J. KSIAM Vol.14, No.2, 79–91, 2010

FRACTIONAL NONLOCAL INTEGRODIFFERENTIAL EQUATIONS AND ITS OPTIMAL CONTROL IN BANACH SPACES

JINRONG WANG[1†], W. WEI[1], AND Y. YANG[2]

[1]COLLEGE OF SCIENCE, GUIZHOU UNIVERSITY, PR CHINA
E-mail address: wjr9668@126.com, wei66@yahoo.com

[2]COLLEGE OF TECHNOLOGY, GUIZHOU UNIVERSITY, PR CHINA
E-mail address: yylong1980@163.com

ABSTRACT. In this paper, a class of fractional integrodifferential equations of mixed type with nonlocal conditions is considered. First, using contraction mapping principle and Krasnoselskii's fixed point theorem via Gronwall's inequailty, the existence and uniqueness of mild solution are given. Second, the existence of optimal pairs of systems governed by fractional integrodifferential equations of mixed type with nonlocal conditions is also presented.

1. INTRODUCTION

The fractional differential equations has recently been proved to be valuable tools in the modeling of many phenomena in various fields of engineering, physics, economy and science. We can find numerous applications in viscoelasticity, electrochemistry, control, porous media, electromagnetic, etc. [11, 13, 14, 15, 24, 25]. In recent years, there has been a significant development in fractional differential equations. One can see the monographs of Kilbas et al. [18], Miller and Ross [23], Podlubny [28], Lakshmikantham et al. [20], and the papers on abstract fractional differential equations [7, 8, 9, 10, 12, 17, 21, 22, 26, 27] and the references therein. On the other hand, the study of initial value problems with nonlocal conditions arises to deal specially with some situations in physics. For the comments and motivations of nonlocal Cauchy problem in different fields, we refer the reader to [1, 6] and the references contained therein.

Very recently, the fractional differential equations with nonlocal conditions on infinite dimensional spaces attracted some authors such as Benchohra, Mophou, N'Guérékata, Sakthivel

Received by the editors December 16 2009; Accepted June 2 2010.

2000 *Mathematics Subject Classification.* 34K05, 34A12, 34A40, 93C25.

Key words and phrases. Fractional integrodifferential equations of mixed type, nonlocal conditions, Krasnoselskii's fixed point theorem, existence, optimal control.

† Corresponding author.

JinRong Wang acknowledge support from the National Natural Science Foundation of Guizhou Province (2010, No.2142) and Introducing Talents Foundation for the Doctor of Guizhou University (2009, No.031). W. Wei acknowledge support from the National Natural Science Foundation of China (No.10961009). YanLong Yang acknowledge support from the Youth Teachers Natural Science Foundation of Guizhou University (2009, No.083).

and etc. However, to our knowledge, the fractional integrodifferential equations of mixed type with nonlocal initial conditions and its optimal control on infinite dimensional Banach spaces are not studied extensively. Motivated by the works [27, 29], the main purpose of this paper is to consider the following more general fractional nonlocal integrodifferential equations of mixed type and its optimal control

$$D^q x(t) = Ax(t) + t^n f\left(t, x(t), (Kx)(t), (Hx)(t)\right),\ t \in J,\ n \in Z^+,\ q \in (0,1), \tag{1.1}$$

$$x(0) = g(x) + x_0 \tag{1.2}$$

where $J = [0, T]$ in a general Banach space $(X, \|\cdot\|)$, where the operator A is the infinitesimal generator of a C_0-semigroup $\{T(t), t \geq 0\}$ on X, $x_0 \in X$, f: $J \times X \times X \times X \to X$ is a nonlinear function, and $g : C(J, X) \to X$ constitutes a nonlocal Cauchy problem. The derivative D^q is understood here in the Riemann-Liouville sense. Operators K and H are nonlinear integral operators given by

$$(Kx)(t) = \int_0^t k(t, s, x(s))ds, \quad (Hx)(t) = \int_0^T h(t, s, x(s))ds.$$

The rest of this paper is organized as follows. In Section 2, we give some preliminaries and introduce the mild solution of system (1.1)-(1.2). In Section 3, we study the existence and uniqueness of mild solutions for system (1.1)-(1.2) using Banach contraction principle and Krasnoselskii's fixed point theorem visa Gronwall's inequality. At last, we introduce a class of admissible controls and an existence result of optimal controls for a Lagrange problem (P) is proved.

2. Preliminaries

Let $L_b(X)$ be the Banach space of all linear and bounded operators on X. $C(J, X)$ be the Banach space of all X-valued continuous functions from J into X endowed with the norm $\|x\|_C = \sup_{t \in J} \|x(t)\|$.

Let us recall the following known definitions. For more details see [28].

Definition 2.1. A real function $f(t)$ is said to be in the space C_α, $\alpha \in R$ if there exists a real number $\kappa > \alpha$, such that $f(t) = t^\kappa g(t)$, where $g \in C[0, \infty)$ and it is said to be in the space C_α^m iff $f^{(m)} \in C_\alpha$, $m \in N$.

Definition 2.2. The Riemann-Liouville fractional integral operator of order $\gamma \geq 0$ of a function $f \in C_\alpha$, $\alpha \geq -1$ is defined as

$$I^\gamma f(t) = \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} f(s) ds$$

where $\Gamma(\cdot)$ is the Euler gamma function.

Definition 2.3. If the function $f \in C_{-1}^m$, $m \in N$, the fractional derivative of order $\gamma > 0$ of a function $f(t)$ is in the Caputo sense is given by

$$\frac{d^\gamma f(t)}{dt^\gamma} = \frac{1}{\Gamma(m-\gamma)} \int_0^t (t-s)^{m-\gamma-1} f^m(s) ds,\ m-1 < \gamma \leq m.$$

Now, we can introduce the mild solution of system (1.1)-(1.2).

Definition 2.4. A mild solution of system (1.1)-(1.2) to be a function in $C(J, X)$ such that

$$x(t) = T(t)(x_0 + g(x)) + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) f\left(s, x(s), (Kx)(s), (Hx)(s)\right) ds.$$

3. MAIN RESULTS

In this section, we give the existence and uniqueness of the mild solutions for system (1.1)-(1.2).

We need the following assumptions.

[HA]: A is the infinitesimal generator of a strongly continuous semigroup $\{T(t), t \geq 0\}$ on X with domain $D(A)$.

[Hf]: $f : J \times X \times X \times X \to X$ is continuous and there exists a function $m_1, m_2, m_3 \in L^1_{Loc}(J, R^+)$ such that

$$\|f(t, x_1, x_2, x_3) - f(t, y_1, y_2, y_3)\| \leq m_1(t)\|x_1 - y_1\| + m_2(t)\|x_2 - y_2\| + m_3(t)\|x_3 - y_3\|$$

for all $x_i, y_i \in X$, $i = 1, 2, 3$ and $t \in J$.

[Hk]: Let $D_k = \{(t, s) \in R^2; 0 \leq s \leq t \leq T\}$. The function $k : D_k \times X \to X$ is continuous and there exists a $m_k(t, s) \in C(D_k, R^+)$ and

$$K^* = \sup_{t \in J} \int_0^t m_k(t, s) ds < \infty$$

such that

$$\|k(t, s, x) - k(t, s, y)\| \leq m_k(t, s)\|x - y\|$$

for each $(t, s) \in D_k$ and $x, y \in X$.

[Hh]: Let $D_h = \{(t, s) \in R^2; 0 \leq s, t \leq T\}$. The function $h : D_h \times X \to X$ is continuous and there exists a $m_h(t, s) \in C(D_h, R^+)$ and

$$H^* = \sup_{t \in J} \int_0^T m_h(t, s) ds < \infty$$

such that

$$\|h(t, s, x) - h(t, s, y)\| \leq m_h(t, s)\|x - y\|$$

for each $(t, s) \in D_h$ and $x, y \in X$.

[Hg]: $g : C(J, X) \to X$ is a continuous and there exists a constant $l_g > 0$ such that

$$\|g(x) - g(y)\| \leq l_g \|x - y\|_C,$$

for arbitrary $x, y \in C(J, X)$, where $\|\cdot\|_C$ denotes $\|\cdot\|_{C(J,X)}$.

[HΩ_n]: The function $\Omega_n : J \to R^+$, $n \in Z^+$, defined by

$$\Omega_n = M\left[l_g + \frac{t^n T^q}{(n+1)\Gamma(q)} \left(\|m_1\|_{L^1_{Loc}(J,R^+)} + K^*\|m_2\|_{L^1_{Loc}(J,R^+)} + H^*\|m_3\|_{L^1_{Loc}(J,R^+)}\right)\right],$$

satisfies $0 < \Omega_n \leq \omega < 1$, for all $t \in J$.

Now we are ready to give our first result which is based on the Banach contraction mapping principle.

Theorem 3.1. *Assume that the conditions [HA], [Hf], [Hk], [Hh], [Hg] and [HΩ_n] are satisfied. Then system (1.1)-(1.2) has a unique mild solution.*

Proof. We consider the operator $\Gamma : C(J,X) \to C(J,X)$ defined by

$$\begin{aligned}(\Gamma x)(t) &= T(t)[x_0 + g(x)] \\ &+ \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) f\left(s, x(s), (Kx)(s), (Hx)(s)\right) ds, \quad (3.1)\end{aligned}$$

for all $t \in J$. Note that Γ is well defined on $C(J,X)$.

Now, take $t \in J$ and $x, y \in C(J,X)$. Then we have

$$\begin{aligned}\|\Gamma x(t) - \Gamma y(t)\| \leq\; & \|T(t)(g(x) - g(y))\| \\ &+ \frac{1}{\Gamma(q)} \left\| \int_0^t (t-s)^{q-1} s^n T(t-s)[f(s, x(s), (Kx)(t), (Hx)(t)) \right. \\ & \left. - f(s, y(s), (Ky)(t), (Hy)(t))] ds \right\|.\end{aligned}$$

Hence, we get

$$\begin{aligned}& \|\Gamma x(t) - \Gamma y(t)\| \\ \leq\; & M\|g(x) - g(y)\| \\ & + M\frac{T^{q-1}}{\Gamma(q)} \int_0^t s^n \|f(s, x(s), (Kx)(t), (Hx)(t)) - f(s, y(s), (Ky)(t), (Hy)(t))\| ds,\end{aligned}$$

where $M = \sup_{t \in J}\{\|T(t)\|_{L_b(X)}\}$ and $L_b(X)$ be the Banach space of all linear and bounded operators on X.

According to [Hf] and [Hg], we obtain

$$\begin{aligned}& \|\Gamma x(t) - \Gamma y(t)\| \\ \leq\; & Ml_g\|x - y\|_C + M\frac{T^{q-1}}{\Gamma(q)} \int_0^t s^n m_1(s) \|x(s)) - y(s)\| ds \\ & + M\frac{T^{q-1}}{\Gamma(q)} \int_0^t s^n m_2(s) \|(Kx)(s)) - (Ky)(s)\| ds \\ & + M\frac{T^{q-1}}{\Gamma(q)} \int_0^t s^n m_3(s) \|(Hx)(s)) - (Hy)(s)\| ds \\ \leq\; & Ml_g\|x - y\|_C + M\frac{T^{q-1}}{\Gamma(q)} \|x - y\|_C \int_0^t s^n [m_1(s) + K^* m_2(s) + H^* m_3(s)] ds.\end{aligned}$$

Therefore, we can deduce that

$$\|\Gamma x(t) - \Gamma y(t)\| \leq \Omega_n(t)\|x - y\|_C.$$

Thus, we obtain

$$\|\Gamma x - \Gamma y\|_C \leq \Omega_n(t)\|x - y\|_C.$$

Hence, assumption [HΩ_n] allows us to conclude in view of the contraction mapping principle, that Γ has a unique fixed point $x \in C(J, X)$, and

$$x(t) = T(t)[x_0 + g(x)] + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds$$

which is the unique mild solution of system (1.1)-(1.2). □

Our second result uses the following Krasnoselskii fixed point theorem.

Theorem 3.2. *Let $\mathfrak{B}$ be a closed convex and nonempty subsets of Banach space X. Let $\mathcal{L}$ and $\mathcal{N}$ be two operators such that*
(1) $\mathcal{L}x + \mathcal{N}y \in \mathfrak{B}$ whenever $x, y \in \mathfrak{B}$;
(2) $\mathcal{L}$ is a contraction mapping;
(3) $\mathcal{N}$ is compact and continuous.
Then there exists $z \in \mathfrak{B}$ such that $z = \mathcal{L}z + \mathcal{N}z$.

Suppose that

[Hf′]: $f : J \times X \times X \times X \to X$, for a.e. $t \in J$, the function $f(t, \cdot, \cdot, \cdot) : X \times X \times X \to X$ is continuous and for all $x, y, z \in X$, the function $f(\cdot, x, y, z) : J \to X$ is measurable. There exists a function $\rho \in L^1_{Loc}(J, R^+)$ such that

$$\|f(t, x, y, z)\| \leq \rho(t)$$

for all $x, y, z \in X$ and $t \in J$.

[Hk′]: Let $D_k = \{(t, s) \in J \times J; 0 \leq s \leq t \leq T\}$. The function $k : D_k \times X \to X$ is continuous and there exists a $m_k(t, s) \in C(D_k, R^+)$ such that

$$\|k(t, s, x)\| \leq m_k(t, s)$$

for each $(t, s) \in D_k$ and $x \in X$.

[Hh′]: Let $D_h = \{(t, s) \in J \times J; 0 \leq s \leq t \leq T\}$. The function $h : D_h \times X \to X$ is continuous and there exists a $m_h(t, s) \in C(D_k, R^+)$ such that

$$\|h(t, s, x)\| \leq m_h(t, s)$$

for each $(t, s) \in D_h$ and $x \in X$.

Now we are ready to state and prove the following existence result.

Theorem 3.3. *Assume that the conditions [HA],[Hf′], [Hk′], [Hh′], [Hg] are satisfied. Then system (1.1)-(1.2) has at least one mild solution on J provided that*

$$Ml_g < 1.$$

Proof. Let us choose

$$r = M(\|x_0\| + G) + M\frac{T^{n+q}}{(n+1)\Gamma(q)}\|\rho\|_{L^1_{Loc}(J,R^+)} + Mc_0 + c_2\frac{T^{n+q}}{n+1}$$

with

$$G = \sup_{x \in C(J,X)} \{\|g(x)\|\}, \tag{3.2}$$

c_0 and c_2 defined respectively by (3.3) and (3.4) below.

Consider the ball

$$B_r = \{x \in C(J, X) \mid \|x\| \leq r\}.$$

Define on B_r the operators Γ_1 and Γ_2 by

$$(\Gamma_1 x)(t) = T(t)[x_0 + g(x)],$$

and

$$(\Gamma_2 x)(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds.$$

Step1. Let us observe that if $x, y \in B_r$ then $\Gamma_1 x + \Gamma_2 y \in B_r$.

In fact,

$$\begin{aligned}
& \|(\Gamma_1 x)(t) + (\Gamma_2 y)(t)\| \\
\leq \; & M\|x_0 + g(x)\| + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n \|f(s, y(s), (Ky)(s), (Hy)(s))\| ds \\
\leq \; & M(\|x_0\| + \|g(x)\|) + M \frac{T^{q-1}}{\Gamma(q)} \int_0^t s^n \|f(s, y(s), (Ky)(s), (Hy)(s))\| ds,
\end{aligned}$$

which according to (3.2), gives

$$\|(\Gamma_1 x)(t) + (\Gamma_2 y)(t)\| \leq M(\|x_0\| + G) + M \frac{T^{n+q}}{(n+1)\Gamma(q)} \|\rho\|_{L^1_{Loc}(J, R^+)} \leq r.$$

Hence, we can deduce that

$$\|\Gamma_1 x + \Gamma_2 x\|_C \leq r.$$

Step 2. We show that Γ_1 is a contraction mapping.

For any $t \in J$, $x, y \in C(J, X)$ we have

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \leq M\|g(x) - g(y)\|$$

which in view of [Hg], gives

$$\|(\Gamma_1 x)(t) - (\Gamma_1 y)(t)\| \leq M l_g \|x - y\|_C,$$

which implies that

$$\|\Gamma_1 x - \Gamma_1 y\| \leq M l_g \|x - y\|_C.$$

Since $Ml_g < 1$, then Γ_1 is a contraction mapping.

Step 3. Let us prove that Γ_2 is continuous and compact.

For this purpose, we assume that $x_n \to x$ in $C(J, X)$. It comes from the continuity of k and h that

$$k(t, s, x_n(s)) \to k(t, s, x(s)) \quad \text{and} \quad \|k(t, s, x_n(s)) - k(t, s, x(s))\| \leq 2K^*,$$

$$h(t, s, x_n(s)) \to h(t, s, x(s)) \quad \text{and} \quad \|h(t, s, x_n(s)) - h(t, s, x(s))\| \leq 2H^*.$$

By the dominated convergence theorem,

$$\int_0^t h(t,s,x_n(s))ds \to \int_0^t h(t,s,x(s))ds, \quad \int_0^T k(t,s,x_n(s))ds \to \int_0^T k(t,s,x(s))ds,$$

as $n \to \infty$. Then by [Hf′], we have

$$f(s,x_n(s),(Kx_n)(s),(Hx_n)(s)) \to f(s,x(s),(Kx)(s),(Hx)(s)) \text{ as } n \to \infty,\ s \in J.$$

$$\|f(s,x_n(s),(Kx_n)(s),(Hx_n)(s))\| \le \rho(s),\ s \in J.$$

By the dominated convergence theorem again, we have

$$\begin{aligned} & \|(\Gamma_2 x_n)(t) - (\Gamma_2 x)(t)\| \\ \le\ & \frac{MT^{n+q}}{(n+1)\Gamma(q)} \int_0^t \|f(s,x_n(s),(Kx_n)(s),(Hx_n)(s)) - f(s,x(s),(Kx)(s),(Hx)(s))\| ds \\ \to\ & 0, \text{ as } n \to \infty, \end{aligned}$$

which implies that Γ_2 is continuous.

To prove that Γ_2 is a compact operator, we observe that Γ_2 is a composition of two operators, that is, $\Gamma_2 = U \circ V$ where

$$(Vx)(s) = T(t-s)f(s,x(s),(Kx)(s),(Hx)(s)), t \in J, 0 < s < t,$$

and

$$(Uy)(t) = \int_0^t (t-s)^{q-1} s^n y(s) ds, t \in J.$$

Since for the same reason as Γ_2, the operator V is also continuous, it suffices to prove that V is uniformly bounded and U is compact to prove that Γ_2 is compact.

Let $x \in B_r$. Then $(Kx)(t) \in B_r^{'} = \{v \in C \mid \|v\|_C \le K^*\}$ and $(Hx)(t) \in B_r^{''} = \{v \in C \mid \|v\|_C \le H^*\}$. In view of [Hf′], f is bounded on the compact set $J \times B_r \times B_r^{'} \times B_r^{''}$. Therefore, we set

$$c_0 = \sup_{(t,x,y,z) \in J \times B_r^{'} \times B_r^{''}} \|f(t,x(t),y(t),z(t))\| < \infty. \tag{3.3}$$

Then, using (3.3), we get

$$\|(Vx)(s)\| \le \|T(t-s)\| \|f(s,x(s),(Kx)(s),(Hx)(s))\| \le Mc_0 \le r,$$

from which we deduce that $\|Vx\|_C \le r$. This means that V is uniformly bounded on B_r.

Since $y \in C(J,X)$, we set

$$c_2 = \sup_{t \in J} \|y(t)\| < \infty. \tag{3.4}$$

Then, on the other hand, we have

$$\|(Uy)(t)\| = \left\| \int_0^t (t-s)^{q-1} s^n y(s) ds \right\| \le c_2 \int_0^t (t-s)^{q-1} s^n ds \le c_2 \frac{T^{n+q}}{n+1} \le r$$

and on the other hand, for $0 < s < t_2 < t_1 < T$,

$$\begin{aligned}
& \|(Uy)(t_1) - (Uy)(t_2)\| \\
= & \left\| \int_0^{t_1} (t_1 - s)^{q-1} s^n y(s) ds - \int_0^{t_2} (t_2 - s)^{q-1} s^n y(s) ds \right\| \\
\leq & \left\| \int_0^{t_2} [(t_1 - s)^{q-1} - (t_2 - s)^{q-1}] s^n y(s) ds \right\| + \left\| \int_{t_1}^{t_2} (t_1 - s)^{q-1} s^n y(s) ds \right\| \\
\leq & \int_0^{t_2} |(t_1 - s)^{q-1} - (t_2 - s)^{q-1}| s^n \|y(s)\| ds + \int_{t_1}^{t_2} (t_1 - s)^{q-1} s^n \|y(s)\| ds \\
\leq & \frac{c_2 T^n}{q} |2(t_1 - t_2)^q + t_2^q - t_1^q| \\
\leq & 2 \frac{c_2 T^n}{q} |t_1 - t_2|^q,
\end{aligned}$$

which does not depend on y. So UB_r is relatively compact. By the Arzela-Ascoli Theorem, U is compact. In short, we have proven that Γ_2 is continuous and compact, Γ_1 is a contraction mapping and $\Gamma_1 x + \Gamma_2 y \in B_r$ if $x, y \in B_r$. Hence, the Krasnoselskii theorem allows us to conclude that system (1.1)-(1.2) has at least one mild solution on J. □

Corollary 3.4. *In addition to the assumptions of Theorem 3.3, assumptions [Hf], [Hk], [Hh] also hold. Then system (1.1)-(1.2) has a unique mild solution on J.*

Proof. To prove the uniqueness of $x(t)$, let $y(t)$ be another mild solution of system (1.1)-(1.2) with nonlocal condition $y_0 + g(y)$. It comes from

$$\begin{aligned}
\|x(t) - y(t)\| \leq & M\|x_0 - y_0\| + Ml_g\|g(x) - g(y)\| \\
& + M \frac{T^{n+q+1}}{(n+1)\Gamma(q)} \int_0^t (m_1(t) + m_2(t)K^* + m_3(t)H^*) \|x(s) - y(s)\| ds
\end{aligned}$$

that

$$\begin{aligned}
& \|x(t) - y(t)\| \\
\leq & \frac{M}{1 - Ml_g} \|x_0 - y_0\| \\
& + MT \frac{T^{n+q}}{(n+1)\Gamma(q)(1 - Ml_g)} \int_0^t (m_1(t) + m_2(t)K^* + m_3(t)H^*) \|x(s) - y(s)\| ds \\
\leq & \frac{M}{1 - Ml_g} \|x_0 - y_0\| + MT \frac{T^{n+q}}{(n+1)\Gamma(q)(1 - Ml_g)} \widehat{M} \int_0^t \|x(s) - y(s)\| ds,
\end{aligned}$$

which implies by Gronwall's inequality

$$\|x(t) - y(t)\| \leq \widetilde{M} \frac{M}{1 - Ml_g} \|x_0 - y_0\|$$

which yield the uniqueness of $x(\cdot)$. □

4. Existence of optimal controls

We suppose that Y is another separable reflexive Banach space from which the controls u take the value. We denote a class of nonempty closed and convex subsets of Y by $W_f(Y)$. The multifunction $\omega : J \longrightarrow W_f(Y)$ is measurable and $\omega(\cdot) \subset E$ where E is bounded set of Y, the admissible control set $U_{ad} = S_\omega^p = \{u \in L^p(E) \mid u(t) \in \omega(t)\ a.e.\}$, $1 < p < \infty$. Then $U_{ad} \neq \emptyset$ (see P142 Proposition 1.7 and P174 Lemma 3.2 of [16]).

Consider the following controlled system

$$\begin{cases} D^q x(t) = Ax(t) + t^n f(t, x(t), (Kx)(t), (Hx)(t)) + C(t)u(t), \\ \qquad\qquad\qquad\qquad t \in J,\, u \in U_{ad},\ n \in Z^+,\ q \in (0,1), \\ x(0) = g(x) + x_0. \end{cases} \tag{4.1}$$

Assumption [HC]: $C \in L_\infty(J; L(Y, X))$.

It is easy to see that $Cu \in L^p(J; X)$ for all $u \in U_{ad}$.

By Theorem 3.3, we have the following result.

Theorem 4.1. *In addition to assumptions of Theorem 3.3, suppose assumption [HC] holds. For every $u \in U_{ad}$, system (4.1) has a mild solution corresponding to u given by the solution of the following integral equation*

$$\begin{aligned} x^u(t) &= T(t)[x_0 + g(x)] + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) f(s, x(s), (Kx)(s), (Hx)(s)) ds \\ &\quad + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) C(s) u(s) ds \end{aligned}$$

Proof. Compared with Theorem 3.3, the key step is to check the term containing control policy. Consider

$$\xi(t) = \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) C(s) u(s) ds,$$

using Hölder inequality, we have

$$\begin{aligned} \|\xi(t)\| &\leq M \frac{T^{q-1}}{\Gamma(q)} \int_0^t s^n \|C(s)u(s)\| ds \\ &\leq M \|C\|_\infty \frac{T^{n+q}}{(n+1)\Gamma(q)} \int_0^t \|u(s)\|_Y ds \\ &\leq M \|C\|_\infty \frac{T^{n+q}}{(n+1)\Gamma(q)} \left(\int_0^t 1^{\frac{p-1}{p}} ds \right)^{\frac{p-1}{p}} \left(\int_0^t \|u(s)\|_Y^p ds \right)^{\frac{1}{p}} \\ &\leq M \|C\|_\infty \frac{T^{n+q}}{(n+1)\Gamma(q)} \|u\|_{L^p(J,Y)}, \end{aligned}$$

where $\|C\|_\infty$ is the norm of operator C in Banach space $L_\infty(J, L(Y, X))$. It is easy to see that $\|T(t-\cdot)C(\cdot)u(\cdot)\|$ is integrable. Hence $\xi(\cdot) \in C(J, X)$. Using Theorem 3.3, one can verify it immediately. $\square$

Assumption [HL]:
[HL1] The functional $l : J \times X \times Y \longrightarrow R \cup \{\infty\}$ is Borel measurable.
[HL2] $l(t, \cdot, \cdot)$ is sequentially lower semicontinuous on $X \times Y$ for almost all $t \in J$.
[HL3] $l(t, x, \cdot)$ is convex on Y for each $x \in X$ and almost all $t \in J$.
[HL4] There exist constants $d \geq 0$, $e > 0$, φ is nonnegative and $\varphi \in L^1(J; R)$ such that

$$l(t, x, u) \geq \varphi(t) + d\|x\| + e\|u\|_Y^p.$$

We consider the Lagrange problem (P):
Find $(x^0, u^0) \in C(J, X) \times U_{ad}$ such that

$$J(x^0, u^0) \leq J(x^u, u), \text{ for all } u \in U_{ad},$$

where

$$J(x^u, u) = \int_0^T l(t, x^u(t), u(t))dt,$$

and $x^u(\cdot, x^*)$ denotes the mild solution of system (4.1) corresponding to the control $u \in U_{ad}$.
In order to obtain the existence of optimal controls we need the following important lemma.

Lemma 4.2. *(See Lemma 4.1 of* [19]*) Suppose that A is the infinitesimal generator of a compact semigroup $\{T(t), t \geq 0\}$ in X. Then the operator $\mathcal{Q} : L^p(J; Y) \longrightarrow C(J; X)$ with $p > 1$, given by*

$$(\mathcal{Q}f)(\cdot) = \int_0^\cdot T(\cdot - s)f(s)ds$$

is strongly continuous.

Now we can give another main result of this paper, the existence of optimal controls for problem (P).

Theorem 4.3. *Suppose X is a separable reflexive Banach space and A is the infinitesimal generator of a compact semigroup $\{T(t), t \geq 0\}$ in X. If the assumption [HL] and the assumptions of Theorem 4.1 holds, then the problem (P) admits at least one optimal pair.*

Proof. If $\inf\{J(x^u, u) \mid u \in U_{ad}\} = +\infty$, there is nothing to prove.
Assume that $\inf\{J(x^u, u) \mid u \in U_{ad}\} = m < +\infty$. Using assumption [HL], we have $m > -\infty$. By definition of infimum there exists a minimizing sequence feasible pair $\{(x^n, u^n)\} \subset A_{ad} \equiv \{(x, u) \mid x \text{ is a mild solution of system (4.1) corresponding to } u \in U_{ad}\}$, such that $J(x^n, u^n) \longrightarrow m$ as $n \longrightarrow +\infty$. Since $\{u_n\} \subseteq U_{ad}$, $\{u_n\}$ is bounded in $L^p(J; Y)$, there exists a subsequence, relabeled as $\{u^n\}$, and $u^0 \in L^p(J; Y)$ such that

$$u^n \xrightarrow{w} u^0 \text{ in } L^p(J; Y).$$

Then, since U_{ad} is closed and convex, thanks to Marzur Lemma, $u^0 \in U_{ad}$.

Suppose x^n is the mild solution of system (4.1) corresponding to u^n ($n = 0, 1, 2, \cdots$), x^n satisfies the following integral equation

$$\begin{aligned} x^n(t) &= T(t)[x_0 + g(x^n)] \\ &+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^nT(t-s)f(s,x^n(s),(Kx^n)(s),(Hx^n)(s))ds \\ &+\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^nT(t-s)C(s)u^n(s)ds \end{aligned}$$

Let $f_n(\theta) \equiv f(\theta, x^n(\theta), (Kx^n)(\theta), (Hx^n)(\theta))$, by assumption [Hf′], we obtain that f_n is a bounded continuous operator from in J into X, hence $f_n(\cdot) \in L^p(J;X)$. Furthermore, $\{f_n(\cdot)\} \subseteq X$, $\{f_n(\cdot)\}$ is bounded in $L^p(J;X)$, there exists a subsequence, relabeled as $\{f_n(\cdot)\}$, and $\widehat{f}(\cdot) \in L^p(J;X)$ such that

$$f_n(\cdot) \xrightarrow{w} \widehat{f}(\cdot) \text{ in } L^p(J;X)$$

By Lemma 4.2, we have

$$\mathcal{Q}f_n \xrightarrow{s} \mathcal{Q}\widehat{f} \text{ in } C(J;X)$$

We consider the following system

$$\begin{cases} D^qx(t) = Ax(t) + t^n\widehat{f}(t) + C(t)u^0(t), t \in J, u \in U_{ad},\ n \in Z^+, \\ x(0) = g(x) + x_0. \end{cases} \tag{4.2}$$

By Theorem 4.1, we know that system (4.2) has a mild solution

$$\begin{aligned} \widehat{x}(t) &= T(t)[x_0 + g(\widehat{x})] + \frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^nT(t-s)\widehat{f}(s)\,ds \\ &\quad +\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^nT(t-s)C(s)u^0(s)ds. \end{aligned}$$

Define

$$\eta_n(t) = \left\|\frac{1}{\Gamma(q)}\int_0^t (t-s)^{q-1}s^nT(t-s)\left[\left(f_n(s)-\widehat{f}(s)\right) + C(s)\left(u^n(s)-u^0(s)\right)\right]ds\right\|,$$

then

$$\eta_n(t) \le \frac{T^{n+q}}{(n+1)\Gamma(q)}\int_0^t \left\|T(t-s)\left[\left(f_n(s)-\widehat{f}(s)\right) + C(s)\left(u^n(s)-u^0(s)\right)\right]\right\| ds.$$

Using Lemma 4.2 again, we have

$$\eta_n \longrightarrow 0 \text{ in } C(J;R) \text{ as } n \longrightarrow \infty.$$

It comes from

$$\begin{aligned}
& \|x^n(t) - \widehat{x}(t)\| \\
\leq \ & \|T(t)[g(x^n) - g(\widehat{x})]\| \\
& + \left\| \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) \left[\left(f_n(s) - \widehat{f}(s) \right) + C(s) \left(u^n(s) - u^0(s) \right) \right] ds \right\| \\
\leq \ & M l_g \|x^n - \widehat{x}\|_C + \eta_n
\end{aligned}$$

and $Ml_g < 1$, one has

$$0 \leq (1 - Ml_g)\|x^n - \widehat{x}\|_C \quad \leq \quad \eta_n.$$

Then we obtain

$$x^n \longrightarrow \widehat{x} \text{ in } C(J;X) \text{ as } n \longrightarrow \infty.$$

Furthermore, using assumptions [Hf′], [Hk′] and [Hh′], we also obtain

$$f_n(\cdot) \to f(\cdot, \widehat{x}(\cdot), (K\widehat{x})(\cdot), (H\widehat{x})(\cdot)) \text{ in } C(J;X) \text{ as } n \longrightarrow \infty.$$

Using the uniqueness of limit, we have

$$\widehat{f}(t) = f(t, \widehat{x}, (K\widehat{x})(t), (H\widehat{x})(t)).$$

Thus, $\widehat{x}$ can be given by

$$\begin{aligned}
\widehat{x}(t) \quad = \quad & T(t)[x_0 + g(\widehat{x})] + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) f(s, \widehat{x}, (K\widehat{x})(s), (H\widehat{x})(s)) ds \\
& + \frac{1}{\Gamma(q)} \int_0^t (t-s)^{q-1} s^n T(t-s) C(s) u^0(s) ds,
\end{aligned}$$

is which just a mild solution of system (4.1) corresponding to u^0.

Since $C(J;X) \hookrightarrow L^1(J;X)$, using the assumption [HL] and Balder's theorem, we can obtain

$$m = \lim_{n \to \infty} \int_0^T l\left(t, x^n(t), u^n(t)\right) dt \geq \int_0^T l\left(t, \widehat{x}(t), u^0(t)\right) dt = J\left(\widehat{x}, u^0\right) \geq m.$$

This show that J attains its minimum at $u^0 \in U_{ad}$. □

References

[1] L. Byszewski; Theorems about the existence and uniqueness of solutions of a semilinear evolution nonlocal Cauchy problem, J. Math. Anal. Appl., 162(1991), 494-505.

[2] L. Byszewski; Existence, uniqueness and asymptotic stability of solutions of abstract nonlocal Cauchy problems, Dynam. Systems Appl., 5(1996), 595-605.

[3] L. Byszewski and H. Akca; Existence of solutions of a semilinear functional differential evolution nonlocal problem, Nonlinear Anal., 34(1998), 65-72.

[4] L. Byszewski and H. Akca; On a mild solution of a semilinear functional-differential evolution nonlocal problem, J. Appl. Math. Stochastic Anal., 10(1997), 265-271.

[5] K. Balachandran and M. Chandrasekaran; The nonlocal Cauchy problem for semilinear integrodifferential equation with devating argument, Proceedings of the Edinburgh Mathematical Society, 44(2001), 63-70.

[6] K. Balachandran and R. R. Kumar; Existence of solutions of integrodifferential evoluition equations with time varying delays, Applied Mathematics E-Notes, 7(2007), 1-8.
[7] K. Balachandran, J. Y. Park; Nonlocal Cauchy problem for abstract fractional semilinear evolution equations, Nonlinear Anal. 71(2009), 4471-4475.
[8] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab; Existence results for fractional order functional differential equations with infinite delay, J. Math. Anal. Appl., 338(2008), 1340-1350.
[9] M. Benchohra, J. Henderson, S. K. Ntouyas, A. Ouahab; Existence results for fractional functional differential inclusions with infinite delay and application to control theory, Fract. Calc. Appl. Anal., 11(2008), 35-56.
[10] Yong-Kui Chang, V. Kavitha, M. Mallika Arjunan; Existence and uniqueness of mild solutions to a semilinear integrodifferential equation of fractional order, Nonlinear Anal., 71(2009), 5551-5559.
[11] Diethelm, A. D. Freed; On the solution of nonlinear fractional order differential equations used in the modeling of viscoelasticity, in: F. Keil, W. Mackens, H. Voss, J. Werther (Eds.), Scientific Computing in Chemical Engineering II-Computational Fluid Dynamics, Reaction Engineering and Molecular Properties, Springer-Verlag, Heidelberg, 1999, pp. 217-224.
[12] M. M. El-Borai; Semigroup and some nonlinear fractional differential equations, Applied Mathematics and Computation, 149(2004), 823-831.
[13] L. Gaul, P. Klein, S. Kempfle; Damping description involving fractional operators, Mech. Syst. Signal Process., 5(1991), 81-88.
[14] W. G. Glockle, T. F. Nonnenmacher; A fractional calculus approach of self-similar protein dynamics, Biophys. J., 68(1995), 46-53.
[15] R. Hilfer; Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[16] S. Hu and N. S. Papageorgiou, Handbook of multivalued Analysis (Theory), Kluwer Academic Publishers, Dordrecht Boston, London, 1997.
[17] Lanying Hu, Yong Ren and R. Sakthivel, Existence and uniqueness of mild solutions for semilinear integro-differential equations of fractional order with nonlocal initial conditions and delays, Semigroup Forum, (in press).
[18] A. A. Kilbas, Hari M. Srivastava, J. Juan Trujillo; Theory and Applications of Fractional Differential Equations, in: North-Holland Mathematics Studies, vol. 204, Elsevier Science B.V, Amsterdam, 2006.
[19] X. Li and J. Yong, Optimal control theory for infinite dimensional systems, Birkhauser Boston, 1995.
[20] V. Lakshmikantham, S. Leela and J. Vasundhara Devi; Theory of Fractional Dynamic Systems, Cambridge Scientific Publishers, 2009.
[21] V. Lakshmikantham; Theory of fractional differential equations, Nonlinear Anal., 60(2008), 3337-3343.
[22] V. Lakshmikantham, A. S. Vatsala; Basic theory of fractional differential equations, Nonlinear Anal., 69(2008), 2677-2682.
[23] K. S. Miller, B. Ross; An Introduction to the Fractional Calculus and Differential Equations, John Wiley, New York, 1993.
[24] F. Mainardi, Fractional calculus; Some basic problems in continuum and statistical mechanics, in: A. Carpinteri, F. Mainardi (Eds.), Fractals and Fractional Calculus in Continuum Mechanics, Springer-Verlag, Wien, 1997, pp. 291-348.
[25] F. Metzler, W. Schick, H. G. Kilian, T. F. Nonnenmache; Relaxation in filled polymers: A fractional calculus approach, J. Chem. Phys., 103(1995), 7180-7186.
[26] G. M. Mophou, G. M. N'Guérékata; Mild solutions for semilinear fractional differential equations, Electron. J. Differ. Equ., 21(2009), 1-9.
[27] G. M. Mophou, G. M. N'Guérékata; Existence of mild solution for some fractional differential equations with nonlocal conditions, Semigroup Forum, 79(2009), 315-322.
[28] I. Podlubny; Fractional Differential Equations, Academic Press, San Diego, 1999.
[29] JinRong Wang, X. Xiang and W. Wei; A class of nonlinear integrodifferential impulsive periodic systems of mixed type and optimal controls on Banach spacs, Journal of Applied Mathematics and Computing, (in press).