Journal of the Korean Data & Information Science Society
2010, **21**(2), 335–343

한국데이터정보과학회지

Noninformative priors for the common scale parameter in Pareto distributions [†]

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Received 27 January 2010, revised 4 March 2010, accepted 15 March 2010

Abstract

In this paper, we develop the reference priors for the common scale parameter in the nonregular Pareto distributions with unequal shape paramters. We derive the reference priors as noninformative prior and prove the propriety of joint posterior distribution under the general prior including the reference priors. Through the simulation study, we show that the proposed reference priors match the target coverage probabilities in a frequentist sense.

Keywords: Nonregular case, pareto distribution, reference prior, scale parameter.

1. Introduction

Consider X and Y are independently distributed random variables according to the Pareto distribution $\mathcal{P}(\alpha_1, \beta)$ with the shape parameter α_1 and the scale parameter β, and the Pareto distribution $\mathcal{P}(\alpha_2, \beta)$ with the shape parameter α_2 and the scale parameter β. Then the Pareto distributions of X and Y are given by

$$f(x|\alpha_1, \beta) = \alpha_1 \beta^{\alpha_1} x^{-(\alpha_1+1)}, x \geq \beta > 0, \alpha_1 > 0, \tag{1.1}$$

and

$$f(y|\alpha_2, \beta) = \alpha_2 \beta^{\alpha_2} y^{-(\alpha_2+1)}, y \geq \beta > 0, \alpha_2 > 0, \tag{1.2}$$

respectively. Here the parameter β is the common scale parameter. The present paper focuses on the reference priors for the common scale parameter.

In recent years, the notion of a noninformative prior has attracted much attention. There are different notions of noninformative prior, the reference prior approach of Bernardo (1979), which extended by Berger and Beranrdo (1989, 1992), and the approach of matching the posterior and frequentist probabilities of confidence intervals. Ghosh and Mukerjee (1992), and Berger and Bernardo (1992) gave a general algorithm to derive a reference prior

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by splitting the parameters into several groups according to their order of inferential importance. The matching idea goes back to Welch and Peers (1963). Interest in such priors revived with the work of Stein (1985) and Tibshirani (1989). Among others, we may any cite work of DiCiccio and Stern (1994), Datta and Ghosh (1995), Datta (1996) and Mukerjee and Ghosh (1997). Although this matching can be justified only asymptotically, simulation results in many works indicate that this is indeed achieved for small or moderate sizes as well.

In the above references, only smooth parametric families were considered. However, nonregular families, such as the uniform or shifted exponential, are also important in many practical problems. So Ghosal and Samanta (1997) developed the reference priors for the case of one parameter families of discontinuous densities. And Ghosal (1997) derived the reference priors for the multiparameter nonregular cases that the family of densities have discontinuities at some points which depend on one component of the parameter, while the family is regular with respect to the other parameters. Also Ghosal (1999) developed the probability matching prior for one parameter and two parameter cases under nonregular families. Quite often reference prior is the probability matching prior.

The Pareto distribution provides a statistical model which has an extensive variety of applications. It has been found in describing distributions of studies of income, property values, insurance risk, stock prices fluctuations, migration, size of cities and firms, word frequencies, occurrences of natural resources, business failures, service time in queuing systems, error clustering in communications circuits and lifetime data, etc (Arnold and Press, 1983; Fernández, 2008). The Pareto distribution has been used by many authors in a Bayesian viewpoint (e.g., Arnold and Press, 1983; Arnold and Press, 1989; Geisser, 1984; Geisser, 1985; Lwin, 1972; Nigm and Hamdy, 1987; Tiwari *et al.*, 1996; Ko and Kim, 1999; Fernández, 2008; Kim *et al.*, 2009). Lee and Lee (2008) studied likelihood based inference for the shape parameter in Pareto distribution. Also for the common scale parameter, Elfessi and Jin (1996) derived a class of improved estimators which uniformly dominates the MLE under a class of convex scale invariant loss functions. However in most of Bayesian works, many authors used the subjective priors such as conjugate prior. So there is a strong necessity for developing an objective priors. In this paper, we develop the reference priors for the common scale parameter of Pareto distributions with unequal shape parameters.

The outline of the remaining sections is as follows. In Section 2, we develop reference priors for the common scale parameter. In Section 3, we provide that the propriety of the posterior distribution for the general prior including the reference priors. In Section 4, simulated frequentist coverage probabilities under the derived priors are given.

2. The reference priors

Reference priors introduced by Bernardo (1979), and extended further by Berger and Bernardo (1992) have become very popular over the years for the development of noninformative priors. Ghosal (1997) derived the reference prior in sense of Bernardo (1979) for multiparameter nonregular cases. In this section, we derive the reference priors for different groups of orderings of $(\beta, \alpha_1, \alpha_2)$ by following Ghosal (1997).

Let $X_i, i = 1, \cdots, n_1$ denote observations from the Pareto distribution $\mathcal{P}(\alpha_1, \beta)$, and $Y_i, i = 1, \cdots, n_2$ denote observations from the Pareto distribution $\mathcal{P}(\alpha_2, \beta)$. Then likelihood function is given by

$$f(\mathbf{x},\mathbf{y}|\beta,\alpha_1,\alpha_2) = \alpha_1^{n_1}\alpha_2^{n_2}\beta^{n_1\alpha_1+n_2\alpha_2}\prod_{i=1}^{n_1} x_i^{-(\alpha_1+1)}\prod_{i=1}^{n_2} y_i^{-(\alpha_2+1)}, x_i \geq \beta, y_i \geq \beta, \quad (2.1)$$

where $\mathbf{x} = (x_1,\cdots,x_{n_1})$, $\mathbf{y} = (y_1,\cdots,y_{n_2})$, $\alpha_1 > 0, \alpha_2 > 0$ and $\beta > 0$.

We firstly derived the reference prior when β is parameter of interest. The reference prior is developed by considering a sequence of compact subsets of the parameter space, and taking the limit of a sequence of priors as these compact subsets fill out of the parameter space. The compact subsets were taken to be Cartesian products of sets of the form

$$\alpha_1 \in [a_1, b_1], \alpha_2 \in [a_2, b_2].$$

In the limit a_1, a_2 will tend to 0 and b_1, b_2 will tend to ∞. Here, and below, a subscripted Q denotes a function that is constant and does not depend on any parameter but any Q may depend on the ranges of the parameters.

From the likelihood function (2.1), the matrix $F(\beta,\alpha_1,\alpha_2)$ is given by

$$F(\beta,\alpha_1,\alpha_2) = Diag\left\{n_1\alpha_1^{-2}, n_2\alpha_2^{-2}\right\},$$

where $F(\beta,\alpha_1,\alpha_2) = \{4J_{jk}(\beta,\alpha_1,\alpha_2)\}, j,k = 1,2,$

$$J_{jk}(\beta,\alpha_1,\alpha_2) = \int\int g_{\alpha_j}(\mathbf{x},\mathbf{y};\beta,\alpha_1,\alpha_2) g_{\alpha_k}(\mathbf{x},\mathbf{y};\beta,\alpha_1,\alpha_2) d\mathbf{x}d\mathbf{y},$$

$g_{\alpha_j} = \partial g/\partial\alpha_j$ and $g = f^{\frac{1}{2}}$. Thus the reference prior for (α_1,α_2) given β is

$$\begin{aligned}\pi(\alpha_1,\alpha_2|\beta) &= [\det F(\beta,\alpha_1,\alpha_2)]^{\frac{1}{2}} \\ &= n_1^{\frac{1}{2}} n_2^{\frac{1}{2}} \alpha_1^{-1}\alpha_2^{-1}. \end{aligned} \quad (2.2)$$

The normalizing constant $K_l(\beta)$ of the reference prior $\pi(\alpha_1,\alpha_2|\beta)$ is given by

$$\begin{aligned}K_l(\beta) &= \left(\int_{a_2}^{b_2}\int_{a_1}^{b_1} [\det F(\beta,\alpha_1,\alpha_2)]^{\frac{1}{2}} d\alpha_1 d\alpha_2\right)^{-1} \\ &= \left(\int_{a_2}^{b_2}\int_{a_1}^{b_1} n_1^{\frac{1}{2}} n_2^{\frac{1}{2}}\alpha_1^{-1}\alpha_2^{-1} d\alpha_1 d\alpha_2\right)^{-1} \\ &= n_1^{-\frac{1}{2}} n_2^{-\frac{1}{2}} [\log(b_1/a_1)\log(b_2/a_2)]^{-1}, \end{aligned} \quad (2.3)$$

and so we obatain

$$p_l(\alpha_1,\alpha_2|\beta) = K_l(\beta)\pi(\alpha_1,\alpha_2|\beta) = [\log(b_1/a_1)\log(b_2/a_2)]^{-1}\alpha_1^{-1}\alpha_2^{-1}. \quad (2.4)$$

Thus the marginal reference prior for β is given by

$$\pi_l(\beta) = \exp\left\{\int_{a_2}^{b_2}\int_{a_1}^{b_1} p_l(\alpha_1,\alpha_2|\beta)\log c(\beta,\alpha_1,\alpha_2) d\alpha_1 d\alpha_2\right\} = \beta^{-1}Q(a_1,b_1,a_2,b_2), (2.5)$$

where $c(\beta,\alpha_1,\alpha_2)=E_{\beta,\alpha_1,\alpha_2}[\partial\log f/\partial\beta]=(n_1\alpha_1+n_2\alpha_2)/\beta$. Therefore the reference prior for $(\beta,\alpha_1,\alpha_2)$, when β is parameter of interest, is given by

$$\begin{aligned}\pi_1(\beta,\alpha_1,\alpha_2)&=\lim_{l\to\infty}\left[\frac{K_l(\beta)\pi_l(\beta)}{K_l(\beta_0)\pi_l(\beta_0)}\right]\pi(\alpha_1,\alpha_2|\beta)\\&\propto\beta^{-1}\alpha_1^{-1}\alpha_2^{-1},\end{aligned}\tag{2.6}$$

where β_0 is a fixed point. Also when both β and (α_1,α_2) are parameters of interest, the reference prior for $(\beta,\alpha_1,\alpha_2)$ is given by

$$\begin{aligned}\pi_2(\beta,\alpha_1,\alpha_2)&=c(\beta,\alpha_1,\alpha_2)[\det F(\beta,\alpha_1,\alpha_2)]^{\frac{1}{2}}\\&\propto\beta^{-1}\alpha_1^{-1}\alpha_2^{-1}(n_1\alpha_1+n_2\alpha_2).\end{aligned}\tag{2.7}$$

When β is parameter of interest, the reference prior for $(\beta,\alpha_1,\alpha_2)$ based on an appropriate penalty term of Ghosh and Mukerjee (1992) (and also see Ghosal, 1997) given by

$$\pi_3(\beta,\alpha_1,\alpha_2)=c(\beta,\alpha_1,\alpha_2)=\beta^{-1}(n_1\alpha_1+n_2\alpha_2).\tag{2.8}$$

3. Implementation of the Bayesian procedure

We investigate the propriety of posteriors for a general class of priors which include the reference priors (2.6), (2.7) and (2.8). We consider the class of priors

$$\pi_g(\beta,\alpha_1,\alpha_2)\propto\beta^{-1}\alpha_1^{-a}\alpha_2^{-b}(n_1\alpha_1+n_2\alpha_2)^c.\tag{3.1}$$

where $a\geq 0, b\geq 0$ and $c\geq 0$. The following general theorem can be proved.

Theorem 3.1 The posterior distribution of $(\beta,\alpha_1,\alpha_2)$ under the general prior (3.1) is proper if $n_1-a+c>0, n_2-b+1>0$ or $n_1-a+1>0, n_2-b+c>0$ when $c<1$, and if $n_1-a+1>0, n_2-b+1>0$ when $c\geq 1$.

Proof: Under the general prior (3.1), the joint posterior for β,α_1,α_2 given $\mathbf{x}$ and $\mathbf{y}$ is

$$\pi(\beta,\alpha_1,\alpha_2|\mathbf{x},\mathbf{y})\propto\alpha_1^{n_1-a}\alpha_2^{n_2-b}(n_1\alpha_1+n_2\alpha_2)^c\beta^{n_1\alpha_1+n_2\alpha_2-1}\prod_{i=1}^{n_1}x_i^{-\alpha_1}\prod_{i=1}^{n_2}y_i^{-\alpha_2}.\tag{3.2}$$

Then integrating with respect to β in (3.2), we have the posterior

$$\pi(\alpha_1,\alpha_2|\mathbf{x},\mathbf{y})\propto\alpha_1^{n_1-a}\alpha_2^{n_2-b}(n_1\alpha_1+n_2\alpha_2)^{c-1}\prod_{i=1}^{n_1}\left(\frac{x_i}{z}\right)^{-\alpha_1}\prod_{i=1}^{n_2}\left(\frac{y_i}{z}\right)^{-\alpha_2},\tag{3.3}$$

where $z=min\{x_1,\cdots,x_{n_1},y_1,\cdots,y_{n_2}\}$. First, we will prove the propriety of posterior density (3.3) for the case $c<1$. Then the posterior density (3.3) is proper if $n_1-a+c>0, n_2-b+1>0$ or $n_1-a+1>0, n_2-b+c>0$. Next, we will prove the propriety of posterior density (3.3) when $c\geq 1$. For $0<\alpha_1<1$ and $0<\alpha_2<1$,

$$\begin{aligned}\pi(\alpha_1,\alpha_2|\mathbf{x},\mathbf{y})&\leq(n_1+n_2)^{c-1}\alpha_1^{n_1-a}\alpha_2^{n_2-b}\prod_{i=1}^{n_1}\left(\frac{x_i}{z}\right)^{-\alpha_1}\prod_{i=1}^{n_2}\left(\frac{y_i}{z}\right)^{-\alpha_2}\\&\equiv\pi'(\alpha_1,\alpha_2|\mathbf{x},\mathbf{y}).\end{aligned}\tag{3.4}$$

Thus the posterior density (3.4) is proper if $n_1 - a + 1 > 0, n_2 - b + 1 > 0$. For $0 < \alpha_1 < 1$ and $\alpha_2 \geq 1$,

$$\begin{aligned}\pi(\alpha_1, \alpha_2|\mathbf{x}, \mathbf{y}) &\leq (n_1 + n_2)^{c-1} \alpha_1^{n_1 - a} \alpha_2^{n_2 - b + c - 1} \prod_{i=1}^{n_1} \left(\frac{x_i}{z}\right)^{-\alpha_1} \prod_{i=1}^{n_2} \left(\frac{y_i}{z}\right)^{-\alpha_2} \\ &\equiv \pi'(\alpha_1, \alpha_2|\mathbf{x}, \mathbf{y}). \end{aligned} \tag{3.5}$$

Thus the posterior density (3.5) is proper if $n_1 - a + 1 > 0, n_2 - b + c > 0$. For $\alpha_1 \geq 1$ and $\alpha_2 \geq 1$,

$$\begin{aligned}\pi(\alpha_1, \alpha_2|\mathbf{x}, \mathbf{y}) &\leq (n_1 + n_2)^{c-1} \alpha_1^{n_1 - a + c - 1} \alpha_2^{n_2 - b + c - 1} \prod_{i=1}^{n_1} \left(\frac{x_i}{z}\right)^{-\alpha_1} \prod_{i=1}^{n_2} \left(\frac{y_i}{z}\right)^{-\alpha_2} \\ &\equiv \pi'(\alpha_1, \alpha_2|\mathbf{x}, \mathbf{y}). \end{aligned} \tag{3.6}$$

Thus the posterior density (3.6) is proper if $n_1 - a + c > 0, n_2 - b + c > 0$. This completes the proof. □

The posterior distributions based on the reference priors in Section 2 are given as follows. Under the reference prior π_1, the marginal posterior density of β is given by

$$\pi(\beta|\mathbf{x}, \mathbf{y}) \propto \beta^{-1} \left[\sum_{i=1}^{n_1} \log x_i - n_1 \log \beta\right]^{-n_1} \left[\sum_{i=1}^{n_2} \log y_i - n_2 \log \beta\right]^{-n_2}. \tag{3.7}$$

Under the reference prior π_2, the marginal posterior density of β is given by

$$\begin{aligned}\pi(\beta|\mathbf{x}, \mathbf{y}) \propto \beta^{-1} &\left[\sum_{i=1}^{n_1} \log x_i - n_1 \log \beta\right]^{-(n_1+1)} \left[\sum_{i=1}^{n_2} \log y_i - n_2 \log \beta\right]^{-(n_2+1)} \\ &\times \left[n_2 \sum_{i=1}^{n_1} \log x_i + n_1 \sum_{i=1}^{n_2} \log y_i - 2 n_1 n_2 \log \beta\right]. \end{aligned} \tag{3.8}$$

Under the reference prior π_3, the marginal posterior density of β is given by

$$\begin{aligned}\pi(\beta|\mathbf{x}, \mathbf{y}) \propto \beta^{-1} &\left[\sum_{i=1}^{n_1} \log x_i - n_1 \log \beta\right]^{-(n_1+2)} \left[\sum_{i=1}^{n_2} \log y_i - n_2 \log \beta\right]^{-(n_2+2)} \\ &\times \left[n_2 \sum_{i=1}^{n_1} \log x_i + n_1 \sum_{i=1}^{n_2} \log y_i - 2 n_1 n_2 \log \beta\right]. \end{aligned} \tag{3.9}$$

Note that normalizing constant for the marginal density of β requires an one dimensional integration. Therefore we can have the marginal posterior density of β and so we compute the marginal moment of β. In Section 4, we investigate the frequentist coverage probabilities for the reference priors π_1, π_2 and π_3, respectively.

4. Numerical study

We investigate the frequentist coverage probability by investigating the credible interval of the marginal posteriors density of β under the noninformative prior π given in Section 3 for several configurations $(\beta, \alpha_1, \alpha_2)$ and (n_1, n_2). That is to say, the frequentist coverage of a $100(1-\alpha)\%th$ posterior quantile should be close to $1-\alpha$. This is done numerically. Table 4.1 and 4.2 give numerical values of the frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for the proposed priors. The computation of these numerical values is based on the following algorithm for any fixed true $(\beta, \alpha_1, \alpha_2)$ and any prespecified value α. Here α is 0.05 (0.95).

Table 4.1 Frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for β

β	α_1, α_2	(n_1, n_2)	π_1	π_2	π_3
0.1	0.1,0.1	5,5	0.070 (0.959)	0.092 (0.962)	0.136 (0.969)
		5,10	0.063 (0.950)	0.078 (0.953)	0.106 (0.959)
		10,10	0.059 (0.952)	0.067 (0.954)	0.085 (0.957)
		10,15	0.059 (0.949)	0.068 (0.951)	0.084 (0.955)
	0.1,0.5	5,5	0.062 (0.952)	0.097 (0.958)	0.141 (0.965)
		5,10	0.056 (0.953)	0.068 (0.956)	0.089 (0.960)
		10,10	0.055 (0.953)	0.071 (0.956)	0.090 (0.959)
		10,15	0.049 (0.951)	0.059 (0.954)	0.070 (0.957)
	0.1,1.0	5,5	0.058 (0.953)	0.090 (0.961)	0.134 (0.968)
		5,10	0.054 (0.951)	0.071 (0.955)	0.089 (0.960)
		10,10	0.053 (0.952)	0.069 (0.956)	0.087 (0.960)
		10,15	0.052 (0.951)	0.061 (0.954)	0.073 (0.956)
	1.0,1.0	5,5	0.069 (0.950)	0.094 (0.956)	0.138 (0.962)
		5,10	0.066 (0.954)	0.079 (0.958)	0.107 (0.964)
		10,10	0.058 (0.953)	0.066 (0.956)	0.086 (0.960)
		10,15	0.054 (0.951)	0.060 (0.953)	0.074 (0.957)
	1.0,5.0	5,5	0.062 (0.954)	0.093 (0.959)	0.142 (0.965)
		5,10	0.055 (0.949)	0.069 (0.952)	0.089 (0.957)
		10,10	0.055 (0.953)	0.067 (0.955)	0.087 (0.959)
		10,15	0.054 (0.952)	0.062 (0.955)	0.074 (0.959)
	1.0,10.0	5,5	0.049 (0.954)	0.083 (0.962)	0.128 (0.968)
		5,10	0.053 (0.948)	0.067 (0.953)	0.086 (0.956)
		10,10	0.058 (0.952)	0.072 (0.955)	0.090 (0.959)
		10,15	0.050 (0.950)	0.059 (0.953)	0.071 (0.956)
1.0	0.1,0.1	5,5	0.069 (0.957)	0.092 (0.961)	0.138 (0.968)
		5,10	0.064 (0.953)	0.079 (0.956)	0.105 (0.962)
		10,10	0.055 (0.955)	0.064 (0.958)	0.082 (0.963)
		10,15	0.054 (0.951)	0.061 (0.954)	0.074 (0.957)
	0.1,0.5	5,5	0.059 (0.953)	0.089 (0.960)	0.135 (0.966)
		5,10	0.052 (0.953)	0.065 (0.957)	0.086 (0.961)
		10,10	0.056 (0.948)	0.069 (0.953)	0.086 (0.957)
		10,15	0.053 (0.953)	0.063 (0.955)	0.075 (0.959)
	0.1,1.0	5,5	0.058 (0.951)	0.095 (0.958)	0.139 (0.965)
		5,10	0.053 (0.953)	0.069 (0.957)	0.090 (0.961)
		10,10	0.053 (0.950)	0.070 (0.954)	0.088 (0.958)
		10,15	0.051 (0.951)	0.058 (0.954)	0.073 (0.957)
	1.0,1.0	5,5	0.066 (0.957)	0.088 (0.960)	0.132 (0.967)
		5,10	0.063 (0.953)	0.075 (0.956)	0.102 (0.960)
		10,10	0.059 (0.953)	0.069 (0.955)	0.090 (0.959)
		10,15	0.054 (0.953)	0.063 (0.954)	0.076 (0.958)
	1.0,5.0	5,5	0.058 (0.954)	0.090 (0.961)	0.137 (0.966)
		5,10	0.057 (0.950)	0.071 (0.954)	0.092 (0.959)
		10,10	0.052 (0.952)	0.066 (0.956)	0.084 (0.960)
		10,15	0.050 (0.954)	0.059 (0.956)	0.070 (0.958)
	1.0,10.0	5,5	0.060 (0.952)	0.095 (0.959)	0.139 (0.964)
		5,10	0.055 (0.951)	0.070 (0.956)	0.089 (0.960)
		10,10	0.054 (0.955)	0.068 (0.958)	0.087 (0.963)
		10,15	0.058 (0.953)	0.067 (0.956)	0.078 (0.959)

Table 4.2 Frequentist coverage probabilities of 0.05 (0.95) posterior quantiles for β

β	α_1, α_2	(n_1, n_2)	π_1	π_2	π_3
5.0	0.1,0.1	5,5	0.063 (0.954)	0.089 (0.959)	0.133 (0.966)
		5,10	0.064 (0.950)	0.080 (0.955)	0.104 (0.961)
		10,10	0.062 (0.954)	0.072 (0.957)	0.093 (0.960)
		10,15	0.057 (0.953)	0.064 (0.955)	0.078 (0.958)
	0.1,0.5	5,5	0.059 (0.951)	0.091 (0.958)	0.139 (0.965)
		5,10	0.055 (0.949)	0.070 (0.954)	0.089 (0.959)
		10,10	0.056 (0.957)	0.069 (0.960)	0.088 (0.963)
		10,15	0.052 (0.946)	0.061 (0.949)	0.073 (0.952)
	0.1,1.0	5,5	0.057 (0.953)	0.089 (0.959)	0.134 (0.965)
		5,10	0.052 (0.950)	0.067 (0.955)	0.087 (0.959)
		10,10	0.053 (0.950)	0.068 (0.953)	0.088 (0.957)
		10,15	0.053 (0.949)	0.062 (0.951)	0.075 (0.955)
	1.0,1.0	5,5	0.068 (0.958)	0.091 (0.964)	0.133 (0.970)
		5,10	0.060 (0.954)	0.074 (0.958)	0.101 (0.963)
		10,10	0.059 (0.951)	0.071 (0.953)	0.090 (0.957)
		10,15	0.056 (0.952)	0.063 (0.953)	0.077 (0.957)
	1.0,5.0	5,5	0.059 (0.952)	0.087 (0.958)	0.131 (0.964)
		5,10	0.052 (0.947)	0.066 (0.951)	0.086 (0.955)
		10,10	0.057 (0.955)	0.069 (0.958)	0.088 (0.961)
		10,15	0.051 (0.951)	0.059 (0.954)	0.072 (0.957)
	1.0,10.0	5,5	0.056 (0.949)	0.093 (0.955)	0.136 (0.963)
		5,10	0.053 (0.953)	0.069 (0.957)	0.090 (0.962)
		10,10	0.051 (0.951)	0.065 (0.955)	0.083 (0.958)
		10,15	0.051 (0.948)	0.061 (0.949)	0.074 (0.952)
10.0	0.1,0.1	5,5	0.072 (0.953)	0.096 (0.959)	0.141 (0.965)
		5,10	0.063 (0.954)	0.077 (0.957)	0.104 (0.963)
		10,10	0.063 (0.957)	0.074 (0.959)	0.094 (0.962)
		10,15	0.058 (0.955)	0.065 (0.957)	0.080 (0.961)
	0.1,0.5	5,5	0.063 (0.954)	0.092 (0.962)	0.135 (0.968)
		5,10	0.058 (0.952)	0.071 (0.956)	0.091 (0.961)
		10,10	0.056 (0.950)	0.069 (0.954)	0.086 (0.957)
		10,15	0.058 (0.950)	0.066 (0.953)	0.079 (0.955)
	0.1,1.0	5,5	0.059 (0.952)	0.094 (0.962)	0.139 (0.968)
		5,10	0.052 (0.956)	0.067 (0.960)	0.087 (0.962)
		10,10	0.051 (0.950)	0.067 (0.954)	0.087 (0.959)
		10,15	0.050 (0.952)	0.059 (0.954)	0.071 (0.957)
	1.0,1.0	5,5	0.065 (0.954)	0.089 (0.959)	0.134 (0.966)
		5,10	0.062 (0.953)	0.075 (0.957)	0.103 (0.962)
		10,10	0.056 (0.953)	0.065 (0.956)	0.083 (0.960)
		10,15	0.058 (0.952)	0.066 (0.954)	0.083 (0.958)
	1.0,5.0	5,5	0.059 (0.955)	0.087 (0.962)	0.131 (0.967)
		5,10	0.052 (0.950)	0.067 (0.954)	0.089 (0.958)
		10,10	0.056 (0.952)	0.068 (0.955)	0.085 (0.961)
		10,15	0.052 (0.950)	0.060 (0.952)	0.073 (0.956)
	1.0,10.0	5,5	0.057 (0.953)	0.092 (0.960)	0.135 (0.966)
		5,10	0.052 (0.951)	0.066 (0.954)	0.085 (0.959)
		10,10	0.052 (0.952)	0.067 (0.956)	0.087 (0.959)
		10,15	0.053 (0.949)	0.062 (0.952)	0.074 (0.955)

Let $\beta^\pi(\alpha|\mathbf{x},\mathbf{y})$ be the posterior α-quantile of β given $\mathbf{x}$ and $\mathbf{y}$. That is to say, $F(\beta^\pi(\alpha|\mathbf{x},\mathbf{y})|\mathbf{x},\mathbf{y}) = \alpha$, where $F(\cdot|\mathbf{x},\mathbf{y})$ is the marginal posterior distribution of β. Then the frequentist coverage probability of this one sided credible interval of β is

$$P_{(\beta,\alpha_1,\alpha_2)}(\alpha,\beta) = P_{(\beta,\alpha_1,\alpha_2)}(0 < \beta < \beta^\pi(\alpha|\mathbf{x},\mathbf{y})). \tag{4.1}$$

The estimated $P_{(\beta,\alpha_1,\alpha_2)}(\alpha,\beta)$ when $\alpha = 0.05(0.95)$ is shown in Tables 4.1 and 4.2. In particular, for fixed $(\beta,\alpha_1,\alpha_2)$, we take 10,000 independent random samples of $\mathbf{X}$ and $\mathbf{Y}$ from the model (2.1).

For the cases presented in Tables 4.1 and 4.2, we see that the reference prior π_1 matches the target coverage probability much more accurately than the reference priors π_2 and π_3 for values of $(\beta,\alpha_1,\alpha_2)$ and values of (n_1,n_2). Note that the reference prior π_1 is the prior when β is parameter of interest, and the results of tables are not much sensitive to change of the values of (α_1, α_2). Thus we recommend to use the reference prior π_1 in the sense of asymptotic frequentist coverage property.

5. Concluding remarks

In the nonregular Pareto distributions, we have found reference priors for the common scale parameter. We derived the reference priors when β is parameter of interest, and both β and (α_1, α_2) are parameters of interest. We showed that the reference prior π_1 when β is parameter of interest performs better than the reference priors π_2 and π_3 in matching the target coverage probabilities. Thus we recommend the use of the reference prior π_1 for the Bayesian inference in two independent Pareto distributions with common scale parameter.

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