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ON COMPLEXITY ANALYSIS OF THE PRIMAL-DUAL INTERIOR-POINT METHOD FOR SECOND-ORDER CONE OPTIMIZATION PROBLEM

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ABSTRACT. The purpose of this paper is to obtain new complexity results for a second-order cone optimization (SOCO) problem. We define a proximity function for the SOCO by a kernel function. Furthermore we formulate an algorithm for a large-update primal-dual interior-point method (IPM) for the SOCO by using the proximity function and give its complexity analysis, and then we show that the new worst-case iteration bound for the IPM is $\mathcal{O}(q\sqrt{N}(\log N)^{\frac{q+1}{q}}\log\frac{N}{\epsilon})$, where $q \geqq 1$.

1. INTRODUCTION

Primal and dual interior-point methods (IPMs) have been well known as the most effective methods for solving wide classes of optimization problems, for example, linear optimization (LO) problem, quadratic optimization problem (QOP), semidefinite optimization (SDO) problem, SOCO problem and convex optimization problem (CP).

The choice of parameter θ, the so-called barrier update parameter, plays an important role in the both in theory and practice of IPMs. Usually, if θ is a constant which is independent of the dimension of a problem, then the algorithm is called a *large-update* method. If it depends on the dimension, then the algorithm is said to be a *small-update* method. Large-update methods are much more efficient than small-update methods in practice ([1]). The gap between theory and practice has been referred to as irony of IPMs ([28]). Recently, many authors have tried to reduce the gap of the worst-case iteration bound between large-update IPM and small-update IPM. Using self-regular proximity functions instead of classical logarithmic barrier functions, Peng et al. ([23, 24, 25, 26, 27]) improved the complexity of large-update IPMs for LO, SDO and SOCO. They obtained the worst-case iteration bound $\mathcal{O}(\sqrt{n}\log n\log\frac{n}{\epsilon})$ for large-update IPMs of LO and SDO, and the worst-case iteration bound $\mathcal{O}(\sqrt{N}\log N\log\frac{N}{\epsilon})$ for SOCO

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([23, 24, 25, 26, 27]). After that, Bai and Wang in [6] and Bai et al. in [7] defined proximity functions of large-update IPMs for SOCO and obtained the bound $\mathcal{O}(\sqrt{N}\log N \log \frac{N}{\epsilon})$.

Recently, Bai et al. ([4]) introduced a new class of kernel functions. The class was defined by some simple conditions on the kernel function and its derivatives, and presented a simple and unified computational scheme for the complexity analysis of kernel functions in the new class. The best iteration bound, which was given by Bai et al. ([4]), is $\mathcal{O}(\sqrt{n}\log n \log \frac{n}{\epsilon})$. Very recently, following the approach of Bai et al. ([4]) for LO, Cho et al. ([8]), Cho and Kim ([9]) and Cho ([10]) used new kernel functions to calculate the iteration bound for a $P_*(\kappa)$ linear complementarity problem ($P_*(\kappa)$ LCP).

The aim of this paper is to obtain new complexity result for an SOCO problem using a new proximity function and following the approach of Bai et al. ([4]).

In this paper, we define a new proximity function for the SOCO by a kernel function which is suggested by Amini and Peyghami for LO in [2], Amini and Peyghami for $P_*(\kappa)$ LCP in [2] and Choi and Lee for SDO in [11]. Using the new proximity function for the SOCO, we formulate an algorithm for a large-update primal-dual IPM for the SOCO and we give its complexity analysis, and then we show that the new worst-case iteration bound for our IPM is

$$\mathcal{O}(q\sqrt{N}(\log N)^{\frac{q+1}{q}} \log \frac{N}{\epsilon}),$$

where $q \geqq 1$.

Now we recall the definition of SOCO, which is the problem of minimizing a linear objective function subject to the constraint set defined by linear equalities and product of second-order cones (see [20] for applications of SOCO). First we give the definitions of second-order cone, its related matrix and induced vector ordering. The set K^j is the second-order cone of dimension n_j defined as

$$K^j \;:=\; \{(x_1^j, \cdots, x_{n_j}^j)^T \in \mathbb{R}^{n_j} \mid (x_1^j)^2 \;-\; \sum_{i=2}^{n_j}(x_i^j)^2 \geqq 0,\; x_1^j \geqq 0\;\},$$

$j = 1, \cdots, N$. Let K_+^j be the interior of K^j. For any $x^j \;=\; (x_1^j, \cdots, x_{n_j}^j)^T \in \mathbb{R}^{n_j}$, let us define a matrix

$$\text{mat}(x^j) \;=\; \begin{pmatrix} x_1^j & x_{2:n_j}^j \\ (x_{2:n_j}^j)^T & x_1^j E_{n_j-1} \end{pmatrix},$$

where $x_{2:n_j}^j = (x_2^j, \cdots, x_{n_j}^j)$ and E_{n_j-1} denotes the identity matrix in $\mathbb{R}^{(n_j-1)\times(n_j-1)}$, $j = 1, \cdots, N$. The vector $x^j \in K^j$ means that mat(x^j) is a symmetric positive semidefinite matrix, that is, mat$(x^j) \succeq 0$. As standard, the notation $x^j \succeq_{K^j} 0$ (or $x^j \succ_{K^j} 0$) means that $x^j \in K^j$ (or $x^j \in K_+^j$). Let $K = K^1 \times \cdots \times K^N$, $K_+ = K_+^1 \times \cdots \times K_+^N$ and $n = \sum_{j=1}^N n_j$. Then for a vector $x = ((x^1)^T, (x^2)^T, \cdots, (x^N)^T)^T \in \mathbb{R}^n$, $x \succeq_K 0$ means that $x^j \in K^j$ for all $j = 1, \cdots, N$.

Consider the following second-order cone optimization problem (shortly, SOCO):

$$\begin{array}{lll} \text{SOCO} & \text{Minimize} & c^T x \\ & \text{subject to} & Ax = b,\ x \succeq_K 0, \end{array}$$

and its dual problem:

$$\begin{array}{lll} \text{SOCD} & \text{Maximize} & b^T y \\ & \text{subject to} & A^T y + s = c,\ s \succeq_K 0, \end{array}$$

where $A \in \mathbb{R}^{m\times n},\ b \in \mathbb{R}^m,\ c \in \mathbb{R}^n, x = ((x^1)^T, (x^2)^T, \cdots, (x^N)^T)^T \in \mathbb{R}^n,$ and $x^j \in \mathbb{R}^{n_j}$, $j = 1, \cdots, N$. The matrix A is assumed to be of full row rank, i.e., rank$A = m$. In the sequel, we assume that both SOCO and SOCD satisfy the interior-point condition (IPC), that is, there exists $(x^0,\ y^0,\ s^0)$ such that $Ax^0 = b,\ \ x^0 \succ_K 0,\ \ A^T y^0 + s^0 = c,\ s^0 \succ_K 0$. Then we have an optimal solution (x, y, s) of SOCO and SOCD.

Jordan-algebraic techniques have proved to be very useful for the analysis of convex optimization problems over symmetric cones. See, e.g., [12, 13, 14, 15, 16, 21, 29, 32]. The Euclidean Jordan algebra for the second order cone K^j is defined by a bilinear operator

$$x^j \circ s^j = ((x^j)^T s^j,\ {x_1}^j s^j_{2:n_j} + {s_1}^j x^j_{2:n_j})^T,$$

where $x^j,\ s^j \in \mathbb{R}^{n_j}$. Obviously, the Jordan product $\circ$ is commutative, i.e., $x^j \circ s^j = s^j \circ x^j$ for each j. But its associativity does not hold in general and the cone K is not closed under the Jordan product. It is also easy to verify that for any $x^j,\ s^j \in \mathbb{R}^{n_j}$ one has $x^j \circ s^j = \mathrm{mat}(x^j)s^j$. We define, $\mathrm{mat}(x) := \mathrm{diag}(\mathrm{mat}(x^1), \cdots, \mathrm{mat}(x^N))$ and $x \circ s := ((x^1 \circ s^1)^T, \cdots, (x^N \circ s^N)^T)^T$. Then we have the following lemma which is well-known:

Lemma 1.1. [18] *The following statements are equivalent:*
(i) $x \succeq_K 0,\ s \succeq_K 0$ *and* $x^T s = 0$;
(ii) $x \succeq_K 0,\ s \succeq_K 0$ *and* $\mathrm{mat}(x)s = 0$;
(iii) $x \succeq_K 0,\ s \succeq_K 0$ *and* $x \circ s = 0$.

Using Lemma 1.1, we can easily check that a pair of optimal solutions of SOCO and SOCD is equivalent to solving the following Newton system:

$$\begin{cases} Ax = b,\ \ x \succeq_K 0, \\ A^T y + s = c,\ \ s \succeq_K 0, \\ x \circ s = 0. \end{cases} \tag{1.1}$$

The basic idea of primal-dual IPMs is to replace the third equation in (1.1), the so-called *complementarity condition* for SOCO and SOCD, by the parameterized equation $x \circ s = \mu\tilde{e}$ with $\mu > 0$, where $\tilde{e} = ((\tilde{e}^1)^T, \cdots, (\tilde{e}^N)^T)^T \in \mathbb{R}^n,\ \tilde{e}^j = (1, 0, \cdots, 0)^T \in \mathbb{R}^{n_j},\ \ j = 1, \cdots, N$. From the system (1.1) , we have the following parameterized system with positive parameter μ:

$$\begin{cases} Ax \;=\; b, \;\; x \succeq_K 0, \\ A^T y \;+\; s \;=\; c, \;\; s \succeq_K 0, \\ x \circ s \;=\; \mu \tilde{e}, \;\; \mu > 0. \end{cases} \tag{1.2}$$

If both SOCO and SOCD satisfy IPC, then for each $\mu > 0$, the parameterized system (1.2) has a unique solution $(x(\mu), y(\mu), s(\mu))$ (see [22], [33]), which is called a μ-center of SOCO and SOCD. The set of μ-centers, that is, $\mathcal{C} = \{(x(\mu),\ y(\mu),\ s(\mu)) \mid \mu > 0\}$, is said to be the *central path* of SOCO and SOCD. The central path converges to the solution pair of SOCO and SOCD as μ reduces to zero ([5, 22, 30]).

In general, IPMs for the SOCO consist of two strategies: The first one, which is called the inner iteration scheme, is to keep the iterative sequence in a certain neighborhood of the central path or to keep the iterative sequence in a certain neighborhood of the μ-center and the second one is called the outer iteration scheme, is to decrease the parameter μ to $\mu_+ := (1-\theta)\mu,$ for some $\theta \in (0, 1)$.

2. Proximity Functions and Search Directions

For any $x^j \in \mathbb{R}^{n_j}$, for each j, we define eigenvalues for x^j in the sense of Jordan algebra;

$$\lambda_{\max}(x^j) \;:=\; x_1{}^j \;+\; \|x^j_{2:n_j}\|, \quad \lambda_{\min}(x^j) \;:=\; x^j_1 \;-\; \|x^j_{2:n_j}\|.$$

The trace of x^j is defined by

$$\text{Tr}(x^j) \;:=\; \lambda_{\max}(x^j) \;+\; \lambda_{\min}(x^j)$$

and the determinant of x^j is given by

$$\det(x^j) \;:=\; \lambda_{\max}(x^j)\lambda_{\min}(x^j).$$

In fact, $\lambda_{\max}(x^j)$ and $\lambda_{\min}(x^j)$ are the maximal and minimal eigenvalues of the matrix mat(x^j), respectively, and every $x^j = (x^j_1,\ x^j_2,\ \cdots,\ x^j_{n_j})^T \in \mathbb{R}^{n_j}$ can be rewritten by the so-called spectral decomposition([17]) in the sense of Jordan algebra:

$$x^j \;=\; \lambda_{\max}(x^j)(\frac{1}{2},\ \frac{x^j_{2:n_j}}{2\|x^j_{2:n_j}\|})^T \;+\; \lambda_{\min}(x^j)(\frac{1}{2},\ -\frac{x^j_{2:n_j}}{2\|x^j_{2:n_j}\|})^T.$$

Here, $\frac{x^j_{2:n_j}}{2\|x^j_{2:n_j}\|} = 0$ if $x^j_{2:n_j} = 0$.

Definition 2.1. [23] *Suppose that $\psi(t)$ is a function from $\mathbb{R}$ to $\mathbb{R}$ and $x^j \in \mathbb{R}^{n_j}$. Then the function $\psi(x^j) : \mathbb{R}^{n_j} \to \mathbb{R}^{n_j}$ associated with the second-order cone K^j is defined as follows:*

$$\psi(x^j) := \frac{1}{2}(\psi(\lambda_{\max}(x^j)) \;+\; \psi(\lambda_{\min}(x^j)),\ \Delta\psi(\lambda_{\max}(x^j),\ \lambda_{\min}(x^j))x^j_{2:n_j})^T,$$

where $\Delta\psi(\lambda_{\max}(x^j), \lambda_{\min}(x^j)) = \frac{\psi(\lambda_{\max}(x^j)) - \psi(\lambda_{\min}(x^j))}{\lambda_{\max}(x^j) - \lambda_{\min}(x^j)}$ *for all* $\lambda_{\max}(x^j) \neq \lambda_{\min}(x^j) \in \mathbb{R}$. *In the special case that* $x^j_{2:n_j} = 0$*, we denote*

$$\psi(x^j) := (\psi(\lambda_{\max}(x^j)), 0, \cdots, 0)^T.$$

We can also define $(x^j)^p$, by the above definition where p is any number in $\mathbb{R}$ and $x^j \in K^j$. The following lemma about general functions associated with the second-order cone can be easily obtained from Definition 2.1.

Lemma 2.2. [23] *Suppose that the function* $\psi : \mathbb{R}^{n_j} \to \mathbb{R}^{n_j}$ *is defined by* Definition 2.1. *Then*

$$\|\psi(x^j)\| = \frac{\sqrt{2}}{2}\sqrt{\psi^2(\lambda_{\max}(x^j)) + \psi^2(\lambda_{\min}(x^j))},$$
$$\mathrm{Tr}\,(\psi(x^j)) = \psi(\lambda_{\max}(x^j)) + \psi(\lambda_{\min}(x^j)),$$
$$\det\,(\psi(x^j)) = \psi(\lambda_{\max}(x^j))\psi(\lambda_{\min}(x^j)) \quad \textit{for each } j.$$

To establish the complexity of the algorithm, we need to know bounds for the derivatives of certain proximity functions in suitable spaces. For SOCO, this requires us to discuss the derivative of the function $\psi(x^j(t))$ where

$$x^j(t) = (x^j_1(t), \cdots, x^j_{n_j}(t))^T$$

is a mapping from $\mathbb{R}$ into $\mathbb{R}^{n_j}$. Let us denote by $(x^j(t))'$ the derivative of $x^j(t)$ with respect to t:

$$(x^j(t))' = ((x^j_1(t))', \cdots, (x^j_{n_j}(t))')^T.$$

Now recalling Definition 2.1, we define

$$\psi'(x^j(t)) := \psi'(\lambda_{\max}(x^j(t)))\Big(\frac{1}{2}, \frac{x^j_{2:n_j}(t)}{2\|x^j_{2:n_j}(t)\|}\Big)^T + \psi'(\lambda_{\min}(x^j(t)))\Big(\frac{1}{2}, -\frac{x^j_{2:n_j}(t)}{2\|x^j_{2:n_j}(t)\|}\Big)^T.$$

Newton's method is a well-known procedure to solve a system of nonlinear equations. Most IPMs for solving SOCO employ different search directions together with suitable strategies for following the central path appropriately. Without loss of generality we assume that $(x(\mu), y(\mu), s(\mu))$ is known for some positive μ. For example, due to the above assumption we may assume this for $\mu^0 = 1$ with $(x^0,\ s^0)$. We then decrease μ to $\mu_+ := (1-\theta)\mu$ for some fixed $\theta \in (0,1)$ and linearize Newton system for (1.2) by replacing $x,\ y,\ s$ with $x_+ := x + \Delta x,\ y_+ := y + \Delta y,\ s_+ := s + \Delta s$, respectively. Finally, we get the following matrix equation:

$$\begin{pmatrix} A & 0 & 0 \\ 0 & E_n & A^T \\ \mathrm{mat}(s) & \mathrm{mat}(x) & 0 \end{pmatrix} \begin{pmatrix} \Delta x \\ \Delta s \\ \Delta y \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \mu_+\tilde{e} - \mathrm{mat}(x)s \end{pmatrix}, \quad x,\ s \succ_K 0. \tag{2.1}$$

This system might not be well defined if its Jacobian matrix is singular. To obtain a Newton-type system that has a unique solution, people usually refer to some scaling schemes. In what follows we will introduce certain variants of such scaling schemes for SOCO and SOCD, as first proposed and studied by Tsuchiya ([31, 32]). Now we are ready to give the definition of a scaling matrix for second-order cones. The scaled vector based on the NT(Nesterov-Todd) scaling scheme, which was given by Tushiya ([31, 32]), is as follows:

For any $x,\ s \succ_K 0$ and $j = 1, 2, \cdots, N$,

$$\tilde{x}_{NT} = \tilde{s}_{NT} := \begin{pmatrix} u_1 W_{NT}^1 x^1 \\ \vdots \\ u_N W_{NT}^N x^N \end{pmatrix} = \begin{pmatrix} u_1^{-1} (W_{NT}^1)^{-1} s^1 \\ \vdots \\ u_N^{-1} (W_{NT}^N)^{-1} s^N \end{pmatrix},$$

where

$$u_j := \Big(\frac{\det(x^j)}{\det(s^j)}\Big)^{-1/4},\ W_{NT}^j = \begin{pmatrix} w_1^j & (w_{2:n_j}^j)^T \\ w_{2:n_j}^j & E_{n_j-1} + \frac{1}{1+w_1^j} w_{2:n_j}^j (w_{2:n_j}^j)^T \end{pmatrix}$$

with $w^j := (w_1^j, (w_{2:n_j}^j)^T)^T := \frac{{u_j}^{-1} s^j + u_j Q^j x^j}{\sqrt{\mathrm{Tr}(x^j \circ s^j) + 2\sqrt{\det(x^j)\det(s^j)}}}$, $Q^j = \mathrm{diag}(1, -1, \cdots, -1)$ as in [25].

To describe our new search direction, we need more notations. To distinguish the NT scaling scheme from many other scaling schemes, we denote by

$\bar{A} := \frac{1}{\sqrt{\mu}} A (U_{NT} W_{NT})^{-1},\ v := \frac{1}{\sqrt{\mu}} U_{NT} W_{NT} x := \frac{1}{\sqrt{\mu}} (U_{NT} W_{NT})^{-1} s,$

$d_x := \frac{1}{\sqrt{\mu}} U_{NT} W_{NT} \Delta x, \quad d_s := \frac{1}{\sqrt{\mu}} (U_{NT} W_{NT})^{-1} \Delta s,$

where $W_{NT} := \mathrm{diag}(W_{NT}^1, \cdots, W_{NT}^N)$, $U_{NT} := \mathrm{diag}(u_1 E_{n_1}, \cdots, u_N E_{n_N})$, $u_1, \cdots, u_N > 0$. Obviously $v \succ_K 0$. The equations (2.1) imply that the NT search direction for SOCO is defined as the unique solution of the system:

$$\begin{cases} \bar{A} d_x = 0, \\ \bar{A}^T \Delta y + d_s = 0, \\ d_x + d_s = v^{-1} - v. \end{cases} \tag{2.2}$$

We can say that $d_x^T d_s = 0$, which is coming from the first and second equations of (2.2) or from the orthogonality of Δx and Δs.

For our IPM, we use the following kernel function ([2, 3, 11]):

$$\psi(t) = \frac{t^2 - 1}{2} + \frac{e^{\frac{1}{t^q} - 1} - 1}{q} \quad \text{for } t > 0,\ q \geqq 1. \tag{2.3}$$

Then, we have

$\psi'(t) = t - \frac{e^{t^{-q}-1}}{t^{q+1}},\ \psi''(t) = 1 + \frac{(q+1)t^q + q}{t^{2q+2}} e^{t^{-q}-1} > 1,$

$$\psi'''(t) = -\left(q^2 t^{-3q-3} + 3q(q+1) t^{-2q-3} + (q+1)(q+2) t^{-q-3}\right) e^{t^{-q}-1} < 0 \tag{2.4}$$

$$\text{and} \qquad t\psi''(t) - \psi'(t) = \left((q+2) t^{-q-1} + q t^{-2q-1}\right) e^{t^{-q}-1} > 0. \tag{2.5}$$

Furthermore, the kernel function (2.3) satisfies

$$\lim_{t \to 0^+} \psi(t) = \lim_{t \to \infty} \psi(t) = \infty.$$

Note that $\psi(1) = \psi'(1) = 0$. Then $\psi(t)$ is determined as follows:

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi.$$

For $v = (v^1, \cdots, v^N) \in \mathbb{R}^n$, we define:

$$\psi(v) := ((\psi(v^1))^T, \cdots, (\psi(v^N))^T)^T, \quad \Psi(v^j) := \text{Tr}(\psi(v^j)) = \psi(\lambda_{\max}(v^j)) + \psi(\lambda_{\min}(v^j)),$$

and the proximity function(measure) for SOCO and SOCD is

$$\Phi(x, s; \mu) \ := \ \Psi(v) = \sum_{j=1}^N \Psi(v^j).$$

Replacing the right hand side of last equation in (2.2) by the kernel function ψ, we have the following system from (2.2):

$$\begin{cases} \bar{A}d_x \ = \ 0, \\ \bar{A}^T \Delta y \ + \ d_s \ = \ 0, \\ d_x \ + \ d_s \ = \ -\psi'(v). \end{cases} \tag{2.6}$$

Let us denote that

$$\sigma^2 := \text{Tr}(\psi'(v) \circ \psi'(v)) = 2(\psi'(v))^T(\psi'(v)) = 2\|\psi'(v)\|^2 = 2(\|d_x\|^2 + \|d_s\|^2).$$

Thus $\sigma = \sqrt{2}\|\psi'(v)\|$, and hence by Lemma 2.2 or the definition of σ^2,

$$\sigma^2 = \ \sum_{j=1}^N \left(\psi'(\lambda_{\max}(v^j))^2 \ + \ \psi'(\lambda_{\min}(v^j))^2\right).$$

Since $\Psi(v)$ is strictly convex and minimal at $v = \tilde{e}$, we have

$$\Psi(\tilde{e}) = \sigma(\tilde{e}) = 0.$$

The following proposition gives a lower bound of σ in terms of $\Psi(v)$.

Proposition 2.3. *For any $v \in K_+$,*

$$\sigma \geqq \sqrt{2\Psi(v)}.$$

Proof. Since $\psi''(t) > 1$,

$$\psi(t) = \ \int_1^t \int_1^\xi \psi''(\zeta)d\zeta d\xi \leqq \ \int_1^t \int_1^\xi \psi''(\xi)\psi''(\zeta)d\zeta d\xi = \ \frac{1}{2}\psi'(t)^2, \ \ t > 0.$$

□

Proposition 2.4. *Let $\varrho : [0, \infty) \to [1, \infty)$ be the inverse function of $\psi(t) := s_\psi$, for $t \geqq 1$. Then*

$$\varrho(s_\psi) \leqq 1 + \sqrt{2s_\psi}.$$

Proof. Since $\psi''(t) > 1$,

$$\psi(t) = \int_1^t \int_1^\xi \psi''(\zeta) d\zeta d\xi \geqq \int_1^t \int_1^\xi d\zeta d\xi = \frac{1}{2}(t-1)^2.$$

□

Our kernel function holds the following lemma which is found in [24].

Lemma 2.5. *Let* $t_1 > 0$ *and* $t_2 > 0$. *Then*

$$\psi(t_1^r t_2^{1-r}) \leqq r\psi(t_1) + (1-r)\psi(t_2), \quad \forall r \in [0, 1].$$

3. Algorithm and its Complexity Analysis

Now we explain our algorithm for the large-update primal-dual IPM for the SOCO. Assuming that a starting point in a certain neighborhood of the central path is available, we can set out from this point. Actually, by using the so-called self-dual embedding model, one can further get the point exactly on the central path corresponding to $\mu = 1$ as an initial point ([19, 25, 33]). Then, we will go to the outer "while loop". If μ satisfies $N\mu \geqq \epsilon$, then it is reduced by the factor $1-\theta$, where $\theta \in (0,1)$. Then, we make use of inner "while loop", and we repeat the procedure until we find iterates that are "close" to $(x(\mu), y(\mu), s(\mu))$, that is, the proximity $\Phi(x, s; \mu) < \tau$. Here, we apply Newton's method targeting at the new μ-centers to decide a search direction $(\Delta x, \Delta y, \Delta s)$. We return to the outer "while loop". The whole process is repeated until μ is small enough, say until $N\mu < \epsilon$.

The choice of the step size α is another crucial issue in the analysis of the algorithm. It has to be taken such that the closeness of the iterates to the current μ-center improves by a sufficient amount. In the algorithm, the inner "while loop" is called the *inner iteration* and the outer "while loop" is called the *outer iteration.* Each outer iteration consists of an update of parameter μ and a sequence of (one or more) inner iterations. The total number of inner iterations is the worst-case iteration bound for our algorithm.

The algorithm for our large-update primal-dual IPM for the SOCO is given as follows:

Primal-Dual Algorithm for SOCO
Inputs
A proximity parameter $\tau > 1$;
an accuracy parameter $\epsilon > 0$;
a variable damping factor α;
a fixed barrier update parameter $\theta \in (0,1)$;
(x^0, s^0) and $\mu^0 = 1$ such that $\Phi(x^0, s^0; \mu^0) \leqq \tau$.
begin
$x := x^0;\ s := s^0;\ \mu := \mu^0;$
while $N\mu \geqq \epsilon$ **do**
begin
$\mu := (1-\theta)\mu;$
while $\Phi(x, s; \mu) \geqq \tau$ **do**

begin
Solve the system (2.6) for $\Delta x,\ \Delta y,\ \Delta s$;
Determine a step size α;
$x := x + \alpha\Delta x$;
$y := y + \alpha\Delta y$;
$s := s + \alpha\Delta s$;
end
end
end

3.1. **Bound of the proximity function after the μ-update.** We have $\Psi(v) \leqq \tau$ before the update of μ with the factor $1-\theta$, at the start of each outer iteration. After updating μ in an outer iteration, the vector v is divided by the factor $\sqrt{1-\theta}$, which in general leads to an increase of the value of $\Psi(v)$. Then during the inner iteration, the value of $\Psi(v)$ decreases until it passes the threshold τ. We will show in the following lemma that an upper bound for $\Psi(\frac{1}{\sqrt{1-\theta}}v)$ is expressed with $\Psi(v)$.

Lemma 3.1. *Let θ be such that $0<\theta<1$. Then, for any $v \in K_+$,*

$$\Psi(\frac{1}{\sqrt{1-\theta}}v) \leqq \frac{2(q+1)\left(\theta\sqrt{N}+\sqrt{\Psi(v)}\right)^2}{1-\theta}.$$

Proof. Our proof follows the method of Theorem 3.2 in [4]. For $\frac{1}{\sqrt{1-\theta}} > 1$, we consider the following maximization problem:

$$\max_v\{\Psi(\frac{1}{\sqrt{1-\theta}}v) : \Psi(v) = z\},$$

where z is any nonnegative number. Then there exist u such that

$$\frac{1}{\sqrt{1-\theta}}\psi'(\frac{1}{\sqrt{1-\theta}}\lambda_{\max}(v^j)) = u\psi'(\lambda_{\max}(v^j)),\ \ j=1,\cdots,N \tag{3.1}$$

and

$$\frac{1}{\sqrt{1-\theta}}\psi'(\frac{1}{\sqrt{1-\theta}}\lambda_{\min}(v^j)) = u\psi'(\lambda_{\min}(v^j)),\ \ j=1,\cdots,N. \tag{3.2}$$

Since $\frac{1}{\sqrt{1-\theta}}\psi'(\frac{1}{\sqrt{1-\theta}}) = u\psi'(1) = 0$, we do not have $\lambda_{\max}(v^j) = 1$ and $\lambda_{\min}(v^j) = 1$ for each j. Let $z^j_{\max}$ be such that $\psi(\lambda_{\max}(v^j)) = z^j_{\max}$ and let $z^j_{\min}$ be such that $\psi(\lambda_{\min}(v^j)) = z^j_{\min}$ for each j. Then these equations have two solutions for each j, respectively, which are $(\lambda_{\max}(v^j))_1 < 1 < (\lambda_{\max}(v^j))_2$, $(\lambda_{\min}(v^j))_1 < 1 < (\lambda_{\min}(v^j))_2$ for each j. So, by Lemma 3.1 of [4], we have $\psi((\lambda_{\max}(v^j))_1) \leqq \psi((\lambda_{\max}(v^j))_2)$ and $\psi((\lambda_{\min}(v^j))_1) \leqq \psi((\lambda_{\min}(v^j))_2)$ for each j. For maximizing $\Psi(\frac{1}{\sqrt{1-\theta}}v)$, we will take $(\lambda_{\max}(v^j))_2 > 1$ and $(\lambda_{\min}(v^j))_2 > 1$ for each j. Then (3.1) and (3.2) imply
$\frac{1}{\sqrt{1-\theta}}\psi'(\frac{1}{\sqrt{1-\theta}}\lambda_{\max}(v^j)) > 0,\ \ \psi'(\lambda_{\max}(v^j)) > 0$ and $u>0,\ \ j=1,\cdots,N$ and

$\frac{1}{\sqrt{1-\theta}}\psi'(\frac{1}{\sqrt{1-\theta}}\lambda_{\max}(v^j)) > 0,\ \ \psi'(\lambda_{\max}(v^j)) > 0$ and $u > 0,\ \ j = 1, \cdots, N$, respectively. We can define g as follows:

$$g(v_i) := \frac{\psi'(v_i)}{\psi'(\beta v_i)},$$

where $v_i,\ i = 1, \cdots, 2N$, is an eigenvalue of v^j greater than 1. and $\beta = \frac{1}{\sqrt{1-\theta}} > 1$. From (3.1) and (3.2), $g(v_i) = \frac{\beta}{u}$ for all i. The function g has

$$g'(v_i) = \frac{\psi''(v_i)\psi'(\beta v_i) - \beta\psi'(v_i)\psi''(\beta v_i)}{(\psi'(\beta v_i))^2}.$$

Let $f(\beta) = \psi''(v_i)\psi'(\beta v_i) - \beta\psi'(v_i)\psi''(\beta v_i)$. By (2.4) and (2.5), we can implies that

$$f'(\beta) > 0. \tag{3.3}$$

Then, by $f(1) = 0$,

$$f(\beta) > 0 \quad \text{for} \quad \beta > 1. \tag{3.4}$$

We can say that, by (3.3) and (3.4),

$$g'(v_i) > 0.$$

Thus $g(v_i)$ is strictly monotonically increasing. Hence, there exist unique v_i for each i, such that $g(v_i) = \frac{\beta}{u}$. It follows that

$$t := \lambda_{\max}(v^1) = \lambda_{\min}(v^1) = \cdots = \lambda_{\max}(v^N) = \lambda_{\min}(v^N) > 1.$$

Hence, $z = \Psi(v) = 2N\psi(t)$. This implies

$$t := \varrho(\frac{z}{2N}) = \varrho(\frac{\Psi(v)}{2N}).$$

So,

$$\Psi(\frac{1}{\sqrt{1-\theta}}v) \leqq 2N\psi(\frac{1}{\sqrt{1-\theta}}\varrho(\frac{\Psi(v)}{2N})),$$

where $\frac{\Psi(v)}{2N}) > 1,\ \ \frac{1}{\sqrt{1-\theta}} > 1$. Since $\psi(t) \leqq \frac{1}{2}\psi''(1)(t-1)^2, \quad t \geqq 1$ (Lemma 2.6 in [4]),

$$\Psi(\frac{1}{\sqrt{1-\theta}}v) \leqq 2N\frac{1}{2}\psi''(1)\left(\frac{1}{\sqrt{1-\theta}}\varrho(\frac{\Psi(v)}{2N}) - 1\right)^2.$$

Since $\psi''(1) = 2q + 2$, it follows from Proposition 2.4 that

$$\Psi(\frac{v}{\sqrt{1-\theta}}) \leqq 2N(q+1)\left(\frac{1}{\sqrt{1-\theta}}\left(1 + \sqrt{\frac{\Psi(v)}{N}}\ \right) - 1\right)^2 \leqq \frac{2(q+1)\left(\theta\sqrt{N} + \sqrt{\Psi(v)}\right)^2}{1-\theta},$$

where the last inequality follows from $1 - \sqrt{1-\theta} \leqq \theta$. □

By the assumption $\Psi(v) \leqq \tau$ just before the update of μ,

$$\Psi(\frac{v}{\sqrt{1-\theta}}) \leqq \frac{2(q+1)\big(\theta\sqrt{N}+\sqrt{\tau}\big)^2}{1-\theta}.$$

We define

$$L(N,\ \theta,\ \tau) = \frac{2(q+1)\big(\theta\sqrt{N}+\sqrt{\tau}\big)^2}{1-\theta}.$$

Since $\tau = \mathcal{O}(N)$ and $\theta = \Theta(1)$,

$$L = \mathcal{O}(N).$$

3.2. **Determining a default step size.** In this section, we compute the feasible step size α such that the proximity function is decreasing and is bound for the decrease during inner iterations; then give our default step size $\bar{\alpha}$; $\bar{\alpha} = \big(1 + 3\sigma(2q+1)(1+\log 3\sigma)^{(q+1)/q}\big)^{-1}$. We will show that the step size not only to keeps the iterates feasible but also to gives rise to a sufficiently large decrease of the barrier function $\Psi(v)$ in each inner iteration. Let us denote the difference between the proximity before and after one step by a function of the step size, that is,

$$g(\alpha) := \Psi(v_+) - \Psi(v).$$

The main task in the rest of this section is to study the decreasing behavior of $g(\alpha)$. Since v_+ is the scaled vector resulting from the NT scaling, which is the scheme to transform the primal and dual vectors, $x + \alpha\Delta x$ and $s + \alpha\Delta s$, to the same vector, from Proposition 6.3.3 in [25], we can get the following: for all $j = 1, \cdots, N$,

$$\det((v_+^j)^2) = \det(v^j + \alpha d_x^j)\det(v^j + \alpha d_s^j) \ \mathrm{Tr}((v_+^j)^2) = \mathrm{Tr}((v^j + \alpha d_x^j) \circ (v^j + \alpha d_s^j)).$$

Using the above two equalities and following the proof techniques in Proposition 6.2.9 in [25], we can prove that there exist $\gamma_1,\ \gamma_2 \in (0,1),\ \gamma_1 + \gamma_2 = 1$ such that

$$\begin{aligned}
&\lambda_{\min}(v_+^j)\\
&= {\lambda_{\min}}^{\gamma_1/2}(v^j + \alpha d_x^j){\lambda_{\min}}^{\gamma_1/2}(v^j + \alpha d_s^j){\lambda_{\max}}^{\gamma_2/2}(v^j + \alpha d_x^j){\lambda_{\max}}^{\gamma_2/2}(v^j + \alpha d_s^j),\\
&\lambda_{\max}(v_+^j)\\
&= {\lambda_{\min}}^{\gamma_2/2}(v^j + \alpha d_x^j){\lambda_{\min}}^{\gamma_2/2}(v^j + \alpha d_s^j){\lambda_{\max}}^{\gamma_1/2}(v^j + \alpha d_x^j){\lambda_{\max}}^{\gamma_1/2}(v^j + \alpha d_s^j).
\end{aligned}$$

Thus, since ${\lambda_{\min}}^{1/2}(v^j + \alpha d_x^j){\lambda_{\min}}^{1/2}(v^j + \alpha d_s^j) > 0$, $\lambda_{\min}(v^j + \alpha d_x^j) > 0$ and $\lambda_{\min}(v^j + \alpha d_s^j) > 0$, from Lemma 2.5, we can induce the following:

$$\Psi(v_+) \leqq \frac{1}{2}(\Psi(v + \alpha d_x) + \Psi(v + \alpha d_s)).$$

So, we have

$$g(\alpha) \ \leqq \ g_1(\alpha) \ := \ \frac{1}{2}(\Psi(v + \alpha d_x) \ + \ \Psi(v + \alpha d_s)) \ - \ \Psi(v).$$

Taking the derivative to α, we get

$$g_1{}'(\alpha) \ = \ \frac{1}{2}\mathrm{Tr}(\psi'(v + \alpha d_x) \circ d_x + \ \psi'(v + \alpha d_s) \circ d_s).$$

This gives $g_1{}'(0) = -\frac{\sigma^2}{2}$ by using the equality $d_x + d_s = -\psi'(v)$.

The next result presents an upper bound for the second derivative of $g_1(\alpha)$, which is usable for establishing the polynomial complexity of the algorithm which is different from the result of Lemma 6.4.3 in [25] for self-regular proximity functions. To facilitate the forthcoming analysis, we also define

$$\lambda_{\min}(v) \;:=\; \min\{\lambda_{\min}(v^j) \;:\; j = 1, \cdots, N\}.$$

Proposition 3.2. *Suppose that the kernel function is defined by* (2.3) . *Then*

$$g_1{}''(\alpha) \leqq \frac{\sigma^2}{2}\psi''(\lambda_{\min}(v) - \alpha\sigma). \tag{3.5}$$

Proof. Following proofs in the Lemma 6.4.3 and Lemma 6.2.10 in [25], we can prove that

$$g_1''(\alpha) = \frac{d}{d\alpha}\left[\frac{1}{2}\mathrm{Tr}\left(\psi'(v+\alpha d_x)\circ d_x + \psi'(v+\alpha d_s)\circ d_s\right)\right].$$

Since $\psi'(v^j + \alpha d_x^j) = \psi'(\lambda_{\max}(v^j + \alpha d_x^j))\left(\frac{1}{2}, \frac{(v^j+\alpha d_x^j)_{2:n_j}}{2||(v^j+\alpha d_x^j)_{2:n_j}||}\right) + \psi'(\lambda_{\min}(v^j + \alpha d_x^j))$ $\left(\frac{1}{2}, -\frac{(v^j+\alpha d_x^j)_{2:n_j}}{2||(v^j+\alpha d_x^j)_{2:n_j}||}\right)$ with $\lambda_{\max}(v^j+\alpha d_x^j) = (v^j+\alpha d_x^j)_1 + ||(v^j+\alpha d_x^j)_{2:n_j}||$ and $\lambda_{\min}(v^j+\alpha d_x^j) = (v^j+\alpha d_x^j)_1 - ||(v^j+\alpha d_x^j)_{2:n_j}||$, we can calculate that, by Cauchy-Schwarz inequality,

$$\begin{aligned}
&\frac{d}{d\alpha}\psi'(v^j+\alpha d_x^j)^T d_x^j \\
=\;& \frac{1}{2}\psi''(\lambda_{\max}(v^j+\alpha d_x^j))\left((d_x^j)_1 + \frac{\left((v^j+\alpha d_x^j)_{2:n_j}{}^T(d_x^j)_{2:n_j}\right)}{||(v^j+\alpha d_x^j)_{2:n_j}||}\right)^2 \\
&+\frac{1}{2}\psi''(\lambda_{\min}(v^j+\alpha d_x^j))\left((d_x^j)_1 - \frac{\left((v^j+\alpha d_x^j)_{2:n_j}{}^T(d_x^j)_{2:n_j}\right)}{||(v^j+\alpha d_x^j)_{2:n_j}||}\right)^2 \\
&+\frac{\psi'(\lambda_{\max}(v^j+\alpha d_x^j)) - \psi'(\lambda_{\min}(v^j+\alpha d_x^j))}{2||(v^j+\alpha d_x^j)_{2:n_j}||} \\
&\qquad\left(||(d_x^j)_{2:n_j}||^2 - \frac{\left((v^j+\alpha d_x^j)_{2:n_j}{}^T(d_x^j)_{2:n_j}\right)^2}{||(v^j+\alpha d_x^j)_{2:n_j}||^2}\right) \\
\leqq\;& \varpi_1||(d_x^j)||^2,
\end{aligned}$$

where $\varpi_1 := \max_{j\in J}\{\psi''(\lambda_{\min}(v^j+\alpha d_x^j)), \psi''(\lambda_{\max}(v^j+\alpha d_x^j)), \frac{\psi'(\lambda_{\max}(v^j+\alpha d_x^j)) - \psi'(\lambda_{\min}(v^j+\alpha d_x^j))}{2||(v^j+\alpha d_x^j)_{2:n_j}||}\}$.
Similarly,

$$\frac{d}{d\alpha}\psi'(v^j+\alpha d_s^j)^T d_s^j \leqq \varpi_2||(d_s^j)||^2,$$

where $\varpi_2 := \max_{j\in J}\{\psi''(\lambda_{\min}(v^j+\alpha d_s^j)), \psi''(\lambda_{\max}(v^j+\alpha d_s^j)), \frac{\psi'(\lambda_{\max}(v^j+\alpha d_s^j))-\psi'(\lambda_{\min}(v^j+\alpha d_s^j))}{2||(v^j+\alpha d_s^j)_{2:n_j}||}\}$.
These imply that

$$g_1''(\alpha) \leqq \sum_{j=1}^{N}(\varpi_1\|d_x^j\|^2 + \varpi_2\|d_s^j\|^2).$$

By the Mean value Theorem, there are $\zeta_x^j \in [\lambda_{\max}(v^j+\alpha d_x^j), \lambda_{\min}(v^j+\alpha d_x^j)]$ and $\zeta_s^j \in [\lambda_{\max}(v^j+\alpha d_s^j),\ \lambda_{\min}(v^j+\alpha d_s^j)]$ satisfing $\psi''(\zeta_x^j) = \frac{\psi'(\lambda_{\max}(v^j+\alpha d_x^j))-\psi'(\lambda_{\min}(v^j+\alpha d_x^j))}{2||(v^j+\alpha d_x^j)_{2:n_j}||}$ and $\psi''(\zeta_s^j) = \frac{\psi'(\lambda_{\max}(v^j+\alpha d_s^j))-\psi'(\lambda_{\min}(v^j+\alpha d_s^j))}{2||(v^j+\alpha d_s^j)_{2:n_j}||}$, respectively. Since $\psi''(\cdot)$ is decreasing, we find a lower bound for $\lambda_{\min}(v^j+\alpha d_x^j)$ and $\lambda_{\min}(v^j+\alpha d_s^j)$ so that we can get a upper bound for $\sum_{j=1}^{N}(\varpi_1\|d_x^j\|^2 + \varpi_2\|d_s^j\|^2)$. For any fixed j,

$$\lambda_{\min}(v^j+\alpha d_x^j) \ \geqq\ \lambda_{\min}(v) - \alpha\sigma \ \text{ and }\ \lambda_{\min}(\mathrm{v}^\mathrm{j}+\alpha \mathrm{d}_\mathrm{s}^\mathrm{j}) \ \geqq\ \lambda_{\min}(\mathrm{v}) - \alpha\sigma.$$

We can claim that the right-hand side of above two inequalities make a maximum value of $\psi''(\cdot)$. Therefore,

$$g_1''(\alpha) \ \leqq\ \sum_{j=1}^{N}\psi''(\lambda_{\min}(v)-\alpha\sigma)(\|d_x^j\|^2+\|d_s^j\|^2) \ =\ \frac{\sigma^2}{2}\psi''(\lambda_{\min}(v)-\alpha\sigma).$$

□

Since $g_1(0) = 0$ and $g_1'(0) = -\frac{\sigma^2}{2}$, by (3.5),

$$\begin{aligned} g(\alpha) \leqq g_1(\alpha) &:= \ g_1(0) + {g_1}'(0)\alpha + \int_0^\alpha\int_0^\xi g_1''(\zeta)d\zeta d\xi \\ \leqq g_2(\alpha) &:= \ g_1(0) + {g_1}'(0)\alpha + \int_0^\alpha\int_0^\xi \frac{\sigma^2}{2}\psi''(\lambda_{\min}(v)-\zeta\sigma)d\zeta d\xi. \end{aligned}$$

Note that $g_2(0) = 0$. Furthermore, since $g_2'(0) = g_1'(0) = -\frac{\sigma^2}{2}$, $g_2'(\alpha) = -\frac{\sigma^2}{2}+\frac{\sigma}{2}\big(\psi'(\lambda_{\min}(v)) -\psi'(\lambda_{\min}(v)-\alpha\sigma)\big)$ and $g_2''(\alpha) = \frac{\sigma^2}{2}\psi''(\lambda_{\min}(v)-\alpha\sigma)$ which is increasing on $\alpha \in [0, \frac{\lambda_{\min}(v)}{\sigma})$. Using $g_1''(\alpha) \leqq g_2''(\alpha)$, we can easily check that

$$g_1'(\alpha) = g_1'(0) + \int_0^\alpha g_1''(\xi)d\xi \leqq g_2'(\alpha).$$

This relation gives that

$$g_1'(\alpha) \leqq 0, \ \text{ if } g_2'(\alpha) \leqq 0.$$

To compute the feasible step size α such that the proximity measure is decreasing when we take a new iterate for fixed μ, we want to calculate the step size α which satisfies that $g_2'(\alpha) \leqq 0$ holds with α as large as possible. Since $g_2''(\alpha) > 0$, that is, $g_2'(\alpha)$ is monotonically increasing at α, the largest possible value at α satisfying $g_2'(\alpha) \leqq 0$ occurs when $g_2'(\alpha) = 0$, that is,

$$-\psi'(\lambda_{\min}(v)-\alpha\sigma) \ +\ \psi'(\lambda_{\min}(v)) = \sigma. \tag{3.6}$$

Since $\psi''(t)$ is monotonically decreasing, the derivative of the left hand-side in (3.6) with respect to $\lambda_{\min}(v)$ is

$$-\psi''(\lambda_{\min}(v) - \alpha\sigma) \; + \; \psi''(\lambda_{\min}(v)) < 0.$$

So, the left-hand side in (3.6) is decreasing at $\lambda_{\min}(v)$. This implies that if $\lambda_{\min}(v)$ gets smaller, then α gets smaller with fixed σ. Note that

$$\sigma = \sqrt{\sum_{j=1}^{N} \left((\psi'(\lambda_{\max}(v^j)))^2 + (\psi'(\lambda_{\min}(v^j)))^2\right)} \geqq |\psi'(\lambda_{\min}(v))| \geqq -\psi'(\lambda_{\min}(v)).$$

Hence, the worse situation for the largest step size occurs when $\lambda_{\min}(v*)$ satisfies

$$-\psi'(\lambda_{\min}(v*)) = \sigma. \tag{3.7}$$

So, we can find out v^* such that $v^* = ((v^*)^1, (v^*)^2, \cdots, (v^*)^N)^T = ((\tilde{e}^1)^T, (\tilde{e}^2)^T, \cdots, 1 \cdot (\frac{1}{2}, \frac{(v^*)^j_{2:n_j}}{2\|(v^*)^j_{2:n_j}\|})^T + \lambda_{\min}(v^*)(\frac{1}{2}, -\frac{(v^*)^j_{2:n_j}}{2\|(v^*)^j_{2:n_j}\|})^T, \cdots, (\tilde{e}^N)^T)^T, 0 < \lambda_{\min}(v*) \leqq 1.$

In that case, the largest α (i.e., $\alpha*$) satisfying (3.6) is minimal. For our purpose, we need to deal with the worse case and so we assume that (3.7) holds.

Let $\rho : [0, \infty) \to (0, 1]$ denote the inverse function of the restriction of $-\psi'(t)$ in the interval $(0, 1]$. Then (3.7) implies

$$\lambda_{\min}(v*) = \rho(\sigma). \tag{3.8}$$

By using (3.6) and (3.7), we immediately obtain

$$-\psi'(\lambda_{\min}(v^*) - \alpha\sigma) \; = \; 2\sigma.$$

By the definition of ρ and (3.8), the largest step size α of the worse case is given as follows:

$$\alpha^* = \frac{\rho(\sigma) - \rho(2\sigma)}{\sigma}. \tag{3.9}$$

For the purpose of finding an upper bound of $g(\alpha)$, we need a default step size $\bar{\alpha}$ that is the lower bound of the α^* and consists of σ.

Lemma 3.3. *Let $\rho : [0, \infty) \to (0, 1]$ be the inverse function of the restriction of $-\psi'(t)$ in the interval $(0, 1]$ and α^* be as defined in* (3.9)*. Then*

$$\alpha^* \geqq \frac{1}{1 + 3\sigma(2q+1)(1 + \log 3\sigma)^{\frac{q+1}{q}}}.$$

Proof. Since $-\psi'(\rho(\sigma)) = \sigma$, taking the derivative of σ at both sides, we get

$$\rho'(\sigma) = -\frac{1}{\psi''(\rho(\sigma))}.$$

Moreover, we have,

$$\alpha^* = \frac{1}{\sigma} \int_{2\sigma}^{\sigma} \rho'(\xi) d\xi \geqq \frac{1}{\sigma} \Big[\frac{\xi}{\psi''(\rho(2\sigma))}\Big]_{\sigma}^{2\sigma} = \frac{1}{\psi''(\rho(2\sigma))},$$

where the inequality follows from $\sigma \leqq \xi \leqq 2\sigma$ and ρ and ψ'' are monotonically decreasing.From $\psi'(t) = t - e^{t^{-q}-1} \cdot t^{-q-1}$, let $-\psi_b'(t) = e^{t^{-q}-1} \cdot t^{-q-1}$and let $\underline{\rho} : [1, \infty) \to (0, 1]$ denote the inverse function of the restriction of $-\psi_b'(t)$ to the interval $(0, 1]$.
Let $\rho(2\sigma) = \tilde{t}$. Then $0 < \tilde{t} \leqq 1$ and $2\sigma = -\psi'(\tilde{t}) = -\tilde{t} - \psi_b'(\tilde{t})$. So, $-\psi_b'(\tilde{t}) = \tilde{t} + 2\sigma \leqq 1 + 2\sigma$. Since $\underline{\rho}$ is decreasing, $\underline{\rho}(-\psi_b'(\tilde{t})) \geqq \underline{\rho}(1 + 2\sigma)$ and hence we have,

$$\rho(2\sigma) \geqq \underline{\rho}(1 + 2\sigma). \tag{3.10}$$

Let $\underline{\rho}(1+2\sigma) = \hat{t}$. Then $1+2\sigma = -\psi_b'(\hat{t}) = e^{(\underline{\rho}(1+2\sigma))^{-q}-1} \cdot (\underline{\rho}(1+2\sigma))^{-q-1}$, $e^{(\underline{\rho}(1+2\sigma))^{-q}-1} = (1 + 2\sigma)(\underline{\rho}(1 + 2\sigma))^{q+1} \leqq 1 + 2\sigma \leqq 3\sigma$. So, $(\underline{\rho}(1 + 2\sigma))^{-q} - 1 \leqq \log 3\sigma$ and hence we have

$$\underline{\rho}(1 + 2\sigma) \geqq (1 + \log 3\sigma)^{-\frac{1}{q}}. \tag{3.11}$$

From (3.10), we have

$$\tilde{\alpha} := \frac{1}{\psi''(\rho(2\sigma))} \geqq \frac{1}{\psi''(\hat{t})}.$$

From (3.11), we have

$$\begin{aligned}
\psi''(\hat{t}) &= 1 + ((q + 1)\hat{t}^q + q)e^{\hat{t}^{-q}-1} \cdot \hat{t}^{-q-1} \cdot \hat{t}^{-q-1} \\
&= 1 + ((q + 1)\hat{t}^q + q)(\psi_b'(\hat{t}))\hat{t}^{-q-1} \\
&= 1 + ((q + 1)\hat{t}^q + q)(1 + 2\sigma)\hat{t}^{-q-1} \\
&\leqq 1 + ((q + 1) \cdot 1 + q)(1 + 2\sigma)(\underline{\rho}(1 + 2\sigma))^{-q-1} \\
&\leqq 1 + 3\sigma(2q + 1)(1 + \log 3\sigma)^{\frac{q+1}{q}}.
\end{aligned}$$

Therefore,

$$\alpha^* \geqq \frac{1}{\psi''(\hat{t})} \geqq \frac{1}{1 + 3\sigma(2q + 1)(1 + \log 3\sigma)^{\frac{q+1}{q}}}.$$

□

Define

$$\bar{\alpha} = \frac{1}{1 + 3\sigma(2q + 1)(1 + \log 3\sigma)^{\frac{q+1}{q}}}. \tag{3.12}$$

We will use $\bar{\alpha}$ as the default step size in the algorithm.

3.3. **Decrease of the proximity function during an inner iteration.** Now we show that our proximity function Ψ with our default step size $\bar{\alpha}$ decreasing. It can be easily established by using the following result:

Lemma 3.4. [24] *Let $h(t)$ be a twice differentiable convex function with $h(0) = 0$, $h'(0) < 0$ and let $h(t)$ attain its (global) minimum at $t^* > 0$. If $h''(t)$ is increasing for $t \in [0, t^*]$, then*

$$h(t) \leqq \frac{th'(0)}{2}, \quad 0 \leqq t \leqq t^*.$$

Since $g_2(\alpha)$ satisfies the conditions of the above Lemma,

$$g(\alpha) \leqq g_1(\alpha) \leqq g_2(\alpha) \leqq \frac{g_2'(0)}{2}\alpha \quad \text{for all } \; 0 \leqq \alpha \leqq \; \alpha^*.$$

Since $g_2'(0) = -\frac{\sigma^2}{2}$, we can obtain the upper bound for the decreasing value of the proximity in the inner iteration by the lemma;

Theorem 3.5. *Let $\bar{\alpha}$ be a step size as defined in* (3.12) *and $\sigma \geqq 1$. Then we have*

$$g(\bar{\alpha}) \leqq -\frac{\Psi^{\frac{1}{2}}}{2 + 6\sqrt{2}(2q+1)\left(1 + \log 3\sqrt{2}\sqrt{\Psi_0}\right)^{\frac{q+1}{q}}}. \tag{3.13}$$

Proof. Since $g_1'(0) = g_2'(0) = -\frac{\sigma^2}{2}$ and $\bar{\alpha} \in [0, \alpha^*]$, we have

$$g(\bar{\alpha}) \leqq -\frac{\sigma^2}{4}\bar{\alpha} \leqq -\frac{1}{4} \cdot \frac{\sigma^2}{1 + 3\sigma(2q+1)(1 + \log 3\sigma)^{\frac{q+1}{q}}}.$$

This expresses the decease in one inner iteration in terms of σ. Since the decrease depends monotonically on σ, we can express the decrease in terms of $\Psi = \Psi(v)$ by Proposition 2.3 as follows:

$$\begin{aligned} g(\bar{\alpha}) &\leqq -\frac{1}{4} \cdot \frac{\frac{\Psi}{2}}{1 + 3\sqrt{2\Psi}(2q+1)\left(1 + \log 3\sqrt{2\Psi}\right)^{\frac{q+1}{q}}} \\ &\leqq -\frac{1}{2} \cdot \frac{\Psi^{\frac{1}{2}}}{1 + 3\sqrt{2}(2q+1)\left(1 + \log 3\sqrt{2}\sqrt{\Psi_0}\right)^{\frac{q+1}{q}}}, \end{aligned}$$

where the second inequality follows from $\Psi_0 \geqq \Psi \geqq \tau \geqq 1$. This result holds the theorem. $\square$

3.4. **Iteration bound.** We need to count how many inner iterations are required to return to the situation where $\Psi(v) \leqq \tau$ after a μ-update. We denote the value of $\Psi(v)$ after μ-update as Ψ_0; the subsequent values in the same outer iteration are denoted as $\Psi_k, \;\; k = 1, \cdots$. If K denotes the total number of inner iterations in the outer iteration, we then have

$$\Psi_0 \;\leqq L = \mathcal{O}(N), \;\; \Psi_{K-1} > \tau, \;\; 0 \leqq \Psi_K \leqq \tau$$

and according to (3.13),

$$\Psi_{k+1} \leqq \Psi_k - \frac{1}{2 + 6\sqrt{2}(2q+1)\left(1 + \log 3\sqrt{2}\sqrt{\Psi_0}\right)^{\frac{q+1}{q}}}\Psi_k^{\frac{1}{2}}.$$

At this stage we invoke the following lemma from Lemma 14 in [24] without proof.

Lemma 3.6. [24] *Let $t_0, \; t_1, \cdots, t_K$ be a sequence of positive numbers such that*

$$t_{k+1} \leqq t_k - \beta t_k^{1-\gamma}, \;\; k = 0, 1, \cdots, K-1,$$

where $\beta > 0$ *and* $0 < \gamma \leqq 1$. *Then*

$$K \leqq \frac{t_0^\gamma}{\beta\gamma}.$$

Letting $t_k = \Psi_k,\ \ \beta = \frac{1}{2+6\sqrt{2}(2q+1)\left(1+\log 3\sqrt{2}\sqrt{\Psi_0}\right)^{\frac{q+1}{q}}}$ and $\gamma = \frac{1}{2}$, we can get the following lemma from Lemma 3.6.

Lemma 3.7. *Let* K *be the total number of inner iterations in the outer iteration. Then we have*

$$K \leqq 2\left(2+6\sqrt{2}(2q+1)\left(1+\log 3\sqrt{2}\sqrt{\Psi_0}\right)^{\frac{q+1}{q}}\right)\Psi_0^{1/2},$$

where Ψ_0 *is the value of* $\Psi(v)$ *after the* $\mu-$*update in outer iteration.*

Now we estimate the total number of iterations of our algorithm.

Theorem 3.8. *If* $\tau \geqq 1$, *the total number of iterations is not more than*

$$\left\lceil 2\left(2+6\sqrt{2}(2q+1)\left(1+\log 3\sqrt{2}\sqrt{\Psi_0}\right)^{\frac{q+1}{q}}\right)\Psi_0^{1/2}\right\rceil \left\lceil \frac{1}{\theta}\log\frac{N}{\epsilon}\right\rceil.$$

Proof. In the algorithm, $N\mu \geqq \epsilon,\ \mu_k := (1-\theta)^k\mu_0$ and $\mu_0 = 1$. By simple computation, we have,

$$k \leqq \frac{1}{\theta}\log\frac{N}{\epsilon}.$$

Therefore, the number of outer iterations is bounded above by

$$\frac{1}{\theta}\log\frac{N}{\epsilon}.$$

Multiplication of this result by the number in the above lemma holds the theorem. □

Since $\Psi_0^{1/2} = \mathcal{O}(\sqrt{N})$, the upper bound for the total number of inner iterations in the outer iteration is

$$\mathcal{O}(q\sqrt{N}(\log N)^{\frac{q+1}{q}}).$$

Also, we take for θ a constant (not depending on N), namely $\frac{1}{\theta} = \Theta(1)$. With $\tau = \mathcal{O}(N)$, the new complexity of the primal-dual interior-point method for second-order cone optimization problem based on a new proximity function is given by

$$\mathcal{O}\left(q\sqrt{N}(\log N)^{\frac{q+1}{q}}\log\frac{N}{\epsilon}\right).$$

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