J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 17, Number 1 (February 2010), Pages 87–92

LIE IDEALS AND DERIVATIONS OF σ-PRIME RINGS

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ABSTRACT. Let R be a 2-torsion free σ-prime ring with an involution σ, U a nonzero square closed σ-Lie ideal, $Z(R)$ the center of R and d a derivation of R. In this paper, it is proved that $d = 0$ or $U \subseteq Z(R)$ if one of the following conditions holds:
(1) $d(xy) - xy \in Z(R)$ or $d(xy) - yx \in Z(R)$ for all $x, y \in U$.
(2) $d(x) \circ d(y) = 0$ or $d(x) \circ d(y) = x \circ y$ for all $x, y \in U$ and d commutes with σ.

1. INTRODUCTION

Throughout the present paper R will denote an associative ring with center $Z(R)$. For any $x, y \in R$, the symbol $[x, y]$ stands for the commutator $xy - yx$ and the symbol $x \circ y$ denotes the anti-commutator $xy + yx$. In all that follows the symbol $Sa_\sigma(R)$, first introduced by L. Oukhtite, will denote the set of symmetric and skew symmetric elements of R, i.e. $Sa_\sigma(R) = \{x \in R \mid \sigma(x) = \pm x\}$. An involution σ of a ring R is an anti-automorphism of order 2 (i.e. an additive mapping satisfying $\sigma(xy) = \sigma(y)\sigma(x)$ and $\sigma(x^2) = x$ for all $x, y \in R$). An additive subgroup U of R is said to be a Lie ideal of R if $[u, r] \in U$ for all $u \in U$ and $r \in R$. A Lie ideal U which satisfies $\sigma(U) = U$ is called a σ-Lie ideal. If U is a Lie (resp. σ-Lie) ideal of R, then U is called a square closed Lie (resp. σ-Lie) ideal if $u^2 \in U$ for all $u \in U$. The fact that $uv + vu = (u + v)^2 - u^2 - v^2 \in U$ together with $uv - vu \in U$ yields that $2uv \in U$ for all $u, v \in U$. Therefore, for all $r \in R$ and $u, v \in U$, we have both $2r[u, v] = 2[u, rv] - 2[u, r]v \in U$ and $2[u, v]r = 2[u, vr] - 2v[u, r] \in U$. This remark will be used freely in the whole paper. An additive mapping $d : R \longrightarrow R$ is called a derivation if $d(xy) = d(x)y + xd(y)$ holds for all $x, y \in R$. A ring R is called 2-torsion free, if whenever $2x = 0$, with $x \in R$, then $x = 0$. Recall that a ring R is

Received by the editors November 10, 2009. Revised December 31, 2009. Accepted January 27, 2010.
2000 *Mathematics Subject Classification.* 16N60, 16U80, 16W25.
Key words and phrases. σ-prime ring, derivation, σ-Lie ideal.
Supported by the Natural Science Research Foundation of Anhui Provincial Education Department(No. KJ2010B200).

prime if for any $a, b \in R$, $aRb = 0$ implies $a = 0$ or $b = 0$. A ring R equipped with an involution σ is said to be a σ-prime ring if for any $a, b \in R$, $aRb = aR\sigma(b) = 0$ implies $a = 0$ or $b = 0$. It is worthwhile to note that every prime ring having an involution σ is σ-prime but the converse is in general not true. Such an example due to L. Oukhtite is as following: Let R be a prime ring, $S = R \times R^{\circ}$ where R° is the opposite ring of R, define $\sigma(x, y) = (y, x)$. From $(0, x)S(x, 0) = 0$, it follows that S is not prime. For the σ-primeness of S, we suppose that $(a, b)S(x, y) = 0$ and $(a, b)S\sigma((x, y)) = 0$, then we get $aRx \times yRb = 0$ and $aRy \times xRb = 0$, and hence $aRx = yRb = aRy = xRb = 0$, or equivalently $(a, b) = 0$ or $(x, y) = 0$.

Recently, L. Oukhtite, S. Salhi and L. Taoufiq extended some results of prime rings to σ-prime rings (see[1-9]). In [10], M. Ashraf and N. Rehman proved that if d is derivation of a prime ring R such that $d(xy) - xy \in Z(R)$ or $d(xy) - yx \in Z(R)$ for all x, y in a nonzero ideal I, then R is commutative. In [11], M. Ashraf and N. Rehman proved that if d is nonzero derivation of a 2-torsion free prime ring R such that $d(x) \circ d(y) = 0$ or $d(x) \circ d(y) = x \circ y$ for all $x, y \in I$, where I is a nonzero ideal of R, then R is commutative. The author [12] extended these results to σ-ideal of σ-prime ring. The purpose of this paper is to extend the above results to some more general settings. Meanwhile, as there were only a few papers on σ-prime rings, it seems that the present paper would develop the study of the subject in this direction.

2. Some Preliminaries

We shall do a great deal of calculation with commutators and anti-commutators, routinely using the following basic identities: For all $x, y, z \in R$;

$$[xy, z] = x[y, z] + [x, z]y \text{ and } [x, yz] = y[x, z] + [x, y]z$$

$$xo(yz) = (xoy)z - y[x, z] = y(xoz) + [x, y]z$$

$$(xy)oz = x(yoz) - [x, z]y = (xoz)y + x[y, z].$$

We shall also make use of several known results, which we now state as lemmas:

Lemma 2.1 ([1, Lemma 4]). *If $U \not\subseteq Z(R)$ is a σ-Lie ideal of a 2-torsion free σ-prime ring R and $a, b \in R$ such that $\sigma(a)Ub = aUb = 0$, then $a = 0$ or $b = 0$.*

Lemma 2.2 ([2, Lemma 2.3]). *Let R be a 2-torsion free σ-prime ring, U a nonzero σ-Lie ideal of R. If $[u, v] = 0$ for all $u, v \in U$, then $U \subseteq Z(R)$.*

Lemma 2.3 ([2, Lemma 2.4]). *Let R be a 2-torsion free σ-prime ring, U a nonzero σ-Lie ideal and d a nonzero derivation of R. If $d(U) \subseteq Z(R)$, then $U \subseteq Z(R)$.*

Lemma 2.4 ([2, Theorem 1.1]). *Let R be a 2-torsion free σ-prime ring, U a nonzero σ-Lie ideal and d a nonzero derivation of R. If $d^2(U) = 0$, then $U \subseteq Z(R)$.*

3. The Main Results

Theorem 3.1. *Let R be a 2-torsion free σ-prime ring with an involution σ, U a nonzero square closed σ-Lie ideal and d a derivation of R. If $d(xy) - xy \in Z(R)$ or $d(xy) - yx \in Z(R)$ for all $x, y \in U$, then $d = 0$ or $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$ and let $G_1 = \{x \in U \mid d(xy) - xy \in Z(R) \ for \ all \ y \in U\}$ and $G_2 = \{x \in U \mid d(xy) - yx \in Z(R) \ for \ all \ y \in U\}$. Then G_1 and G_2 are additive subgroups of U and $U = G_1 \bigcup G_2$. But a group can't be a union of two of its proper subgroups, hence $U = G_1$ or $U = G_2$. Suppose that $U = G_1$, then

$$d(x)y + xd(y) - xy \in Z(R) \text{ for all } x, y \in U. \tag{1}$$

Replacing y by $2yz$ in (1) we get

$$(d(x)y + xd(y) - xy)z + xyd(z) \in Z(R) \text{ for all } x, y, z \in U.$$

And therefore $0 = [(d(x)y + xd(y) - xy)z + xyd(z), z] = [xyd(z), z]$, hence

$$xy[d(z), z] + x[y, z]d(z) + [x, z]yd(z) = 0 \text{ for all } x, y, z \in U. \tag{2}$$

Replacing x by $2wx$ in (2) and using (2), then we have $[w, z]xyd(z) = 0$ and therefore

$$[w, z]Uyd(z) = 0 \text{ for all } w, y, z \in U. \tag{3}$$

If $z \in U \bigcap Sa_\sigma(R)$, then (3) yields $[w, z]Uyd(z) = 0 = \sigma([w, z])Uyd(z)$ whence it follows $yd(z) = 0$ by Lemma 2.1, in which case $d(z)Ud(z) = 0 = \sigma(d(z))Ud(z)$ and thus $d(z) = 0$, or $[w, z] = 0$. Accordingly, $d(z) = 0$, or $[U, z] = 0$ for all $z \in U \bigcap Sa_\sigma(R)$. Let $u \in U$, as $u - \sigma(u) \in U \bigcap Sa_\sigma(R)$, then $d(u - \sigma(u)) = 0$ or $[U, u - \sigma(u)] = 0$. If $d(u - \sigma(u)) = 0$, replacing z by $\sigma(u)$ in (3) we find that $\sigma([w, u])Uyd(u) = 0$, which leads to $d(u) = 0$ or $[U, u] = 0$. If $[U, u - \sigma(u)] = 0$, then $[w, u] = [w, \sigma(u)]$ for all $w \in U$ which gives, because of (3), $\sigma([w, u])Uyd(u) = 0$, whence it follows that $[w, u] = 0$ or $d(u) = 0$. In conclusion we find that $d(u) = 0$ or $[U, u] = 0$ for all $u \in U$. Consequently, U is a union of two additive subgroups U_1 and U_2, where $U_1 = \{u \in U \mid d(u) = 0\}$ and $U_2 = \{u \in U \mid [U, u] = 0\}$. But a group can't be a union of two of its proper subgroups and thus $U = U_1$ or $U = U_2$.

The fact that $U \not\subseteq Z(R)$ forces, because of Lemma 2.2, $U = U_1$. We therefore have $d(U) = 0$, whence it follows, according to Lemma 2.3, that $d = 0$. Now suppose that $U = G_2$, then we have $d(x)y + xd(y) - yx \in Z(R)$ for all $x, y \in U$. Using the similar techniques as used in [10, Theorem 2.3], we obtain

$$[x, z]xyd(x) = 0 \text{ for all } x, y, z \in U. \tag{4}$$

Replacing z by $2zu$ in (4) we get $[x, z]uxyd(x) = 0$ and therefore

$$[x, z]Uxyd(x) = 0 \text{ for all } x, y, z \in U. \tag{5}$$

For all $x \in U \bigcap Sa_\sigma(R)$, from (5) $\sigma([x, z])Uxyd(x) = 0$ and hence $xUd(x) = 0$, in which case $x = 0$ or $d(x) = 0$, or $[x, z] = 0$. Consequently $d(x) = 0$, or $[x, U] = 0$ for all $x \in U \bigcap Sa_\sigma(R)$. Using the same techniques as used above, we get $d = 0$. □

Theorem 3.2. *Let R be a 2-torsion free σ-prime ring with an involution σ, U a nonzero square closed σ-Lie ideal and d a derivation of R which commutes with σ. If $d(x) \circ d(y) = 0$ or $d(x) \circ d(y) = x \circ y$ for all $x, y \in U$, then $d = 0$ or $U \subseteq Z(R)$.*

Proof. Assume that $U \not\subseteq Z(R)$. Using similar arguments as in the beginning of the proof of Theorem 3.1, we get $d(x) \circ d(y) = 0$ for all $x, y \in U$ or $d(x) \circ d(y) = x \circ y$ for all $x, y \in U$. Suppose that

$$d(x) \circ d(y) = 0 \text{ for all } x, y \in U. \tag{6}$$

Replacing y by $2yz$ in (6) we get

$$[d(x), y]d(z) - d(y)[d(x), z] = 0 \text{ for all } x, y, z \in U. \tag{7}$$

Substituting $2[u, v]d(x)$ for y in the above relation, we obtain

$$[d(x), [u, v]]d(x)d(z) - d([u, v])d(x)[d(x), z] - [u, v]d^2(x)[d(x), z] = 0$$

According to (7), one can replace in the above relation $d(x)[d(x), z]$ by $[d(x), x]d(z)$ which gives

$$[d(x), [u, v]]d(x)d(z) - d([u, v])[d(x), x]d(z) - [u, v]d^2(x)[d(x), z] = 0$$

Since $[d(x), [u, v]]d(x) - d([u, v])[d(x), x] = 0$ by (7), we then get

$$[u, v]d^2(x)[d(x), z] = 0 \text{ for all } u, v, x, z \in U. \tag{8}$$

Replacing v by $2vw$ in (8) and using (8), we obtain $[u, v]wd^2(x)[d(x), z] = 0$ and therefore

$$[u, v]Ud^2(x)[d(x), z] = 0 \text{ for all } u, v, x, z \in U. \tag{9}$$

As $U \not\subseteq Z(R)$, then $[U,U] \neq 0$ by Lemma 2.2, and from (9) it follows that

$$d^2(x)[d(x),z] = 0 \text{ for all } x,z \in U. \tag{10}$$

Putting $2yz$ for z in (10), we arrive at $d^2(x)y[d(x),z] = 0$ and therefore

$$d^2(x)U[d(x),z] = 0 \text{ for all } x,z \in U. \tag{11}$$

For all $x \in U \bigcap Sa_\sigma(R)$, since d commutes with σ, from (11) it follows that $d^2(x) = 0$ or $[d(x),z] = 0$ for all $z \in U$. Using the same techniques as used in the proof of Theorem 3.1, we conclude that $d^2(x) = 0$ or $[d(x),U] = 0$ for all $x \in U$. Let $U_3 = \{u \in U \mid d^2(u) = 0\}$ and $U_4 = \{u \in U \mid [d(u),U] = 0\}$, it is clear that U_3 and U_4 are additive subgroups of U and $U = U_3 \bigcup U_4$ and hence $U = U_3$ or $U = U_4$. If $U = U_3$, then $d^2(U) = 0$ and Lemma 2.4 forces $d = 0$. If $U = U_4$, then $[d(x),y] = 0$ for all $x,y \in U$. Taking $2r[y,z]$ instead of y, where $r \in R$, we obtain $[d(x),r][y,z] = 0$ for all $x,y,z \in U$ and $r \in R$. The substitution of rs for r in the above relation gives $[d(x),r]s[y,z] = 0$ and therefore

$$[d(x),r]R[y,z] = 0 \text{ for all } x,y,z \in U \textit{ and } r \in R. \tag{12}$$

Since U is a σ-Lie ideal, then $[d(x),r]R\sigma([y,z]) = 0$ and (12) yields that $[d(x),r] = 0$ or $[y,z] = 0$ for all $x,y,z \in U$ and $r \in R$. As $U \not\subseteq Z(R)$, then $[U,U] \neq 0$ and thus $[d(x),r] = 0$. Accordingly, $d(U) \subseteq Z(R)$, whence it follows, applying Lemma 2.3, that $d = 0$. Now suppose that

$$d(x) \circ d(y) = x \circ y \text{ for all } x,y \in U. \tag{13}$$

Substituting $2yz$ for y in (13) we get

$$(d(x) \circ y)d(z) - d(y)[d(x),z] - y[d(x),d(z)] + y[x,z] = 0 \text{ for all } x,y,z \in U. \tag{14}$$

Replacing y by $2d(x)[u,v]$ in (14) and using (14) we obtain

$$d^2(x)[u,v][d(x),z] = 0 \text{ for all } u,v,x,z \in U. \tag{15}$$

Substituting $2wz$ for z in (15) and using (15), we have $d^2(x)[u,v]w[d(x),z] = 0$ and therefore

$$d^2(x)[u,v]U[d(x),z] = 0 \text{ for all } u,v,x,z \in U. \tag{16}$$

For all $x \in U \bigcap Sa_\sigma(R)$, since d commutes with σ, from (16) it follows that $d^2(x) = 0$ or $[d(x),z] = 0$ for all $z \in U$. Reasoning as above, we conclude that $d = 0$. □

Acknowledgments. The author would like thank the referee for pointing out several misprints and rearranging the manuscript which clarify the whole paper. Thanks are also due to Professor N. Rehman for his valuable advice and encouragement.

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