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Default Bayesian testing for normal mean with known coefficient of variation

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Abstract

This article deals with the problem of testing mean when the coefficient of variation in normal distribution is known. We propose Bayesian hypothesis testing procedures for the normal mean under the noninformative prior. The noninformative prior is usually improper which yields a calibration problem that makes the Bayes factor to be defined up to a multiplicative constant. So we propose the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor under the reference prior. Specially, we develop intrinsic priors which give asymptotically same Bayes factor with the intrinsic Bayes factor under the reference prior. Simulation study and a real data example are provided.

Keywords: Coefficient of variation, fractional Bayes factor, intrinsic Bayes factor, intrinsic prior, normal mean, reference prior.

1. Introduction

Normal distribution has been widely used to model various phenomena in agricultural, biological, environmental and physical sciences. A problem of interest is the inference concerning normal mean when the coefficient of variation is known. The assumption of a known coefficient of variation is actually common in many clinical chemistry and pharmaceutical sciences (Gleser and Healy, 1976).

For example, when batches of some substance or chemicals are to be analyzed, if sufficient batches of the substances are analyzed, their coefficient of variation will be known. In agricultural experiments, it is customary to conduct multi-locational trials. When the results of a few centers are available, the coefficient of variation is known and it can be used for the inferential purpose of an experiment to be conducted in a new location. In environmental studies, such situations arise when the standard deviation of a pollutant is directly related

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to the mean (Bhat and Rao, 2007). Many Researches considered the problem of estimation of the normal mean when the coefficient of variation is known (Gleser and Healy, 1976; Soofi and Gokhale, 1991; Arnholt and Hebert, 1995; Guo and Pal, 2003; and the references cited therein).

Although the problem of estimation of the normal mean has drawn the interest of many researchers, the tests for normal mean has received a few attention. Hinkley (1977) shows that the usual t statistic is ancillary for the mean and developed a conditional test for the one-sided hypothesis. Although Hinkley (1977) advocated the use of the conditional test in lieu of the locally most powerful one-sided test, the conditional test does not seem to be popular with the practitioners. So Bhat and Rao (2007) derived the likelihood ratio test and Wald test for the one-sided and two-sided alternatives, as well as the two-sided version of the locally most powerful test. They showed that for the two-sided alternatives, the likelihood ratio test and the Wald test are more powerful than other tests.

However, there is a little work in this problem from the viewpoint of Bayesian framework. In Bayesian model selection or testing problem, the Bayes factor under proper priors or informative priors have been very successful. However, limited information and time constraints often require the use of noninformative priors. Since noninformative priors such as Jeffreys' prior or reference prior (Berger and Bernardo, 1989, 1992) are typically improper so that such priors are only defined up to arbitrary constants which affects the values of Bayes factors. Spiegelhalter and Smith (1982), O'Hagan (1995) and Berger and Pericchi (1996) have made efforts to compensate for that arbitrariness.

Spiegelhalter and Smith (1982) used the device of imaginary training samples in the context of linear model comparisons to choose the arbitrary constants. But the choice of imaginary training sample depends on the models under comparison, and so, there is no guarantee that the Bayes factor of Spiegelhalter and Smith (1982) is coherent for multiple model comparisons. Berger and Pericchi (1996) introduced the intrinsic Bayes factor using a data-splitting idea, which would eliminate the arbitrariness of improper priors. O'Hagan (1995) proposed the fractional Bayes factor. For removing the arbitrariness he used to a portion of the likelihood with a so-called the fraction b. These approaches have shown to be quite useful in many statistical areas (Kang, *et al.* 2005, 2006, 2007). An excellent exposition of the objective Bayesian method to model selection is Berger and Pericchi (2001).

In this paper, we propose the objective Bayesian hypothesis testing procedures based on the Bayes factors for the mean of the normal distribution with known coefficient of variation. The outline of the remaining sections is as follows. In Section 2, we introduce the Bayesian hypothesis testing based on the Bayes factor. In Section 3, using the reference prior, we provide the Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor. In Section 4, simulation study and a real data example are given.

2. Intrinsic and fractional Bayes factors

Suppose that hypotheses $H_1, H_2, \cdots, H_q$ are under consideration, with the data $\mathbf{x} = (x_1, x_2, \cdots, x_n)$ having probability density function $f_i(\mathbf{x}|\theta_i)$ under hypothesis H_i. The parameter vector θ_i is unknown. Let $\pi_i(\theta_i)$ be the prior distributions of hypothesis H_i, and let p_i be the prior probability of hypothesis H_i, $i = 1, 2, \cdots, q$. Then the posterior probability that the hypothesis H_i is true is

$$P(H_i|\mathbf{x}) = \left(\sum_{j=1}^{q} \frac{p_j}{p_i} \cdot B_{ji} \right)^{-1}, \tag{2.1}$$

where B_{ji} is the Bayes factor of hypothesis H_j to hypothesis H_i defined by

$$B_{ji} = \frac{\int f_j(\mathbf{x}|\theta_j)\pi_j(\theta_j)d\theta_j}{\int f_i(\mathbf{x}|\theta_i)\pi_i(\theta_i)d\theta_i} = \frac{m_j(\mathbf{x})}{m_i(\mathbf{x})}. \tag{2.2}$$

The B_{ji} interpreted as the comparative support of the data for H_j versus H_i. The computation of B_{ji} needs specification of the prior distribution $\pi_i(\theta_i)$ and $\pi_j(\theta_j)$. Often in Bayesian analysis, one can use noninformative priors π_i^N. Common choices are the uniform prior, Jeffreys' prior and the reference prior. The noninformative prior π_i^N is typically improper. Hence the use of noninformative prior π_i^N in (2.2) causes the B_{ji} to contain unspecified constants. To solve this problem, Berger and Pericchi (1996) proposed the intrinsic Bayes factor, and O'Hagan (1995) proposed the fractional Bayes factor.

One solution to this indeterminacy problem is to use part of the data as a training sample. Let $\mathbf{x}(l)$ denote the part of the data to be so used and let $\mathbf{x}(-l)$ be the remainder of the data, such that

$$0 < m_i^N(\mathbf{x}(l)) < \infty, i = 1, \cdots, q. \tag{2.3}$$

In view (2.3), the posteriors $\pi_i^N(\theta_i|\mathbf{x}(l))$ are well defined. Now, consider the Bayes factor,$B_{ji}(l)$, with the remainder of the data $\mathbf{x}(-l)$, using $\pi_i^N(\theta_i|\mathbf{x}(l))$ as the priors:

$$B_{ji}(l) = \frac{\int f(\mathbf{x}(-l)|\theta_j, \mathbf{x}(l))\pi_j^N(\theta_j|\mathbf{x}(l))d\theta_j}{\int f(\mathbf{x}(-l)|\theta_i, \mathbf{x}(l))\pi_i^N(\theta_i|\mathbf{x}(l))d\theta_i} = B_{ji}^N \cdot B_{ij}^N(\mathbf{x}(l)) \tag{2.4}$$

where

$$B_{ji}^N = B_{ji}^N(\mathbf{x}) = \frac{m_j^N(\mathbf{x})}{m_i^N(\mathbf{x})}$$

and

$$B_{ij}^N(\mathbf{x}(l)) = \frac{m_i^N(\mathbf{x}(l))}{m_j^N(\mathbf{x}(l))}$$

are the Bayes factors that would be obtained for the full data $\mathbf{x}$ and training samples $\mathbf{x}(l)$, respectively.

Berger and Pericchi (1996) proposed the use of a minimal training sample to compute $B_{ij}^N(\mathbf{x}(l))$. Then, an average over all the possible minimal training samples contained in the sample is computed. Thus the arithmetic intrinsic Bayes factor (AIBF) of H_j to H_i is

$$B_{ji}^{AI} = B_{ji}^N \times \frac{1}{L} \sum_{l=1}^{L} B_{ij}^N(\mathbf{x}(l)), \tag{2.5}$$

where L is the number of all possible minimal training samples. Also the median intrinsic Bayes factor (MIBF) by Berger and Pericchi (1998) of H_j to H_i is

$$B_{ji}^{MI} = B_{ji}^N \times ME[B_{ij}^N(\mathbf{x}(l))], \tag{2.6}$$

where ME indicates the median for all the training sample Bayes factors.

Therefore we can also calculate the posterior probability of H_i using (2.1), where B_{ji} is replaced by B_{ji}^{AI} and B_{ji}^{MI} from (2.5)and (2.6), respectively.

The fractional Bayes factor (O'Hagan, 1995) is based on a similar intuition to that behind the intrinsic Bayes factor but, instead of using part of the data to turn noninformative priors into proper priors, it uses a fraction,b ,of each likelihood function, $L(\theta_i) = f_i(\mathbf{x}|\theta_i)$, with the remaining $1-b$ fraction of the likelihood used for model discrimination. Then the fractional Bayes factor (FBF) of hypothesis H_j versus hypothesis H_i is

$$B_{ji}^F = B_{ji}^N \cdot \frac{\int L^b(\mathbf{x}|\theta_i)\pi_i^N(\theta_i)d\theta_i}{\int L^b(\mathbf{x}|\theta_j)\pi_j^N(\theta_j)d\theta_j} = B_{ji}^N \cdot \frac{m_i^b(\mathbf{x})}{m_j^b(\mathbf{x})}. \tag{2.7}$$

O'Hagan (1995) proposed three ways for the choice of the fraction b. One common choice of b is $b = m/n$, where m is the size of the minimal training sample, assuming that this number is uniquely defined. (O'Hagan, 1995, 1997; the discussion by Berger and Mortera in O'Hagan, 1995).

3. Bayesian hypothesis testing procedures

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed random variables from a normal distribution $N(\mu, c^2\mu^2)$, where c is the coefficient of variation. Then the joint probability density function is

$$f(\mathbf{x}|\mu) = (2\pi)^{-n/2}(|c\mu|)^{-n} \exp\left\{-\frac{S^2 + n(\overline{x}-\mu)^2}{2c^2\mu^2}\right\}, \tag{3.1}$$

where $\mathbf{x} = (x_1, x_2, \cdots, x_n)$, $\overline{x} = \sum_{i=1}^n x_i/n$ and $S^2 = \sum_{i=1}^n (x_i - \overline{x})^2$.

We are interested in testing the hypotheses $H_1 : \mu = \mu_0$ versus $H_2 : \mu \neq \mu_0$ based on the fractional Bayes factor and the intrinsic Bayes factor. From the likelihood function (3.1), the reference prior for is

$$\pi^N(\mu) \propto |\mu|^{-1}$$

and the posterior under this reference prior is proper (See Appendix 1).

3.1. Bayesian hypothesis testing based on the fractional bayes factor

The likelihood function under the hypothesis $H_1 : \mu = \mu_0$ is

$$L(\mu_0|\mathbf{x}) = (2\pi)^{-n/2}(|c\mu_0|)^{-n} \exp\left\{-\frac{S^2 + n(\overline{x}-\mu_0)^2}{2c^2\mu_0^2}\right\}. \tag{3.2}$$

Then from the likelihood (3.2), the element of the FBF under H_1 is given by

$$m_1^b(\mathbf{x}) = (2\pi)^{-bn/2}(|c\mu_0|)^{-bn} \exp\left\{-\frac{b[S^2 + n(\overline{x}-\mu_0)^2]}{2c^2\mu_0^2}\right\}.$$

For the hypothesis $H_2 : \mu \neq \mu_0$, the reference prior for μ is

$$\pi_2^N(\mu) \propto |\mu|^{-1}. \tag{3.3}$$

The likelihood function under the hypothesis $H_2 : \mu \neq \mu_0$ is

$$L(\mu|\mathbf{x}) = (2\pi)^{-n/2}(|c\mu|)^{-n} \exp\left\{-\frac{S^2 + n(\overline{x}-\mu)^2}{2c^2\mu^2}\right\}. \tag{3.4}$$

Thus from the likelihood (3.4) and the reference prior (3.3), the element of FBF under $H_2 : \mu \neq \mu_0$ is given as follows.

$$\begin{aligned} m_2^b(\mathbf{x}) &= \int_{-\infty}^{\infty} L^b(\mu|\mathbf{x})\pi_2^N(\mu)d\mu \\ &= (2\pi)^{-\frac{nb}{2}} 2^{\frac{nb}{2}} \Gamma\left(\frac{nb}{2}\right)[b(S^2+n\overline{x}^2)]^{-nb/2}\exp\{-\frac{nb}{2c^2}\} \\ &\quad \times Hypergeometric_1F_1\left[\frac{nb}{2}, \frac{1}{2}, \frac{n^2b\overline{x}^2}{2c^2(S^2+n\overline{x}^2)}\right], \end{aligned}$$

where $Hypergeometric_1F_1[a,b,c]$ is the Kummer confluent hypergeometric function. Therefore the element B_{21}^N of FBF is given by

$$B_{21}^N = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}, \tag{3.5}$$

where

$$S_1(\mathbf{x}) = (|c\mu_0|)^{-n}\exp\left\{-\frac{S^2+n(\overline{x}-\mu_0)^2}{2c^2\mu_0^2}\right\}$$

and

$$S_2(\mathbf{x}) = 2^{\frac{n}{2}}\Gamma(\frac{n}{2})(S^2+n\overline{x}^2)^{-n/2}\exp\{-\frac{n}{2c^2}\}Hypergeometric_1F_1\left[\frac{n}{2}, \frac{1}{2}, \frac{n^2\overline{x}^2}{2c^2(S^2+n\overline{x}^2)}\right].$$

And the ratio of marginal densities with fraction b is

$$\frac{m_1^b(\mathbf{x})}{m_2^b(\mathbf{x})} = \frac{S_1(\mathbf{x};b)}{S_2(\mathbf{x};b)},$$

where

$$S_1(\mathbf{x};b) = (|c\mu_0|)^{-nb}\exp\left\{-\frac{b[S^2+n(\overline{x}-\mu_0)^2]}{2c^2\mu_0^2}\right\}$$

and

$$S_2(\mathbf{x};b) = 2^{\frac{nb}{2}}\Gamma[\frac{nb}{2}][b(S^2+n\overline{x}^2)]^{-\frac{nb}{2}}\exp\{-\frac{nb}{2c^2}\}Hypergeometric_1F_1\left[\frac{nb}{2}, \frac{1}{2}, \frac{n^2b\overline{x}^2}{2c^2(S^2+n\overline{x}^2)}\right].$$

Thus the FBF of H_2 versus H_1 is given by

$$B_{21}^F = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}\cdot\frac{S_1(\mathbf{x};b)}{S_2(\mathbf{x};b)}. \tag{3.6}$$

Note that the calculations of the FBF of H_2 versus H_1 requires only one dimensional integration.

3.2. Bayesian hypothesis testing based on the intrinsic Bayes factor

The element B_{21}^N of the intrinsic Bayes factor is computed in the fractional Bayes factor. So under minimal training sample, we only calculate the marginal densities for the hypotheses H_1 and H_2, respectively. The marginal density of X_j is finite for all j,$j = 1, 2, \cdots, n$ under each hypothesis (See Appendix). Thus we conclude that any training sample of size 1 is a minimal training sample.

The marginal density $m_1^N(x_j)$ under H_1 is given by

$$m_1^N(x_j) = (2\pi)^{-1/2}(|c\mu_0|)^{-1}\exp\left\{-\frac{(x_j-\mu_0)^2}{2c^2\mu_0^2}\right\}.$$

And the marginal density $m_2^N(x_j)$ under H_2 is given by

$$\begin{aligned} m_2^N(x_j) &= \int_{-\infty}^{\infty} f(x_j|\mu)\pi_2^N(\mu)d\mu \\ &= \int_{-\infty}^{\infty} \frac{\exp\{-\frac{(x_j-\mu)^2}{2c^2\mu^2}\}}{\sqrt{2\pi}|c\mu|}\frac{1}{|\mu|}d\mu \\ &= (2\pi)^{-\frac{1}{2}}\frac{\sqrt{2\pi}}{|x_j|} \\ &= \frac{1}{|x_j|}. \end{aligned}$$

Therefore the AIBF of H_2 versus H_1 is given by

$$B_{21}^{AI} = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}\left[\frac{1}{n}\sum_{j=1}^{n}\frac{T_1(x_j)}{T_2(x_j)}\right], \tag{3.7}$$

where

$$T_1(x_j) = (|c\mu_0|)^{-1}\exp\left\{-\frac{(x_j-\mu_0)^2}{2c^2\mu_0^2}\right\}$$

and

$$T_2(x_j) = \frac{\sqrt{2\pi}}{|x_j|}.$$

And also the MIBF of H_2 versus H_1 is given by

$$B_{21}^{MI} = \frac{S_2(\mathbf{x})}{S_1(\mathbf{x})}ME\left[\frac{T_1(x_j)}{T_2(x_j)}\right]. \tag{3.8}$$

Note that the calculations of the AIBF and the MIBF of H_2 versus H_1 require only one dimensional integration.

3.3. Bayesian hypothesis testing based on intrinsic prior

Berger and Pericchi (1996) introduced intrinsic prior which gives asymptotically equivalent to IBF. It is very useful in many ways like dealing with the problems with AIBF such as unstableness and coherence problems. But in spite of its merits, on the developing these prior, there is a problem that the solution to get intrinsic prior may have many solutions.

Moreno, et. al. (1998) proposed an intrinsic limiting procedure to determine an unique intrinsic prior. Following theirs steps, we develop intrinsic prior for our problem.

It essentially consists in considering a fixed point θ_1 and finding the conditional intrinsic prior $\pi_2^I(\theta_2|\theta_1)$ which is unique and given by

$$\pi_2^I(\theta_2|\theta_1) = \pi_2^N(\theta_2) E_{x(l)|\theta_2}^{H_2}[\tilde{B}_{12}^N(x(l))],$$

where

$$\tilde{B}_{12}^N(x(l)) = \frac{f_1(x(l)|\theta_1)}{m_2^N(x(l))}.$$

This prior does not depend on any arbitrary constant and it is a probability density function. Then the arbitrary point of θ_1 is integrated out by $\pi_1^N(\theta_1)$, that is

$$\pi_2^I(\theta_2) = \int \pi_2^N(\theta_2) E_{x(l)|\theta_2}^{M_2}[\tilde{B}_{12}^N(x(l))]\pi_1^N(\theta_1)d\theta_1.$$

Note that the pair $\pi_1^N(\theta_1)$ and $\pi_2^I(\theta_2)$, although improper, are well-calibrated in the sense that both depends on the same arbitrary constant. Further, they are justified by using a limiting argumentation: even when they are not proper the associated Bayes factor is a well-defined limit of actual Bayes factors for priors which are probability densities (Moreno *et al.*, 1998). Interestingly, the intrinsic priors $\left\{\pi_1^N(\theta_1), \pi_2^I(\theta_2)\right\}$ have proved to behave extremely well in a wide variety of problems involving nested models (Bertolino *et al.*, 2000; Moreno *et al.*, 1999, 2005; Moreno and Liseo, 2003).

From now on, we develop intrinsic priors $\left\{\pi_1^N(\theta_1), \pi_2^I(\theta_2)\right\}$ for our problem. Since,

$$E_{x(l)|\mu_1}^{H_2}[\tilde{B}_{12}^N(x(l))] = (2\pi)^{-1/2}|c\mu_0|^{-1} E_{x(l)|\mu_1}^{H_2}\left[|X_j| \exp\left\{-\frac{(X_j-\mu_0)^2}{2c^2\mu_0^2}\right\}\right],$$

after some calculation, we can find that

$$E_{x(l)|\mu}^{H_2}\left[|X_j| \exp\left\{\frac{-(X_j-\mu_0)^2}{2c^2\mu_0^2}\right\}\right] =$$
$$\frac{1}{\sqrt{2\pi}|c\mu|}\left\{\frac{c^2e^{-c^{-2}}(\mu\mu_0)^2}{\mu^2+\mu_0^2}\left(2+e^{\frac{(\mu+\mu_0)^2}{2c^2(\mu^2+\mu_0^2)}}\sqrt{2\pi}\sqrt{\frac{(\mu+\mu_0)^2}{c^2(\mu^2+\mu_0^2)}}Erf\left[\sqrt{\frac{(\mu+\mu_0)^2}{2c^2(\mu^2+\mu_0^2)}}\right]\right)\right\},$$

where

$$Erf(a) = \frac{2}{\sqrt{\pi}}\int_0^a e^{-t^2}dt.$$

And the conditional intrinsic prior is given by

$$\begin{aligned}\pi_2^I(\mu|\mu_0) &= \pi_2^N(\mu) \times E_{x(l)|\mu}^{H_2}[\tilde{B}_{12}^N(x(l))] \\ &= \sqrt{\frac{2}{\pi}}\frac{|c|(\mu\mu_0)^2}{(\mu^2+\mu_0^2)}e^{-\frac{1}{c^2}} + \frac{(\mu\mu_0)^2|\mu+\mu_0|}{(\mu^2+\mu_0^2)^{3/2}}e^{-\frac{(\mu-\mu_0)^2}{2c^2(\mu^2+\mu_0^2)}}Erf\left[\sqrt{\frac{(\mu+\mu_0)^2}{2c^2(\mu^2+\mu_0^2)}}\right].\end{aligned}$$

Since, $\pi_1^N(\mu) = I(\mu = \mu_0)$, then $\pi_2^I(\mu_1) = \pi_2^I(\mu_1|\mu_0)$.

Now, we apply these intrinsic priors to calculate Bayes factor for our testing problem. For testing H_2 verses H_1, the Bayes factor under intrinsic priors is

$$B_{21}^{IP} = \frac{m_2^{IP}(\mathbf{x})}{m_1^{IP}(\mathbf{x})}, \tag{3.9}$$

where

$$m_1^{IP}(\mathbf{x}) = (2\pi)^{-n/2}(|c\mu_0|)^{-n} \exp\left\{-\frac{s^2 + n(\overline{x} - \mu_0)^2}{2c^2\mu_0^2}\right\}$$

and

$$m_2^{IP}(\mathbf{x}) = \int_{-\infty}^{\infty} (2\pi)^{-n/2}(|c\mu|)^{-n} e^{-\frac{s^2+n(\overline{x}-\mu)^2}{2c^2\mu^2}} \times \left(\sqrt{\frac{2}{\pi}}\frac{|c|(\mu\mu_0)^2}{(\mu^2+\mu_0^2)}e^{-\frac{1}{c^2}} + \frac{(\mu\mu_0)^2|\mu+\mu_0|}{(\mu^2+\mu_0^2)^{3/2}}e^{-\frac{(\mu-\mu_0)^2}{2c^2(\mu^2+\mu_0^2)}} Erf\left[\sqrt{\frac{(\mu+\mu_0)^2}{2c^2(\mu^2+\mu_0^2)}}\right]\right) d\mu.$$

It needs only one dimensional integration to evaluate the Bayes factor based on intrinsic priors.

4. Numerical studies

In order to assess the Bayesian hypothesis testing procedures, we evaluate the posterior probability for several configurations of (c, μ) and n. In particular, for fixed (c, μ), we take 1,000 independent random samples of X_i with sample size n from the model (3.1). We want to test the hypotheses $H_1 : \mu = 5$ versus $H_2 : \mu \neq 5$. The posterior probabilities of H_1 being true are computed assuming equal prior probabilities. Tables 4.1 and 4.2 show the results of the averages and the standard deviations in parentheses of posterior probabilities. From Tables 4.1 and 4.2, the FBF, the AIBF, the MIBF and intrinsic priors give fairly reasonable answers for all configurations. Also the AIBF, the MIBF and intrinsic priors give a similar behavior for all sample sizes, and the FBF favors the hypothesis H_2 than the AIBF, the MIBF and intrinsic priors.

Example. In a bioequivalence study of two formulations of a drug product presented in Wu and Jiang (2001), we now consider the period 1 data (Cmax data) presented in their Table 3. Wu and Jiang (2001) showed that the Cmax data follow the normal distribution. These data are as follows;

18.25, 37.99, 24.09, 36.47, 24.60, 29.25, 28.27, 32.77 25.79, 32.50, 32.41, 19.52, 31.13

For this data, the sample mean is 28.695, the sample standard deviation are 5.803 and coefficient of variation is 0.202. So we put $c = 0.202$. We want to test the hypotheses $H_1 : \mu = 29$ versus $H_2 : \mu \neq 29$. The p-value based on the likelihood ratio test (Bhat and Rao, 2007) and the values of the Bayes factor and the posterior probabilities of H_1 are given in Table 4.2. In Table 4.2, $P^F(\cdot)$,$P^{AI}(\cdot)$, $P^{MI}(\cdot)$ and $P^{IP}(\cdot)$ are the posterior probabilities of the hypothesis H_1 being true based on FBF, AIBF, MIBF and intrinsic priors, respectively. From the results of Table 4.2, the p-value and posterior probabilities based on various Bayes factors give the same answer, and select the hypothesis H_1. The

Table 4.1 The averages and the standard deviations in parentheses of posterior probabilities

μ	n	$P^F(H_1\|\mathbf{x})$	$P^{AI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$	$P^{IP}(H_1\|\mathbf{x})$
			$c = 0.1$		
	5	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
1.5	10	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	15	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	20	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	5	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3.5	10	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	15	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	20	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	5	0.255 (0.203)	0.277 (0.228)	0.277 (0.221)	0.283 (0.233)
4.5	10	0.101 (0.149)	0.115 (0.170)	0.111 (0.163)	0.117 (0.172)
	15	0.032 (0.079)	0.037 (0.091)	0.035 (0.087)	0.038 (0.093)
	20	0.011 (0.045)	0.013 (0.054)	0.012 (0.051)	0.013 (0.053)
	5	0.606 (0.115)	0.664 (0.132)	0.651 (0.138)	0.680 (0.127)
5.0	10	0.669 (0.126)	0.728 (0.128)	0.712 (0.131)	0.735 (0.127)
	15	0.704 (0.135)	0.760 (0.134)	0.744 (0.137)	0.765 (0.133)
	20	0.727 (0.136)	0.781 (0.133)	0.765 (0.136)	0.784 (0.131)
	5	0.287 (0.233)	0.309 (0.259)	0.305 (0.254)	0.319 (0.267)
5.5	10	0.125 (0.185)	0.141 (0.208)	0.135 (0.201)	0.143 (0.210)
	15	0.054 (0.123)	0.063 (0.140)	0.060 (0.135)	0.064 (0.142)
	20	0.025 (0.081)	0.030 (0.095)	0.028 (0.090)	0.030 (0.095)
	5	0.000 (0.001)	0.000 (0.001)	0.000 (0.002)	0.000 (0.001)
6.5	10	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	15	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	20	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	5	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
8.5	10	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	15	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	20	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
			$c = 0.5$		
	5	0.004 (0.006)	0.006 (0.011)	0.009 (0.029)	0.004 (0.007)
1.5	10	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	15	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	20	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
	5	0.390 (0.194)	0.444 (0.215)	0.454 (0.212)	0.454 (0.221)
3.5	10	0.235 (0.205)	0.281 (0.234)	0.281 (0.228)	0.283 (0.234)
	15	0.131 (0.160)	0.164 (0.191)	0.161 (0.186)	0.164 (0.190)
	20	0.072 (0.115)	0.094 (0.142)	0.092 (0.139)	0.093 (0.141)
	5	0.597 (0.150)	0.667 (0.165)	0.660 (0.166)	0.683 (0.165)
4.5	10	0.632 (0.179)	0.703 (0.184)	0.689 (0.183)	0.709 (0.184)
	15	0.643 (0.195)	0.711 (0.197)	0.696 (0.197)	0.716 (0.196)
	20	0.631 (0.214)	0.699 (0.214)	0.682 (0.215)	0.703 (0.213)
	5	0.616 (0.128)	0.688 (0.140)	0.679 (0.141)	0.705 (0.138)
5.0	10	0.681 (0.142)	0.751 (0.144)	0.737 (0.145)	0.759 (0.142)
	15	0.720 (0.140)	0.787 (0.136)	0.772 (0.137)	0.791 (0.135)
	20	0.751 (0.133)	0.814 (0.126)	0.800 (0.129)	0.817 (0.125)
	5	0.587 (0.163)	0.661 (0.181)	0.656 (0.180)	0.674 (0.181)
5.5	10	0.620 (0.193)	0.693 (0.199)	0.680 (0.199)	0.699 (0.199)
	15	0.631 (0.220)	0.701 (0.224)	0.687 (0.224)	0.705 (0.224)
	20	0.629 (0.238)	0.696 (0.241)	0.680 (0.242)	0.699 (0.241)
	5	0.431 (0.242)	0.491 (0.275)	0.492 (0.273)	0.500 (0.278)
6.5	10	0.334 (0.277)	0.386 (0.307)	0.378 (0.303)	0.391 (0.309)
	15	0.251 (0.272)	0.294 (0.304)	0.285 (0.298)	0.296 (0.305)
	20	0.188 (0.259)	0.222 (0.290)	0.214 (0.283)	0.223 (0.291)
	5	0.149 (0.215)	0.170 (0.249)	0.180 (0.256)	0.171 (0.252)
8.5	10	0.029 (0.102)	0.035 (0.120)	0.035 (0.118)	0.035 (0.119)
	15	0.003 (0.024)	0.004 (0.030)	0.004 (0.031)	0.004 (0.031)
	20	0.001 (0.018)	0.002 (0.023)	0.002 (0.023)	0.002 (0.022)

Table 4.2 The averages and the standard deviations in parentheses of posterior probabilities

μ	n	$P^F(H_1\|\mathbf{x})$	$P^{AI}(H_1\|\mathbf{x})$	$P^{MI}(H_1\|\mathbf{x})$	$P^{IP}(H_1\|\mathbf{x})$
			$c=1$		
1.5	5	0.067 (0.073)	0.091 (0.097)	0.120 (0.138)	0.083 (0.086)
	10	0.002 (0.007)	0.003 (0.012)	0.005 (0.020)	0.003 (0.008)
	15	0.000 (0.000)	0.000 (0.000)	0.000 (0.001)	0.000 (0.000)
	20	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3.5	5	0.533 (0.189)	0.616 (0.203)	0.638 (0.206)	0.628 (0.206)
	10	0.454 (0.232)	0.538 (0.250)	0.552 (0.251)	0.542 (0.250)
	15	0.376 (0.248)	0.457 (0.272)	0.464 (0.271)	0.460 (0.271)
	20	0.291 (0.236)	0.365 (0.268)	0.370 (0.269)	0.366 (0.267)
4.5	5	0.634 (0.137)	0.722 (0.145)	0.724 (0.148)	0.740 (0.147)
	10	0.676 (0.176)	0.760 (0.176)	0.759 (0.171)	0.765 (0.175)
	15	0.712 (0.168)	0.792 (0.163)	0.789 (0.162)	0.795 (0.162)
	20	0.713 (0.188)	0.791 (0.179)	0.786 (0.180)	0.794 (0.178)
5.0	5	0.642 (0.143)	0.735 (0.152)	0.744 (0.159)	0.749 (0.151)
	10	0.716 (0.129)	0.800 (0.123)	0.794 (0.126)	0.809 (0.119)
	15	0.757 (0.124)	0.835 (0.113)	0.829 (0.115)	0.839 (0.111)
	20	0.775 (0.125)	0.849 (0.110)	0.843 (0.110)	0.852 (0.110)
5.5	5	0.630 (0.144)	0.728 (0.150)	0.732 (0.156)	0.744 (0.149)
	10	0.680 (0.172)	0.772 (0.166)	0.770 (0.166)	0.776 (0.166)
	15	0.703 (0.183)	0.790 (0.173)	0.786 (0.174)	0.792 (0.173)
	20	0.715 (0.197)	0.796 (0.187)	0.790 (0.188)	0.799 (0.186)
6.5	5	0.538 (0.218)	0.633 (0.239)	0.647 (0.239)	0.647 (0.242)
	10	0.509 (0.277)	0.598 (0.295)	0.603 (0.293)	0.602 (0.296)
	15	0.434 (0.302)	0.518 (0.327)	0.521 (0.325)	0.520 (0.328)
	20	0.403 (0.317)	0.481 (0.343)	0.481 (0.340)	0.482 (0.344)
8.5	5	0.308 (0.278)	0.374 (0.327)	0.401 (0.333)	0.378 (0.330)
	10	0.162 (0.246)	0.202 (0.289)	0.208 (0.292)	0.204 (0.290)
	15	0.075 (0.169)	0.099 (0.208)	0.105 (0.215)	0.099 (0.208)
	20	0.028 (0.102)	0.039 (0.128)	0.041 (0.132)	0.039 (0.128)
			$c=5$		
1.5	5	0.120 (0.108)	0.149 (0.130)	0.166 (0.145)	0.149 (0.135)
	10	0.010 (0.021)	0.014 (0.028)	0.016 (0.031)	0.014 (0.029)
	15	0.001 (0.002)	0.001 (0.003)	0.001 (0.004)	0.001 (0.003)
	20	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)	0.000 (0.000)
3.5	5	0.508 (0.182)	0.589 (0.200)	0.597 (0.200)	0.608 (0.207)
	10	0.500 (0.229)	0.585 (0.242)	0.586 (0.240)	0.595 (0.244)
	15	0.475 (0.261)	0.557 (0.274)	0.555 (0.274)	0.565 (0.276)
	20	0.422 (0.272)	0.502 (0.291)	0.500 (0.290)	0.508 (0.292)
4.5	5	0.595 (0.143)	0.689 (0.151)	0.690 (0.152)	0.709 (0.157)
	10	0.659 (0.181)	0.745 (0.179)	0.744 (0.178)	0.756 (0.180)
	15	0.703 (0.189)	0.784 (0.180)	0.782 (0.181)	0.790 (0.179)
	20	0.728 (0.188)	0.804 (0.176)	0.801 (0.178)	0.809 (0.175)
5.0	5	0.593 (0.142)	0.692 (0.151)	0.693 (0.154)	0.709 (0.155)
	10	0.677 (0.167)	0.768 (0.160)	0.766 (0.162)	0.777 (0.161)
	15	0.725 (0.163)	0.808 (0.147)	0.805 (0.149)	0.814 (0.147)
	20	0.758 (0.160)	0.833 (0.144)	0.830 (0.146)	0.838 (0.143)
5.5	5	0.594 (0.146)	0.696 (0.153)	0.697 (0.156)	0.716 (0.155)
	10	0.650 (0.187)	0.745 (0.181)	0.743 (0.181)	0.754 (0.181)
	15	0.701 (0.188)	0.789 (0.175)	0.786 (0.177)	0.794 (0.174)
	20	0.725 (0.193)	0.806 (0.178)	0.804 (0.178)	0.811 (0.178)
6.5	5	0.521 (0.202)	0.621 (0.223)	0.622 (0.224)	0.639 (0.227)
	10	0.542 (0.252)	0.638 (0.264)	0.637 (0.264)	0.647 (0.264)
	15	0.518 (0.292)	0.607 (0.306)	0.605 (0.306)	0.612 (0.307)
	20	0.495 (0.313)	0.579 (0.327)	0.575 (0.327)	0.584 (0.327)
8.5	5	0.332 (0.257)	0.408 (0.302)	0.411 (0.304)	0.419 (0.310)
	10	0.236 (0.269)	0.296 (0.313)	0.296 (0.312)	0.301 (0.316)
	15	0.159 (0.243)	0.202 (0.286)	0.201 (0.285)	0.205 (0.288)
	20	0.100 (0.200)	0.130 (0.239)	0.129 (0.239)	0.132 (0.241)

Table 4.3 p-value, Bayes factor and posterior probabilities of $H_1 : \mu = 29$ being true

p-value	B_{21}^F	$P^F(H_1\|\mathbf{x})$	B_{21}^{AI}	$P^{AI}(H_1\|\mathbf{x})$	B_{21}^{MI}	$P^{MI}(H_1\|\mathbf{x})$	B_{21}^{IP}	$P^{IP}(H_1\|\mathbf{x})$
0.847	0.126	0.888	0.028	0.973	0.031	0.970	0.027	0.974

FBF has the smallest posterior probability than any other posterior probabilities based on the AIBF, the MIBF and intrinsic priors. The FBF seems to favor the complex hypothesis. As we expected, Bayes factors and posterior probabilities based on AIBF and intrinsic prior are similar.

5. Concluding remarks

In this paper, we developed the objective Bayesian hypothesis testing procedures based on the fractional Bayes factor and the intrinsic Bayes factor for the normal mean with known coefficient of variation under the reference prior. From our numerical results, the developed hypothesis testing procedures give fairly reasonable answers for all parameter configurations. However the FBF slightly favours the hypothesis H_2 than the AIBF, the MIBF and intrinsic priors. Therefore from our simulation and example, we recommend the use of the AIBF, the MIBF and intrinsic priors than the FBF in practical application.

Appendix: The propriety of posterior distribution

We prove the propriety of posterior distribution based on the reference prior. Under the reference prior, the posterior for μ given $\mathbf{x}$ is

$$\pi(\mu|\mathbf{x}) \propto (|c\mu|)^{-n}|\mu|^{-1} \exp\left\{-\frac{S^2 + n(\overline{x}-\mu)^2}{2c^2\mu^2}\right\},$$

where $\overline{x} = \sum_{i=1}^n x_i/n$. Then

$$\int_{-\infty}^{\infty} \pi(\mu|\mathbf{x})d\mu \propto 2^{\frac{n}{2}}\Gamma\left(\frac{n}{2}\right)\left[S^2 + n\overline{x}^2\right]^{-\frac{n}{2}} \exp\left\{-\frac{n}{2c^2}\right\}$$
$$\times Hypergeometric_1F_1\left[\frac{n}{2}, \frac{1}{2}, \frac{n^2\overline{x}^2}{2c^2(S^2+n\overline{x}^2)}\right],$$

where $Hypergeometric_1F_1[a,b,c]$ is the Kummer confluent hypergeometric function and this value of integral is proper if $n \geq 1$. This completes the proof.

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