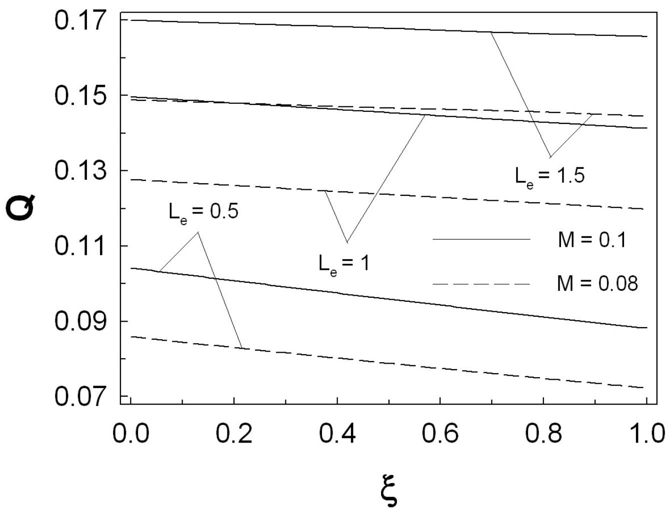

FIGURE 3. Heat loss as a function of the fin shape factor (L_b=0.1, L_h=0.1)

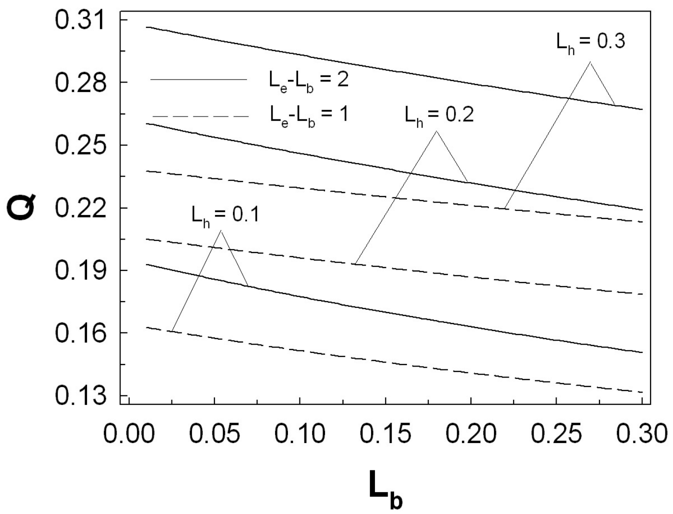

FIGURE 4. Heat loss as a function of the fin base thickness ($M = 0.1, \xi = 0.5$)

shape factor increases due to the decrease of heat transfer area and this phenomenon is more remarkable as the fin tip length decreases. Therefore it can be inferred that the effect of ξ on the heat loss disappears for the long fin enough. It is observed that the heat loss decreases almost linearly with the increase of ξ for all given values of L_e and M.

The effect of dimensionless fin base thickness on the heat loss is presented in Fig. 4. The fin base thickness has a significant influence on the heat loss because the fin base temperature decreases due to the increased thermal resistance as the fin base thickness increases. It is found that the increase of fin base thickness will reduce the heat loss almost linearly even though the actual fin length (i.e. $L_e - L_b$) remains constant. It also shows that the effect of the fin base thickness on the heat loss becomes a little larger as the fin length increases from 1 to 2 for fixed fin base height.

Figure 5 depicts the dimensionless heat loss as a function of the dimensionless fin base height for two different fin length and three different convection characteristic numbers. For all given convection characteristic numbers, the heat loss increases parabolically first and then increases linearly for L_e-L_b=2 while that increases linearly for L_e-L_b=0.5 as the fin base height increases.

Figure 6 shows the relationship between dimensionless fin tip length and fin base height for equal amounts of heat loss based on the value of L_e=0.8 and L_h=0.2 in the case of M=0.01, 0.05 and 0.1. The value of fin tip length decreases almost linearly in the case of M=0.01 while that decreases parabolically for M=0.1 as the fin base height increases from 0.1 to 0.2. It also shows that the fin base height increases as M increases in the range of $L_h < 0.2$ while that decreases with the increase of M for $L_h > 0.2$ when the fin tip length is fixed.

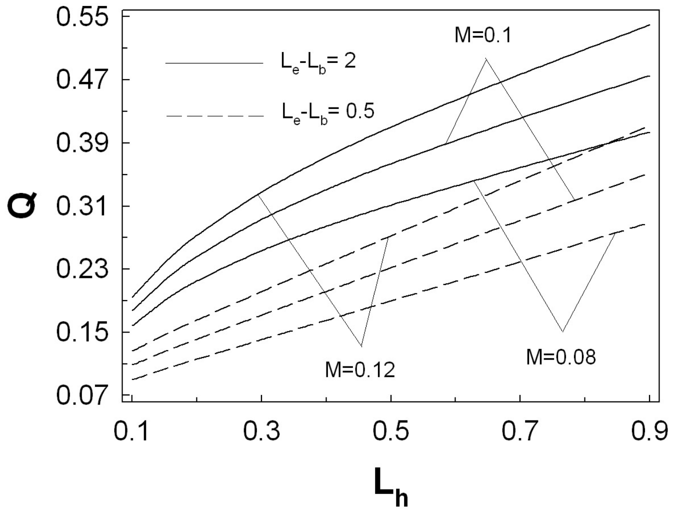

FIGURE 5. Heat loss as a function of the fin base height (ξ=0.5, L_b=0.1)

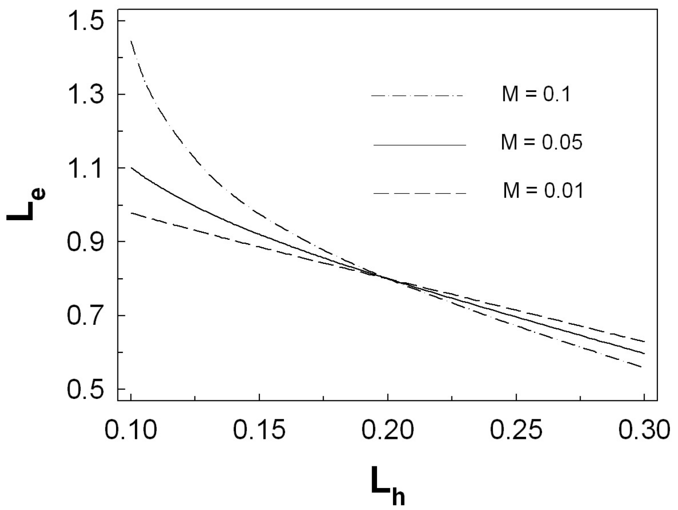

FIGURE 6. Relationship between the fin tip length and fin base height for equal amounts of heat loss (ξ=0.5, L_b=0.1)

Figure 7 presents the relationship between the fin tip length and the convection characteristic number for equal amounts of heat loss based on the value of L_e=2 and M=0.05 for three values of the fin shape factor. The value of L_e decreases parabolically as M increases for all three values of the fin shape factor. It can be noted that the slope of the curve steeps as the fin shape factor decreases from 1 to 0. It means physically that the fin length varies more remarkably

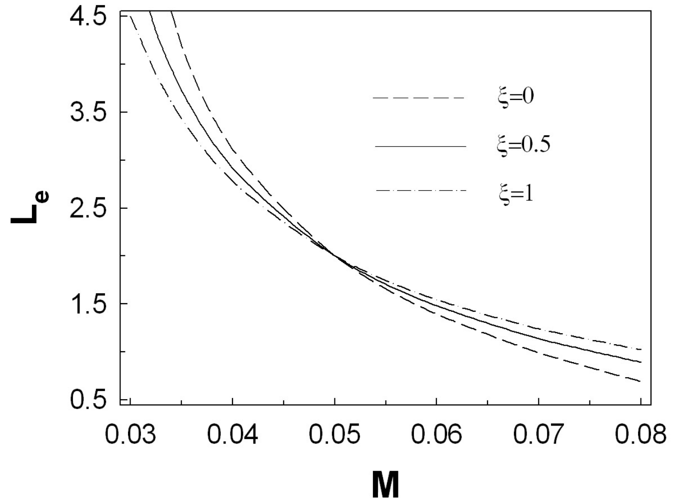

FIGURE 7. Relationship between the fin tip length and convection characteristic number for equal amounts of heat loss (L_b=0.1, L_h=0.4)

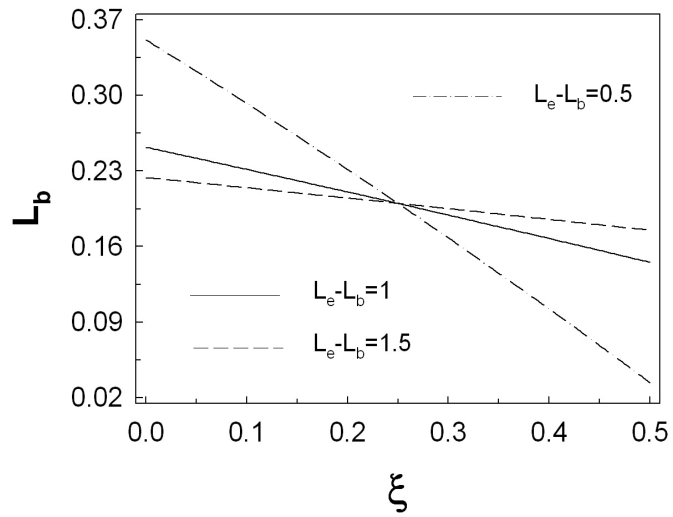

FIGURE 8. Relationship between the fin base thickness and fin shape factor for equal amounts of heat loss ($M = 0.1, L_h = 0.2$)

with the variation of the convection characteristic number to produce the equal amount of heat loss as the shape of the fin changes from the rectangular fin through the reversed trapezoidal fin with short fin tip height to that with long fin tip height.

The relationship between the fin base thickness and the fin shape factor for equal amounts of heat loss based on the value of L_b=0.2 and ξ=0.25 for three values of fin length is depicted in Fig. 8. The value of L_b decreases almost linearly with the increase of ξ for all three values of the fin length. It shows that the variation of L_b with the variation of ξ becomes less remarkable as the fin length increases. From this phenomenon, physically it can be inferred that the fin shape has no effect on the fin base thickness for equal amounts of heat loss when the fin length is very long.

4. Conclusions

From this two-dimensional analysis of a reversed trapezoidal fin, the following conclusions can be drawn:

(1) Heat loss from the reversed trapezoidal fin decreases linearly as the fin shape factor and/or fin base thickness increases.
(2) When other variables are fixed, fin tip length decreases parabolically as the convection characteristic number increases for equal amounts of heat loss.
(3) For the relatively short fin, the fin base thickness decreases remarkably with the increase of the fin shape factor to transfer equal amounts of heat loss.

Nomenclature

h : heat transfer coefficient over the fin [W/m^2 $^\circ C$]
k : thermal conductivity of the fin material [W/m $^\circ C$]
l_b : fin base thickness [m]
L_b : dimensionless fin base thickness, l_b/l_c
l_c : characteristic length [m]
l_e : fin tip length [m]
L_e : dimensionless fin tip length, l_e/l_c
l_h : one half fin base height [m]
L_h : dimensionless one half fin base height, l_h/l_c
l_w : fin width [m]
M : convection characteristic number, $(h\, l_c)/k$
q : heat loss from the fin [W]
Q : dimensionless heat loss from the fin, $q/(k\, l_w \varphi_i)$
s : fin lateral surface slope, $\{(1-\xi)l_h\}/(l_e - l_b)$
T : fin temperature [$^\circ C$]
T_b : fin base temperature [$^\circ C$]
T_i : inside wall temperature [$^\circ C$]
T_∞ : ambient temperature [$^\circ C$]
x : length directional variable [m]
X : dimensionless length directional variable, x/l_c
y : height directional variable [m]
Y : dimensionless height directional variable, y/l_c

Greek symbol

θ : dimensionless temperature, $(T - T_\infty)/(T_i - T_\infty)$
λ_n : eigenvalues $(n = 1, 2, 3, \cdots)$
φ_i : adjusted temperature of inside wall [$^\circ C$], $(T_i - T_\infty)$
ξ : fin shape factor, $(0 \leq \xi \leq 1)$

Subscript

b: fin base
c: characteristic
e: fin tip
h: fin base height
i: inside wall
w: fin width
∞: surrounding

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