J. Korean Soc. Math. Educ. Ser. B: Pure Appl. Math. ISSN 1226-0657
Volume 17, Number 1 (February 2010), Pages 81–86

WEAKLY CANCELLATIVE ELEMENTS IN SEMIGROUPS

JONG MOON SHIN

ABSTRACT. This paper gives some sorts of weakly cancellative of elements which are to be or not to be left magnifying elements in certain semigroups and gives a semilattice congruence in a weakly separative semigroup.

1. INTRODUCTION

The notion of a left (right) magnifying element of a semigroup was introduced by E. S. Ljapin[6]. In [1, 2, 3, 8, 10] and related papers authers determined many properties of magnifying elements of semigroups. In [12], K. Tolo introduced the notion of strongly left (right) magnifying elements of a semigroup. This kind of magnifying elements also arises in connection with another interesting problem, the possible factorization of semigroups. It is well known that any cancellative semigroup can have no left (right) magnifying element. In this paper, we discuss some sorts of weakly cancellative elements, which are to be or not to be left (right) magnifying elements in certain semigroups. And we give analogues of some of results in the separative semigroups. For undefined terms used in this paper, we referred to [11].

2. SEMICANCELLABLE ELEMENTS

We recall that an element a of a semigroup S is said to be left magnifying if there exists a proper subset M of S such that $aM = S$.

Dually, a right magnifying element is defined.

Note that any idempotent element of a semigroup is not a left (right) magnifying element and that any cancellative semigroup can have no a left (right) magnifying element [6].

Received by the editors September 16, 2009. Accepted December 14, 2009.
2000 *Mathematics Subject Classification.* 20D99.
Key words and phrases. magnifying elements, semicancellable elements, weakly separative semigroup.

We recall that an element of a semigroup S is said to be strongly left magnifying if there exists a proper subsemigroup T of S such that $aT = S$. Dually, a strongly right magnifying element is defined. This notion is due to K. Tolo [12].

Note that every strongly left magnifying element of a semigroup S is a left magnifying element.

Definition. A semigroup S is said to be factorizable if $S = AB$, where A and B are proper subsemigroups of S.

Example ([8]). If a is a strongly left maginifying element of the semigroup S, then there exists a proper subsemigroup T of S such that $aT = S$. Thus $S = < a > T$, where $< a > \neq S$ since a commutative semigroup contains no magnifying element. Hence S is factorizable.

We recall that a proper subset M of a semigroup S is said to be minimal for the left magnifying element a if $aM = S$ and $a(M - \{m\}) \neq S$ for any $m \in S$. This notion and the following Lemma are due to F. Miglinorini [10].

Lemma 2.1 ([10]). *Let a be a left magnifying element of a semigroup S. Then there exists a proper subset M of S which is minimal for a such that $aM = S$.*

Let a be a left maginifying element of a semigroup S. Is it possible that among the proper minimal subsets M for $aM = S$, a certain set could be found, which is subsemigroup of S?. This is possible, as the following Lemma shows.

Lemma 2.2 ([3]). *Let a be a left magnifying element of a semigroup S with left identity. Then there exists a minimal right ideal T of S which is minimal for a such that $aT = S$.*

Definition. An element a of a semigroup S is said to be left semicancellable if $as = at$, $s, t \in S$, there exists $r \in S$ such that $ar = a$ and $rs = rt$.

Definition. An element a of a semigroup S is said to be left e-cancellable if there exists $e \in S$ such that $as = at, s, t \in S$, implies $ae = a$ and $es = et$.

Note that any idempotent $e \in S$ is left e-cancellable and every left e-cancellable element is left semicancellable [5].

Remark. If a is a left e-cancellable element of a semigroup S and $e^2 = e$ in S, then a may be a left magnifying element of S.

Example. Let $S = B(p,q)$ be the bicyclic semigroup generated by two elements p and q with the relation $pq = 1$. Then the elements of S can be displayed as follows;

$$\begin{array}{ccccc} 1 & p & p^2 & p^3 & \cdots \\ q & qp & qp^2 & qp^3 & \cdots \\ q^2 & q^2p & q^2p^2 & q^2p^3 & \cdots \\ q^3 & q^3p & q^3p^2 & q^3p^3 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \end{array}$$

Let $e = qp$. Then e is an idempotent of S. For $ps = pt$ for some $s, t \in S$, $pe = p(qp) = p$. From $ps = pt$, $qps = qpt$. So, $es = et$. Thus p is a left e-cancellable element of S. Since the set qS does not contain p, qS is a proper subset of S. And $p(qS) = (pq)S = S$. Hence p is a left magnifying element of S.

Theorem 2.1. *If a left semicancellable element a of a semigroup S with left identity is a left magnifying element, then a is a left e - cancellable element of S and $e^2 = e$.*

Proof. Let $a \in S$ be left semicancellable element. Suppose that a is a left magnifying element of S. Then by the Lemma 2.2., there exists a minimal right ideal T of S which is minimal for a such that $aT = S$. Let $s \in S - T$. Then $as = at$ for some $t \in T$. Since $aT = S$, there exists $v \in T$ such that $a = av$. Since a is left semicancellable, there exists $s' \in S$ such that $as' = (av)s' = a$ and $s's = s't$. Thus $vs's = vs't$. We put $e = vs'$, then $e = vs' \in TS \subset T$ since T is a minimal right ideal of S. Thus $e \in T$ and $a = ae$ and $es = et$. Here e is not a left identity of S by the choice of s and t. Moreover, $ae = ae^2$. Now assume that $e \neq e^2$. Then $a(T - \{e^2\}) = aT = S$, and $T - \{e^2\} \neq T$. This contradict to the minimality of T. Thus $e^2 = e$. Hence a is a left e-cancellable element of S and $e^2 = e$. □

3. Weakly Separative Semigroup

We recall that a semigroup S is said to be weakly separative if for any $a, b \in S$, $a^2 = ab = b^2$ implies $a = b$.

This notion is a weak version of a cancellative semigroup.

Definition. A semigroup S is said to be medial if for any $a, b, x, y \in S$,

$$a\,x\,y\,b = a\,y\,x\,b.$$

We recall that an element a of a semigroup S is said to be completely regular if there exists $x \in S$ such that $a = axa$ and $ax = xa$.

Lemma 3.1 ([3]). *If a is a completely regular element of a semigroup S, then a is not a left magnifying element of S.*

Lemma 3.2 ([11]). em Let S be a semigroup. Then a is a completely regular element of S if and only if $a \in a^2S \cap Sa^2$.

Theorem 3.1. *If S is a medial weakly separative semigroup. Then S can have no left magnifying element.*

Proof. Suppose that S is a medial weakly separative semigroup. Assume that a is a left magnifying element of S. Then there exists a proper subset M of S such that $aM = S$. Also, $aS = S$. So, $a^2S = aS = S$. Thus there exists $x \in S$ such that $a = a^2x$. This implies, by using the mediality of S, $a^2 = a^2xa = a(axa) = axaa = (ax)(a^2x)a = (axa)^2$. Since S is weakly separative, $a = axa$. Since $x \in S = aS$, $x = ay$ for some $y \in S$. Thus $xa^2 = aya^2$. This implies, by using the mediality of S, $(xa^2)^2 = (xa^2)(xa^2) = (xa^2)(aya^2) = (xa^2)(ayaa) = (xa^2)(aaya) = (xa^2)(axa) = (xa^2)a$, and $(xa^2)a = (aya^2)a = a(ya)a^2 == a(ay)a^2 = axa^2 = (axa)a = a^2$. So, $(xa^2)^2 = (xa^2)a = a^2$. Since S is weakly separative, $xa^2 = a$. Thus $a \in a^2S \cap Sa^2$. By Lemma 3.2, a is a completely regular element of S. By Lemma 3.1, a is not a left magnifying element of S. This is a contradiction. □

We recall that a semigroup S is (1,2)–commutative if for any a, b, $c \in S, a(bc) = (bc)a$. This notion is a weak version of a commutative semigroup.

A semigroup S is left (right) separative if for any a, $b \in S, ab = a^2$ and $ba = b^2$ ($ab = b^2$ and $ba = a^2$) imply $a = b$. A semigroup is separative if it is both left and right separative. Note that a semigroup is separative implies it is weak separative. the converse holds in a commutative semigroup.

The following results are some analogues in the separative semigroups [11].

Lemma 3.3. *Let S be an $(1,2)$-commutative weakly separative semigroup. Then the following statements hold. For any a, b, x, $y \in S$,*

(1) *$xa = xb$ if and only if $ax = bx$.*

(2) *$x^2a = x^2b$ if and only if $xa = xb$.*

(3) *$xya = xyb$ if and only if $yxa = yxb$.*

Proof. (1) Suppose that $xa == xb$ for all $x, a, b \in S$. Then $(ax)^2 = (ax)(ax) = a(xa)x = a(xb)x = a(xbx) = a(bxx) = axxb = ax^2b$ by using the (1,2) - commutativity of S. Similarly, $(bx)^2 = b(xb)x = b(xa)x = (xa)xb = (xax)b = (axx)b = ax^2b$. Thus $(ax)^2 = (ax)(bx) = ax^2b = (bx)^2$ for all $x, a, b \in S$. Since S is weakly separative, $ax = bx$. The opposite implication follows by symmetry.

(2) Suppose that $x^2a = x^2b$ for all $x, a, b \in S$. Then $x(xa) = x(xb)$. So, $xax = xbx$ by (1,2) - commutativity of S. by using the (1,2) - commutativity of S, $(ax)^2 = a(xax) = a(xbx) = (ax)(bx) = a(xbx) = a(bxx) = axxb = ax^2b$, and $(bx)^2 = b(xbx) = b(xax) = b(axx) = bax^2 = ax^2b$. Thus $(ax)^2 = (ax)(bx) = (bx)^2$. Since S is weakly separative, $ax = bx$. By (1), $xa = xb$.

(3) Suppose that $xya = xyb$ for all $a, b, x, y \in S$. Then $xyay = xyby$. By (1), $yayx = ybyx$. This implies $yayxa = ybyxa$. By (1), $ayxay = byxay$ and $ayxby = byxby$. So, $ayxayx = byxayx$ and $ayxbyx = byxbyx$. Thus $(ayx)^2 = (byx)(ayx)$ and $(byx)^2 = (ayx)(byx)$. By using the (1,2) - commutativity of S, $(ayx)(byx) = (ayx)b(yx) = b(yx)(ayx) = (byx)(ayx)$. So, $(ayx)^2 = (ayx)(byx) = (byx)^2$. Since S is weakly separative, $ayx = byx$. By (1), $yxa = yxb$. The opposite implication follows by symmetry. □

Definition. Let S be a semigroup, and let σ is a congruence on S. σ is called a semilattice congruence if S/σ is a semilattice.

Theorem 3.2. *Let S be an $(1,2)$ - commutative weakly separative semigroup. Define a relation σ on S by $x \,\sigma\, y$ if for any $a, b \in S$, $xa = xb$ if and only if $ya = yb$. Then σ is a semilattice congruence.*

Proof. We can easily prove that σ is an equivalence relation on S. Let $x \,\sigma\, y$ and $z \in S$. Then $xa = xb$ if and only if $ya = yb$ for any $a, b, \in S$. Since a and b are arbitrary, $x(za) = x(zb)$ if and only if $y(za) = y(zb)$. So, $(xz)a = (xz)b$ if and only if $(yz)a = (yz)b$ for any $a, b, \in S$. Thus $xz \,\sigma\, yz$. Next, if $(zx)a = (zx)b$, then $z(xa) = z(xb)$. By (1) of Lemma 3.3., $(xa)z = (xb)z$. So, $x(az) = x(bz)$. By definition of σ, $y(az) = y(bz)$. By (1) of Lemma 3.3., $(az)y = (bz)y$. So, $a(zy) = b(zy)$. By (1) of Lemma 3.3., $(zy)a = (zy)b$. Again if $(zy)a = (zy)b$, then $z(ya) = z(yb)$. By (1) of Lemma 3.3., $(ya)z = (yb)z$. So, $y(az) = y(bz)$.

By definition of σ, $x(az) = x(bz)$. By (1) of Lemma 3.3, $(az)x = (bz)x$. So, $a(zx) = b(zx)$. By (1) of Lemma 3.3, $(zx)a = (zx)b$. Thus $(zx)a = (zx)b$ if and only if $(zy)a = (zy)b$. Hence $zx \,\sigma\, zy$, and hence σ is a congruence on S. Let $\sigma_x \in S/\sigma$.

Then by (2) of Lemma 3.3., $x^2a = x^2b$ if and only if $xa = xb$ for any $a, b \in S$. So, $x^2 \sigma x$. Thus $(\sigma_x)^2 = \sigma_x\sigma_x = \sigma_{x^2} = \sigma_x$. Hence S/σ is a band. Let $\sigma_x, \sigma_y \in S/\sigma$. Then by (3) of Lemma 3.3., $xya = xyb$ if and only if $yxa = yxb$. So, $xy \sigma yx$. Thus $\sigma_{xy} = \sigma_{yx}$. Hence $\sigma_x\sigma_y = \sigma_{xy} = \sigma_{yx} = \sigma_y\sigma_x$, and hence S/σ is commutative. Therefore S/σ is a semilattice. □

Theorem 3.3. *Every idempotent of an* $(1, 2)$ *- commutative weakly separative semigroup* S *is central.*

Proof. Let $a \in S$, and let e is an idempotent in S. Then by using the (1,2) - commutativity of S, $(ea)^2 = (ea)(ea) = (ea)(eea) = (ea)(eae) = (ea)(aee) = (ea)(ae) = (eea)(ae) = (eae)(ae) = (aee)(ae) = (ae)^2$. Thus $(ea)^2 = (ea)(ae) = (ae)^2$. Since S is weakly separative, $ea = ae$. Hence e is central. □

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DEPARTMENT OF MATHEMATICS, DONGGUK UNIVERSITY, GYEONGJU, GYEONGBUK 780-714, KOREA
Email address: shinjm@dongguk.ac.kr