SOLUTION OF TENTH AND NINTH-ORDER BOUNDARY VALUE PROBLEMS BY HOMOTOPY PERTURBATION METHOD

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ABSTRACT. In this paper, we apply homotopy perturbation method (HPM) for solving ninth and tenth-order boundary value problems. The suggested algorithm is quite efficient and is practically well suited for use in these problems. The proposed iterative scheme finds the solution without any discretization, linearization or restrictive assumptions. Several examples are given to verify the reliability and efficiency of the method. The fact that the proposed homotopy perturbation method solves nonlinear problems without using Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method.

1. INTRODUCTION

In the last two decades, with the rapid development of nonlinear science, there has appeared ever-increasing interest of scientists, physicists and engineers in the analytical techniques for nonlinear problems. It is well known, that perturbation methods provide the most versatile tools available in nonlinear analysis of engineering problems, see [7-14, 22-29, 42] and the references therein. The Perturbation methods, like other nonlinear analytical techniques, have their own limitations. At first, almost all perturbation methods are based on an assumption that a small parameter must exist in the equation. This so-called small parameter assumption greatly restricts applications of perturbation techniques. As is well known, an overwhelming majority of nonlinear problems have no small parameters at all. Secondly, the determination of small parameter seems to be a special art requiring special techniques. An appropriate choice of small parameters leads to the ideal results but, an unsuitable choice may create serious problems. Furthermore, the approximate solutions solved by perturbation methods are valid, in most cases, only for the small values of the parameters. It is obvious that all these limitations come from the small parameter assumption. These facts have motivated to suggest alternate techniques such as, variational iteration [1-3, 14-21, 25-35], decomposition [39, 40], variation

of parameters [36] and exp-function [37, 38]. In order to overcome these drawbacks, combining the standard homotopy method and perturbation, which is called the homotopy perturbation, modifies the homotopy method.

This paper is devoted to the study of boundary value problems of tenth and ninth-order which are known to arise in the study of astrophysics, hydrodynamic and hydro magnetic stability, see [4-6, 25, 34, 35, 39, 40]. A class of characteristic-value problems of higher order (as higher as twenty four) is known to arise in hydrodynamic and hydro magnetic stability. In addition, it is well known that when a layer of fluid is heated from below and is subject to the action of rotation, instability may set in as ordinary convection which may be modeled by a tenth-order boundary value problem, see [4-6, 34, 25, 35, 39, 40] and the references therein. The boundary value problems of higher-order have been investigated due to their mathematical importance and the potential for applications in diversified applied sciences. Several techniques including the finite-difference, decomposition, variational iteration and modified variational iteration have been employed for solving such problems, see [6, 25, 34, 35, 39, 40] and the references therein. He [9-14] developed the homotopy perturbation method for solving linear, nonlinear, initial and boundary value problems. Moreover, He realized the physical significance of the homotopy perturbation method, its compatibility with the physical problems and applied this promising technique to a wide class of linear and nonlinear, ordinary, partial, deterministic or stochastic differential equation; see [9-14]. In this method the solution is given in an infinite series usually converging to an accurate solution [7-14, 22-29, 42]. The basic motivation of this paper is to apply the homotopy perturbation method for solving boundary value problems of tenth and ninth-order. It is worth mentioning that the suggested method is applied without any discretization, restrictive assumption or transformation and is free from round off errors. The selection of initial value is done very carefully because the approximants are heavily dependant upon the initial value. Unlike the method of separation of variables that require initial and boundary conditions, the method provides an analytical solution by using the initial conditions only. The proposed method work efficiently and the results are very encouraging and reliable. The fact that the proposed HPM solves nonlinear problems without using the Adomian's polynomials can be considered as a clear advantage of this technique over the decomposition method. Several examples are given to verify the reliability and efficiency of the homotopy perturbation method.

2. Homotopy Perturbation Method

To explain the homotopy perturbation method, we consider a general equation of the type,

$$L(u) = 0, \tag{2.1}$$

where L is any integral or differential operator. We define a convex homotopy $H(u, p)$ by

$$H(u, p) = (1 - p)F(u) + pL(u), \tag{2.2}$$

where $F(u)$ is a functional operator with known solutions v_0, which can be obtained easily. It is clear that, for

$$H(u, p) = 0, \tag{2.3}$$

we have

$$H(u,0) = F(u), \quad H(u,1) = L(u).$$

This shows that $H(u,p)$ continuously traces an implicitly defined curve from a starting point $H(v_0,0)$ to a solution function $H(f,1)$. The embedding parameter monotonically increases from zero to unit as the trivial problem $F(u) = 0$ is continuously deforms the original problem $L(u) = 0$. The embedding parameter $p \in (0,1]$ can be considered as an expanding parameter [7-14, 22-29, 42]. The homotopy perturbation method uses the homotopy parameter p as an expanding parameter [9-14] to obtain

$$u = \sum_{i=0}^{\infty} p^i u_i = u_0 + pu_1 + p^2 u_2 + p^3 u_3 + \cdots, \tag{2.4}$$

if $p \to 1$, then (2.4) corresponds to (2.2) and becomes the approximate solution of the form,

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i. \tag{2.5}$$

It is well known that series (2.5) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$, see [9-14]. We assume that (2.5) has a unique solution. The comparisons of like powers of p give solutions of various orders.

3. NUMERICAL APPLICATIONS

In this section, we apply the homotopy perturbation method (HPM) for solving the boundary value problems of tenth and ninth-order. The selection of initial value is done carefully because the approximants are heavily dependant upon initial value.

Example 3.1 [25, 34, 35, 39] Consider the following nonlinear boundary value problem of tenth-order

$$y^{(x)}(x) = e^{-x} y^2(x), 0 < x < 1,$$

with boundary condition

$$\begin{aligned} y(0) &= 1, & y''(0) &= y^{(iv)}(0) &= y^{(vi)}(0) &= y^{(viii)}(0) &= 1, \\ y(1) &= e, & y''(1) &= y^{(iv)}(1) &= y^{(vi)}(1) &= y^{(viii)}(1) &= e. \end{aligned}$$

The exact solution of the problem is

$$y(x) = e^x.$$

Applying the convex homotopy method

$$y_0^{(x)}(x) + py_1^{(x)}(x) + p^2 y_2^{(x)}(x) + \cdots = p\left(e^{-x}\left(y_0(x) + py_1(x) + p^2 y_2(x) + \cdots\right)^2\right).$$

Comparing the co-efficient of like powers of p

$$p^{(0)} : y_0(x) = 1,$$

$$p^{(1)} : y_1(x) = Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7$$
$$+ \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}x^{10} + \frac{1}{11!}x^{11} + \frac{1}{12!}x^{12} + \cdots,$$

$$p^{(2)} : y_2(x) = \frac{2}{11!}Ax^{11} + \left(-\frac{4}{12!}A + \frac{1}{239500800}\right)x^{12} + \cdots,$$

$$\vdots \qquad .$$

The series solution is given as:

$$y(x) = 1 + Ax + \frac{1}{2!}x^2 + \frac{1}{3!}Bx^3 + \frac{1}{4!}x^4 + \frac{1}{5!}Cx^5 + \frac{1}{6!}x^6 + \frac{1}{7!}Dx^7$$
$$+ \frac{1}{8!}x^8 + \frac{1}{9!}Ex^9 + \frac{1}{10!}Dx^{10} + \left(-\frac{1}{19958400}A + \frac{1}{39916800}\right)x^{11}$$
$$+ \left(-\frac{1}{119750400}A + \frac{1}{159667200}\right)x^{12} + O(x^{13}),$$

where
$$A = y'(0), \ B = y^{(3)}(0), \ C = y^{(5)}(0), \ D = y^{(7)}(0), \ E = y^{(9)}(0).$$

Imposing the boundary conditions at $x = 1$, we obtain

$$A = 1.00001436, \quad B = 0.999858964, \quad C = 1.001365775,$$
$$D = 0.987457318, \quad E = 1.0932797434.$$

The series solution is given as:

$$y(x) = 1 + 1.00001436\,x + \frac{1}{2!}x^2 + 0.1666431607\,x^3 + \frac{1}{4!}x^4 + 0.008344714791\,x^5$$
$$+ \frac{1}{6!}x^6 + 0.00019524071\,x^7 + \frac{1}{8!}x^8 + 3.013 \times 10^{-6}x^9 + \frac{1}{10!}x^{10}$$
$$+ 2.51 \times 10^{-8}x^{11} - 2.087 \times 10^{-9}x^{12} + \cdots.$$

Table 1 exhibits the exact solution and the series solution along with the errors obtained by using the homotopy perturbation method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

Figure 1 clearly indicates the accuracy of the proposed homotopy perturbation method (HPM).

Example 3.2 [34, 35, 40] Consider the following linear boundary value problem of tenth-order

$$y^{(x)}(x) = -8e^x + y''(x), 0 < x < 1,$$

TABLE 1. (Error estimates) Error = Exact solution - Series solution.

| x	Exact solution	Series solution	Errors
0.0	1.000000000	1.000000000	0.00000
0.1	1.105170918	1.10517233	-1.41E-6
0.2	1.221402758	1.221405446	-2.69E-6
0.3	1.349858808	1.349862509	-3.70E-6
0.4	1.491824698	1.49182905	-4.35E-6
0.5	1.648721271	1.648725849	-4.58E-6
0.6	1.822118800	1.822123158	-4.36E-6
0.7	2.013752707	2.013756415	-3.71E-6
0.8	2.225540928	2.225543623	-2.69E-6
0.9	2.459603111	0.459604528	-1.42E-6
1.0	2.718281828	2.7182830	2.00E-9
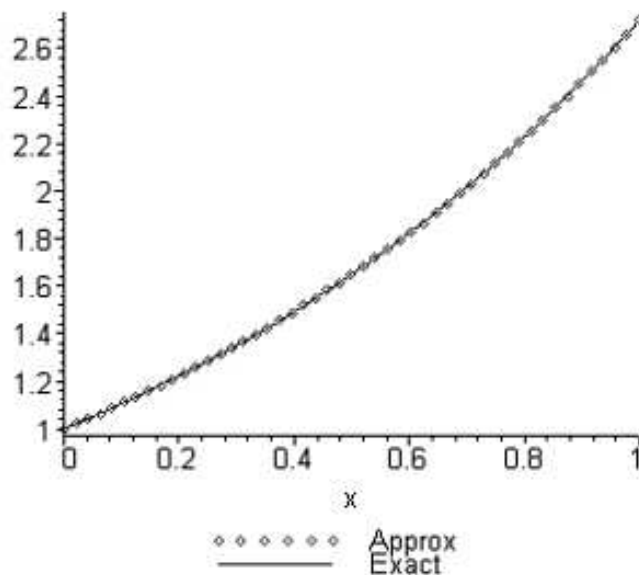
FIGURE 1. Graphical comparison between approximate solution and exact solution

with boundary conditions

$$y(0) = 1, \ y'(0) = 0, \ y''(0) = -1, \ y'''(0) = -2, \ y^{(iv)}(0) = -3,$$
$$y(1) = 0, \ y'(1) = -e, \ y''(1) = -2e, \ y'''(1) = -3e, \ y^{(iv)}(1) = -4\,e.$$

The exact solution of the problem is

$$y(x) = (1 - x)e^x.$$

Applying the convex homotopy method

$$y_0^{(x)}(x) + py_1^{(x)}(x) + p^2 y_2^{(x)}(x) + \cdots = p\left((y_0''(x) + py_1''(x) + p^2 y_2''(x) + \cdots) - 8e^x\right).$$

Comparing the co-efficient of like powers of p

$$p^{(0)} : y_0(x) = 1,$$

$$p^{(1)} : y_1(x) = -8\,e^x + 8 + 8\,x + \frac{7}{2!}x^2 + x^3 + \frac{5}{4!}x^4 + \left(\frac{1}{15} + \frac{1}{5!}A\right)x^5$$
$$+ \left(\frac{1}{90} + \frac{1}{6!}B\right)x^6 + \left(\frac{1}{630} + \frac{1}{7!}C\right)x^7 + \left(\frac{1}{7!} + \frac{1}{8!}D\right)x^8$$
$$+ \left(\frac{1}{45360} + \frac{1}{9!}E\right)x^9,$$

$$p^{(2)} : y_2(x) = \frac{1}{518400}x^{10} + \frac{1}{6652800}x^{11} + \frac{1}{95800320}x^{12}$$
$$+ \left(\frac{1}{778377600} + \frac{1}{6227020800}A\right)x^{13} + \cdots,$$

$$\vdots$$

The series solution is given by

$$y(x) = 17 - 16\,e^x + 16x + \frac{15}{2!}x^2 + \frac{7}{3}x^3 + \frac{13}{4!}x^4 + \left(\frac{2}{15} + \frac{1}{15!}\right)x^5$$
$$+ \left(\frac{1}{45} + \frac{1}{6!}B\right)x^6 + \left(\frac{1}{315} + \frac{1}{7!}C\right)x^7 + \left(\frac{2}{7!} + \frac{1}{8!}D\right)x^8$$
$$+ \frac{1}{9!}(8 + E)x^9 + \frac{7}{10!}x^{10} + \frac{6}{11!}x^{11} + \frac{5}{12!}x^{12} + \cdots.$$

Imposing the boundary conditions at $x = 1$ yields

$$A = -4.00002,\ B = -4.99999999,\ C = -6.00100,\ D = -7.00000,\ E = -8.010000.$$

The series solution is given by

$$y(x) = 17 - 16\,e^x + 16x + \frac{15}{2!}x^2 + \frac{7}{3}x^3 + \frac{13}{4!}x^4 + 0.99999999999997x^5$$
$$+ 0.15277791666666666x^6 + 0.00198392857142857x^7$$
$$+ 0.000233214285714286\,x^8 - 2.75573192239859 \times 10^{-8}\,x^9$$
$$+ \frac{7}{518400}x^{10} + \frac{1}{6652800}x^{11} + \frac{1}{9580320}x^{12} + \cdots.$$

Table 2 exhibits the exact solution and the series solution along with the errors obtained by using the homotopy perturbation method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

TABLE 2. (Error estimates) Error = Exact solution - series solution.

| x	Exact solution	Series solution	Errors
0.00	1.0000000000	1.0000000000	0.0000000000
0.10	.9946538263	.9946538263	0.0000000000
0.20	.9771222065	.9771222065	0.0000000000
0.30	.9449011653	.9449011648	0.0000000005
0.40	.8950948186	.8950948125	0.0000000061
0.50	.8243606354	.8243605909	0.0000000444
0.60	.7288475202	.7288472928	0.0000002274
0.70	.6041258122	.6041249068	0.0000009055
0.80	.4451081857	.4451051850	0.0000030006
0.90	.2459603111	.2459516717	0.0000086394
1.00	0.0000000000	-.0000222586	0.0000222586
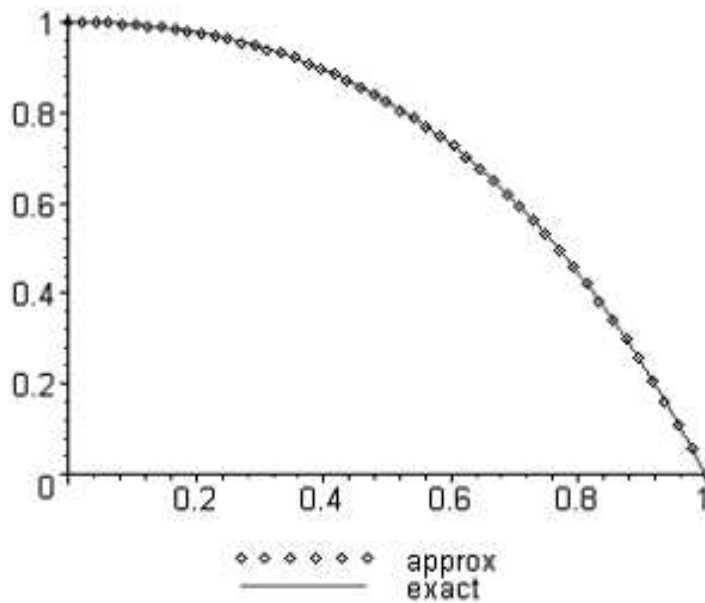
FIGURE 2. Graphical comparison between approximate solution and exact solution

Figure 2 clearly indicates the accuracy of the proposed homotopy perturbation method (HPM).

Example 3.3 [39] Consider the following ninth order boundary value problem

$$y^{(ix)} = -9e^x + y(x), 0 < x < 1$$

with boundary conditions

$$y(0) = 1, \ y^{(1)}(0) = 0, \ y^{(2)}(0) = -1, \ y^{(3)}(0) = -2, \ y^{(4)}(0) = -3,$$
$$y(1) = 0, \ y^{(1)}(1) = -e, \ y^{(2)}(1) = -2e, \ y^{(3)}(1) = -3e.$$

The exact solution of the problem is

$$y(x) = (1 - x)e^x.$$

Applying the convex homotopy method

$$y_0^{(ix)}(x) + py_1^{(ix)}(x) + p^2 y_2^{(ix)}(x) + \cdots = p\left(\left(y_0(x) + py_1(x) + p^2 y_2(x) + \cdots\right) - 9e^x\right).$$

Comparing the co-efficient of like powers of p

$$p^{(0)} : y_0(x) = 1,$$
$$p^{(1)} : y_1(x) = -\frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}A\,x^5 + \frac{1}{6!}B\,x^6 + \frac{1}{7!}C\,x^7 + \frac{1}{8!}D\,x^8$$
$$- \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots,$$
$$\vdots \qquad .$$

The series solution is given by

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 - \frac{1}{8}x^4 + \frac{1}{5!}A\,x^5 + \frac{1}{6!}B\,x^6 + \frac{1}{7!}C\,x^7 + \frac{1}{8!}D\,x^8$$
$$- \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots.$$

Imposing the boundary condition at $x = 1$ gives

$$A = -3.999992, \ B = -5.00017, \ C = -5.9985, \ D = -7.005.$$

The series solution is given as

$$y(x) = 1 - \frac{1}{2}x^2 - \frac{1}{3}x^3 + \frac{1}{8}x^4 - 0.03333326667x^5 - 0.006944680556x^6$$
$$- 0.001190178571x^7 - 0.000173735119x^8$$
$$- \frac{8}{9!}x^9 - \frac{9}{10!}x^{10} - \frac{10}{11!}x^{11} - \frac{11}{12!}x^{12} + \cdots.$$

Table 3 exhibits the exact solution and the series solution along with the errors obtained by using the homotopy perturbation method. It is obvious that the errors can be reduced further and higher accuracy can be obtained by evaluating more components of $y(x)$.

Figure 3 clearly indicates the accuracy of the proposed homotopy perturbation method (HPM).

TABLE 3. (Error estimates) Error = Exact solution - Series solution.

| xx	Exact solution	Series solution	Errors
0.0	1.00000000	1.0000000000	0.000000
0.1	0.99465383	0.9946538264	-2.0E-10
0.2	0.97712221	0.9771222066	-2.0E-10
0.3	0.94490117	0.9449011654	-2.0E-10
0.4	0.89509482	0.8950948186	-2.0E-10
0.5	0.82436064	0.8243606355	-2.0E-10
0.6	0.72884752	0.7288475206	-6.0E-10
0.7	0.60412581	0.6041258131	-1.0E-9
0.8	0.44510819	0.4451081876	-2.0E-9
0.9	0.24596031	0.2459603145	-3.4E-9
1.0	0.00000000	0.0000000000	0.000000
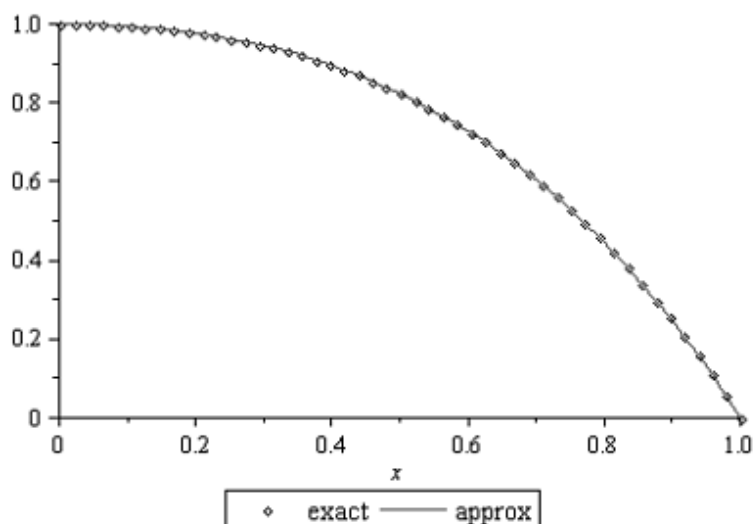
FIGURE 3. Graphical comparison between approximate solution and exact solution

4. CONCLUSIONS

In this paper, we applied the homotopy perturbation method (HPM) for finding the solution of ninth and tenth-order boundary value problems. The method is applied in a direct way without using linearization, transformation, discretization or restrictive assumptions. It may be concluded that HPM is very powerful and efficient in finding the analytical solutions for a wide class of boundary value problems. The method gives more realistic series solutions that

converge very rapidly in physical problems. It is worth mentioning that the method is capable of reducing the volume of the computational work as compare to the classical methods while still maintaining the high accuracy of the numerical result. The fact that the HPM solves nonlinear problems without using the Adomian's polynomials is a clear advantage of this technique over the decomposition method.

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