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REAL HALF LIGHTLIKE SUBMANIFOLDS WITH TOTALLY UMBILICAL PROPERTIES

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ABSTRACT. In this paper, we prove two characterization theorems for real half lightlike submanifold $(M, g, S(TM))$ of an indefinite Kaehler manifold $\bar{M}$ or an indefinite complex space form $\bar{M}(c)$ subject to the conditions : (a) M is totally umbilical in $\bar{M}$, or (b) its screen distribution $S(TM)$ is totally umbilical in M.

1. INTRODUCTION

It is well known that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of the half lightlike submanifolds M of a semi-Rimannian manifold $(\bar{M}, \bar{g})$ of codimension 2 is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank 1. Then there exists complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which called the *screen* and *co-screen distribution* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \; TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{1.1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E (same notation for any other vector bundle) over M. Choose $L \in \Gamma(S(TM^{\perp}))$ as a unit vector field with $\bar{g}(L, L) = \epsilon = \pm 1$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly ξ and L belong to $\Gamma(S(TM)^{\perp})$. Hence we have the following orthogonal decomposition

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

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where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined null vector field $N \in \Gamma(ltr(TM))$ [1] satisfying

$$\bar{g}(\xi, N) = 1, \; \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, L) = 0, \; \forall\, X \in \Gamma(S(TM)). \tag{1.2}$$

We call N, $ltr(TM)$ and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector field, lightlike transversal vector bundle* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Therefore the tangent space $T\bar{M}$ of the ambient manifold $\bar{M}$ is decomposed as follows:

$$\begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM) \oplus_{orth} S(TM^{\perp}). \end{aligned} \tag{1.3}$$

The purpose of this paper is to study the geometry of real half lightlike submanifolds $(M, g, S(TM))$ of an indefinite Kaehler manifold $\bar{M}$ or an indefinite complex space form $\bar{M}(c)$ subject to the conditions : (a) M is totally umbilical in $\bar{M}$ or (b) its screen $S(TM)$ is totally umbilical in M. In section 3, we prove a characterization theorem for totally umbilical real half lightlike submanifolds M of an indefinite Kaehler manifold $\bar{M}$. This theorem shows that the second fundamental forms B and D of such a real half lightlike submanifold M satisfy $B = D = 0$, i.e., M is totally geodesic (Theorem 3.1). Furthermore, if $\bar{M} = \bar{M}(c)$, then we have also $c = 0$, i.e., $\bar{M}$ is a semi-Euclidean space (Theorem 3.3). In section 4, we prove a characterization theorem for real half lightlike submanifolds M of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in M. This theorem shows that the second fundamental form C of $S(TM)$ and the constant holomorphic sectional curvature c satisfy $C = c = 0$, i.e., $S(TM)$ is totally geodesic and $\bar{M}$ is a semi-Euclidean space (Theorem 4.2). Using these theorems, we prove several additional theorems for real half lightlike submanifold M of $\bar{M}(c)$ such that M is totally umbilical or $S(TM)$ is totally umbilical in M. Recall the following structure equations:

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and P the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). Then the local Gauss and Weingarten formulas are given by

$$\bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)L, \tag{1.4}$$

$$\bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)L, \tag{1.5}$$

$$\bar{\nabla}_X L = -A_L X + \phi(X)N, \tag{1.6}$$

$$\nabla_X PY = \nabla^*_X PY + C(X, PY)\xi, \tag{1.7}$$

$$\nabla_X \xi = -A^*_\xi X - \tau(X)\xi, \tag{1.8}$$

for all X, $Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, B and D are called the *local second fundamental forms* of M, C is called the *local second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_L are linear operators on TM and τ, ρ and ϕ are 1-forms on TM. We say that $h(X,Y) = B(X,Y)N + D(X,Y)L$ is the *second fundamental tensor* of M.

Since $\bar{\nabla}$ is torsion-free, ∇ is also torsion-free and both B and D are symmetric. From the facts $B(X,Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X,Y) = \epsilon\bar{g}(\bar{\nabla}_X Y, L)$ for all X, $Y \in \Gamma(TM)$, we know that B and D are independent of the choice of a $S(TM)$ and

$$B(X,\xi) = 0, \quad D(X,\xi) = -\epsilon\phi(X), \tag{1.9}$$

for all $X \in \Gamma(TM)$. The induced connection ∇ of M is not metric and satisfies

$$(\nabla_X g)(Y,Z) = B(X,Y)\,\eta(Z) + B(X,Z)\,\eta(Y), \tag{1.10}$$

for all X, Y, $Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X,N), \tag{1.11}$$

for all $X \in \Gamma(TM)$. But the connection ∇^* on $S(TM)$ is metric. The above three local second fundamental forms are related to their shape operators by

$$B(X,Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.12}$$

$$C(X,PY) = g(A_N X, PY), \qquad \bar{g}(A_N X, N) = 0, \tag{1.13}$$

$$\epsilon D(X,PY) = g(A_L X, PY), \qquad \bar{g}(A_L X, N) = \epsilon\rho(X), \tag{1.14}$$

$$\epsilon D(X,Y) = g(A_L X, Y) - \phi(X)\eta(Y), \ \forall X, Y \in \Gamma(TM). \tag{1.15}$$

By (1.12) and (1.13), we show that A^*_ξ and A_N are $\Gamma(S(TM))$-valued shape operators related to B and C respectively and A^*_ξ is self-adjoint on TM and

$$A^*_\xi \xi = 0. \tag{1.16}$$

But A_N and A_L are not self-adjoint on $S(TM)$ and TM respectively.

Denote by $\bar{R}$, R and R^* the curvature tensors of $\bar{\nabla}$, ∇ and ∇^* respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, for all $X, Y, Z, W \in \Gamma(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\bar{g}(\bar{R}(X,Y)Z,\, PW) = g(R(X,Y)Z,\, PW) + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) + \epsilon\{D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW)\}, \tag{1.17}$$

$$\bar{g}(\bar{R}(X,Y)Z,\, \xi) = (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), \tag{1.18}$$

$$\bar{g}(\bar{R}(X,Y)Z,\, N) = \bar{g}(R(X,Y)Z,\, N) + \epsilon\{D(X,Z)\rho(Y) - D(Y,Z)\rho(X)\}, \tag{1.19}$$

$$\bar{g}(\bar{R}(X,Y)Z,\, L) = \epsilon\{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + B(Y,Z)\rho(X) - B(X,Z)\rho(Y)\}, \tag{1.20}$$

$$\bar{g}(R(X,Y)PZ,\, PW) = g(R^*(X,Y)PZ,\, PW) + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), \tag{1.21}$$

$$g(R(X,Y)PZ,\, N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X). \tag{1.22}$$

2. Real Half Lightlike Submanifolds

Let $\bar{M} = (\bar{M}, J, \bar{g})$ be a real $2m$-dimensional indefinite Kaehler manifold, where $\bar{g}$ is a semi-Riemannian metric of index $q = 2v$, $0 < v < m$ and J is an almost complex structure on $\bar{M}$ satisfying, for all $X,\, Y \in \Gamma(T\bar{M})$,

$$J^2 = -I, \quad \bar{g}(JX, JY) = \bar{g}(X,Y), \quad (\bar{\nabla}_X J)Y = 0. \tag{2.1}$$

An indefinite complex space form, denoted by $\bar{M}(c)$, is a connected indefinite Kaehler manifold of constant holomorphic sectional curvature c such that

$$\bar{R}(X,Y)Z = \frac{c}{4}\{\bar{g}(Y,Z)X - \bar{g}(X,Z)Y + \bar{g}(JY,Z)JX - \bar{g}(JX,Z)JY + 2\bar{g}(X,JY)JZ\},\ \forall X,\, Y,\, Z \in \Gamma(TM). \tag{2.2}$$

Definition 1. Let $(M, g, S(TM))$ be a real lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. We say that M is a *CR-lightlike submanifold*[2] of $\bar{M}$ if the following two conditions are fulfilled:

(A) $J(Rad(TM))$ is a distribution on M such that

$$Rad(TM) \cap J(Rad(TM)) = \{0\}.$$

(B) There exist vector bundles H_o and H' over M such that

$$S(TM) = \{J(Rad(TM)) \oplus H'\} \oplus_{orth} H_o; \ J(H_o) = H_o; \ J(H') = K_1 \oplus_{orth} K_2,$$

where H_o is a non-degenerate almost complex distribution on M, and K_1 and K_2 are vector subbundles of $ltr(TM)$ and $S(TM^{\perp})$ respectively.

Theorem 2.1 ([7]). *Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is a CR-lightlike submanifold of $\bar{M}$.*

Proof. Let ξ, N and L be local sections of $Rad(TM)$, $ltr(TM)$ and $S(TM^{\perp})$ respectively. From $\bar{g}(J\xi, \xi) = 0$ and $Rad(TM) \cap J(Rad(TM)) = \{0\}$, we show that $J(Rad(TM))$ is a vector subbundle of $S(TM)$ or $S(TM^{\perp})$ of rank 1. Also, from $\bar{g}(JN, N) = 0$ and $\bar{g}(JN, \xi) = -\bar{g}(N, J\xi) = 0$, $J(ltr(TM))$ is also a vector subbundle of $S(TM)$ or $S(TM^{\perp})$ of rank 1. Since $J\xi$ and JN are null vector fields satisfying $\bar{g}(J\xi, JN) = 1$ and both $S(TM)$ and $S(TM^{\perp})$ are non-degenerate, we see that either $\{J\xi, JN\} \in \Gamma(S(TM))$ or $\{J\xi, JN\} \in \Gamma(S(TM^{\perp})$. If $\{J\xi, JN\} \in \Gamma(S(TM^{\perp})$, since $J(Rad(TM))$, $J(ltr(TM))$ and $S(TM^{\perp})$ are non-degenerate of rank 1, we have $J(Rad(TM)) = J(ltr(TM)) = S(TM^{\perp})$. It is contradiction. Thus we choose a screen distribution $S(TM)$ that contains $J(Rad(TM))$ and $J(ltr(TM))$. For $L \in \Gamma(S(TM^{\perp})$, as $\bar{g}(JL, L) = 0$, $\bar{g}(JL, \xi) = -\bar{g}(L, J\xi) = 0$ and $\bar{g}(JL, N) = -\bar{g}(L, JN) = 0$, $J(S(TM^{\perp})$ is also a vector subbundle of $S(TM)$ such that

$$J(S(TM^{\perp})) \oplus_{orth} \{J(Rad(TM)) \oplus J(ltr(TM))\}.$$

We choose $S(TM)$ to contain $J(S(TM^{\perp}))$ too. Thus the screen distribution $S(TM)$ is expressed as follow:

$$S(TM) = \{J(Rad(TM)) \oplus J(ltr(TM))\} \oplus_{orth} J(S(TM^{\perp}) \oplus_{orth} H_o, \tag{2.3}$$

where H_o is a non-degenerate almost complex distribution on M with respect to J, i.e., $J(H_o) = H_o$. Denote $H' = J(ltr(TM)) \oplus_{orth} J(S(TM^{\perp}))$. Thus (2.3) gives $S(TM)$ as in condition (B) and $J(H') = K_1 \oplus_{orth} K_2$, where $L_1 = ltr(TM)$ and $K_2 = S(TM^{\perp})$. Hence M is a CR-lightlike submanifold of $\bar{M}$. □

From Theorem 2.1, the general decompositions (1.1) and (1.3) reduce to

$$TM = H \oplus H', \quad T\bar{M} = H \oplus H' \oplus tr(TM), \tag{2.4}$$

where H is a 2-lightlike almost complex distribution on M such that

$$H = Rad(TM) \oplus_{orth} J(Rad(TM)) \oplus_{orth} H_o.$$

Consider null vector fields $\{U, V\}$ and non-null vector field W such that

$$U = -JN, \quad V = -J\xi, \quad W = -JL. \tag{2.5}$$

Denote by S the projection morphism of TM on H. Then, by the first equation of (2.4)[denote (2.4)-1], any $X \in \Gamma(TM)$ is expressed as follows

$$X = SX + u(X)U + w(X)W, \quad JX = FX + u(X)N + w(X)L, \tag{2.6}$$

where u, v and w are 1-forms locally defined on M by

$$u(X) = g(X, V), \quad v(X) = g(X, U), \quad w(X) = \epsilon\, g(X, W) \tag{2.7}$$

and F is a tensor field of type $(1, 1)$ globally defined on M by

$$FX = JSX, \quad {}^{\forall} X \in \Gamma(TM).$$

Differentiating (2.5) with $X \in \Gamma(TM)$ and using the local Gauss and Weingartan formulas (1.4)~(1.8), (2.1), (2.6) and (2.7), we have

$$B(X, U) = C(X, V), \; C(X, W) = \epsilon D(X, U), \; B(X, W) = \epsilon D(X, V). \tag{2.8}$$

We say that two vectors X and Y on M are *conjugate* with respect to the second fundamental tensor h if $h(X, Y) = 0$. A self-conjugate vector is said to be an *asymptotic vector field.* Then by (2.8) we get

Theorem 2.2. *Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then the vector fields ξ and V are conjugate with respect to the second fundamental forms C and D.*

Proof. Replacing X with ξ in the first and third equations in (2.8) by turns and using the equation (1.9), we have $C(\xi, V) = 0$ and $D(\xi, V) = 0$. □

Definition 2. A half lightlike submanifold $(M, g, S(TM))$ is said to be *irrotational*[8] if $\bar{\nabla}_X \xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

Note 1. Since $B(X, \xi) = 0$ due to the first equation of (1.9), the above definition is equivalent to $D(X, \xi) = 0 = \phi(X)$ for all $X \in \Gamma(TM)$.

Theorem 2.3. *Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is irrotational. Moreover, if M is totally geodesic,* *i.e., $h = 0$, then H is an integrable and parallel distribution with respect to the induced connection ∇ on M.*

Proof. Take $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$. Then we show that $FY = JY \in \Gamma(H)$ due to $u(Y) = w(Y) = 0$. Apply J to (1.4) and use (1.4), (2.1), (2.5) and (2.6), we have

$$\begin{aligned} &\nabla_X FY + B(X, FY)N + D(X, FY)L \\ &= F(\nabla_X Y) + u(\nabla_X Y)N + w(\nabla_X Y)L - B(X, Y)U - D(X, Y)W. \end{aligned}$$

Taking the scalar product with ξ and L in this equation, we have

$$B(X, FY) = g(\nabla_X Y, V), \quad D(X, FY) = \epsilon g(\nabla_X Y, W), \tag{2.9}$$

$$(\nabla_X F)Y = -D(X, Y)U - D(X, Y)W. \tag{2.10}$$

Apply the operator $\bar{\nabla}_X$ to (2.7) and then, to (2.6)-2 and use (1.12)~(1.14), (2.1), (2.6)-2 and (2.7) and Gauss-Weingarten equations for M, we deduce

$$(\nabla_X u)(Y) = -u(Y)\tau(X) - w(Y)\phi(X) - B(X, FY), \tag{2.11}$$

$$(\nabla_X v)(Y) = v(Y)\tau(X) + \epsilon\, w(Y)\rho(X) - g(A_N X, FY), \tag{2.12}$$

$$(\nabla_X w)(Y) = -u(Y)\rho(X) + \epsilon v(Y)\phi(X) - D(X, FY), \tag{2.13}$$

$$\begin{aligned} (\nabla_X F)(Y) = {} & u(Y)A_N X + w(Y)A_L X - B(X, Y)U - D(X, Y)W \\ & - \epsilon\, v(Y)\phi(X)L, \quad \forall X,\, Y \in \Gamma(TM). \end{aligned} \tag{2.14}$$

Take $Y \in \Gamma(H)$ in (2.14) and use (2.10), we have $v(Y)\phi(X) = 0$ for all $X \in \Gamma(TM)$ and $Y \in \Gamma(H)$. Replace Y by V in this equation, we have $\phi(X) = 0$ for all $X \in \Gamma(TM)$. Thus M is irrotational. Moreover, if M is totally geodesic, then, by (2.9), H is an integrable and parallel distribution with respect to ∇.

3. Totally Umbilical Half Lightlike Submanifolds

Definition 3. We say that M is *totally umbilical*[3] in $\bar{M}$ if, on any coordinate neighborhood $\mathcal{U}$, there is a smooth vector field $\mathcal{H} \in \Gamma(tr(TM))$ such that

$$h(X, Y) = \mathcal{H}\, g(X, Y), \quad \forall X,\, Y \in \Gamma(TM). \tag{3.1}$$

In case $\mathcal{H} \neq 0$ on $\mathcal{U}$, we say that M is *proper totally umbilical.*

It is easy to see that M is totally umbilical if and only if, on each coordinate neighborhood $\mathcal{U}$, there exist smooth functions β and δ such that

$$B(X, Y) = \beta g(X, Y), \quad D(X, Y) = \delta g(X, Y), \quad \forall X,\, Y \in \Gamma(TM). \tag{3.2}$$

Theorem 3.1. *Let $(M, g, S(TM))$ be a totally umbilical real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then M is totally geodesic.*

Proof. From the third equation of (2.8) and (3.2), we show that

$$\beta g(X, W) = \epsilon\delta g(X, V), \quad \forall X \in \Gamma(TM).$$

Replacing X by W and U in this equation by turns, we have $\beta = 0$ and $\delta = 0$ respectively. Thus $B = D = 0$ and M is totally geodesic in $\bar{M}$. □

Corollary 1. *We have the following assertions:*

(1) *There exist no proper totally umbilical real half lightlike submanifolds of an indefinite Kaehler manifold $\bar{M}$.*
(2) *The second fundamental form C of the screen distribution $S(TM)$ is degenerate on $\Gamma(S(TM))$.*
(3) *The vector fields V and W are conjugate to any vector field on M with respect to C. In particular, V and W are asymptotic vector fields.*
(4) *A_N is $\Gamma(J(Rad(TM)) \oplus_{orth} H_o)$-valued shape operator related to C.*

Proof. From the first two equations of (2.8) and the fact that $B = D = 0$, we have $C(X, V) = C(X, W) = 0$ for any $X \in \Gamma(TM)$. Therefore we have (2) and (3). From these equations and (1.13), we get $g(A_N X, V) = g(A_N X, W) = 0$ for all $X \in \Gamma(TM)$, which proves the assertion (4). □

Combining Theorem 2.3 and 3.1, we have the following theorem:

Theorem 3.2. *Let $(M, g, S(TM))$ be a totally umbilical real half lightlike submanifold of an indefinite Kaehler manifold $\bar{M}$. Then H is a parallel distribution with respect to ∇ and M is locally a product manifold $L_u \times L_w \times M^\sharp$, where L_u and L_w are null and non-null curves tangent to $J(ltr(TM))$ and $J(S(TM^\perp))$ respectively and $M^\sharp$ is a leaf of H.*

Theorem 3.3. *Let $(M, g, S(TM))$ be a totally umbilical real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then we have $c = 0$.*

Proof. Using (1.18) and the fact that $B = D = 0$, we get

$$\frac{c}{4}\{u(X)\bar{g}(JY, Z) - u(Y)\bar{g}(JX, Z) + 2u(Z)\bar{g}(X, JY)\} = 0,$$

for all $X, Y, Z \in \Gamma(TM)$. Replace Y by ξ and use (2.5) and (2.7), we show that $\frac{3c}{4}u(X)u(Z) = 0$ for all X, $Z \in \Gamma(TM)$. Take $X = Z = U$, we get $c = 0$. □

Corollary 2. *There exist no totally umbilical real half lightlike submanifolds of an indefinite complex space form $\bar{M}(c)$ with $c \neq 0$.* □

Theorem 3.4. *Let $(M, g, S(TM))$ be a totally umbilical real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then M and each leaf M^* of $S(TM)$ are spaces of constant curvature 0.*

Proof. Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$. Using this quasi-orthonormal frame field, we obtain

$$R(X,Y)Z = \sum_{a=1}^{2m-3} \epsilon_a g(R(X,Y)Z, W_a)W_a + g(R(X,Y)Z, N)\xi, \tag{3.3}$$

for any X, $Y \in \Gamma(TM)$ and $\epsilon_a = g(W_a, W_a)$. Using (1.17), (1.19) and the last equation, we have $R(X,Y)Z = 0$ for any $X, Y, Z \in \Gamma(TM)$, due to the facts that $c = 0$ and $B = D = 0$. Thus M is a lightlike manifold of constant curvature 0. Also, from (1.17) and (1.21), we also have $R^*(X,Y)Z = 0$ for any $X, Y, Z \in \Gamma(S(TM))$. Thus M^* is also a semi-Euclidean space. □

Combining Theorem 3.2 and 3.4, we have the following theorem:

Theorem 3.5. *Let $(M, g, S(TM))$ be a totally umbilical real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. Then M is locally a product manifold $L_u \times L_w \times M^\sharp$, where L_u and L_w are null and non-null curves respectively and $M^\sharp$ is a 2-lightlike manifold of constant curvature 0.*

4. Totally Umbilical Screen Distributions

Definition 4. We say that (each leaf M^* of) $S(TM)$ is *totally umbilical*[3] in M if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ such that $A_N X = \gamma PX$ for any $X \in \Gamma(TM)$, or equivalently,

$$C(X, PY) = \gamma\, g(X,Y), \quad \forall X, Y \in \Gamma(TM). \tag{4.1}$$

In case $\gamma = 0$ (or $\gamma \neq 0$) on $\mathcal{U}$, we say that (each leaf M^* of) $S(TM)$ is *totally geodesic* (or *proper totally umbilical*) in M.

In general, $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$:

Theorem 4.1 ([1]). *Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M}, \bar{g})$. Then the following are equivalent:*

(1) *$S(TM)$ is integrable.*

(2) *C is symmetric on $\Gamma(S(TM))$.*

(3) *A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.*

Note 2. If $S(TM)$ is totally umbilical in M, then C is symmetric on $\Gamma(S(TM))$. Thus $S(TM)$ is integrable and M is locally a product manifold $L_\xi \times M^*$, where L_ξ is a null curve tangent to $Rad(TM)$ and M^* is a leaf of $S(TM)$ [2].

Theorem 4.2. *Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$. If $S(TM)$ is totally umbilical in M, then we have $c = 0$ and $C = 0$, on any coordinate neighborhood $\mathcal{U} \subset M$. Moreover,*

(1) *$c = 0$ implies the ambient space $\bar{M}(c)$ is a semi-Euclidean space,*

(2) *$C = 0$, on any $\mathcal{U} \subset M$, implies $S(TM)$ is totally geodesic in M.*

Proof. Using the first two equations of (2.8) and (4.1), we have

$$B(X, U) = \gamma\, g(X, V), \quad D(X, U) = \epsilon\, \gamma\, g(X, W), \tag{4.2}$$

for all $X \in \Gamma(TM)$. Using (1.19), (1.22), (2.2), (2.5), (2.7) and (4.1), we get

$$\begin{aligned} &\gamma\,\{B(Y, PZ)\eta(X) - B(X, PZ)\eta(Y)\} \\ &\quad + \epsilon\,\{D(Y, PZ)\rho(X) - D(X, PZ)\rho(Y)\} \\ &= \{X[\gamma] - \gamma\tau(X) - \frac{c}{4}\eta(X)\}g(Y, PZ) \\ &\quad - \{Y[\gamma] - \gamma\tau(Y) - \frac{c}{4}\eta(Y)\}g(X, PZ) \\ &\quad + \frac{c}{4}\{\bar{g}(JX, PZ)v(Y) - \bar{g}(JY, PZ)v(X) - 2\bar{g}(X, JY)v(PZ)\}, \end{aligned} \tag{4.3}$$

for any X, Y, $Z \in \Gamma(TM)$. Replacing X by ξ in this equation, we have

$$\begin{aligned} &\gamma B(Y, PZ) + \epsilon D(Y, PZ)\rho(\xi) \\ &= \{\xi[\gamma] - \gamma\tau(\xi) - \frac{c}{4}\}g(Y, PZ) - \frac{c}{4}\{u(PZ)v(Y) + 2u(Y)v(PZ)\}, \end{aligned} \tag{4.4}$$

for all Y, $Z \in \Gamma(TM)$. Taking $Y = U$, $PZ = V$; $Y = V$, $PZ = U$ and $Y = PZ = U$ in (4.4) by turns and using (2.7) and (4.2), we have

$$\xi[\gamma] - \gamma\tau(\xi) - \frac{3c}{4} = 0, \quad \xi[\gamma] - \gamma\tau(\xi) - \frac{c}{2} = 0, \quad \gamma^2 = 0,$$

respectively. This shows that $\gamma = 0$ and $c = 0$. Thus we have our theorem. □

Corollary 3. *We have the following assertions:*

(1) *There exist no real half lightlike submanifolds of $\bar{M}(c)$ with $c \neq 0$ such that $S(TM)$ is totally umbilical in M.*

(2) *There exist no real half lightlike submanifolds of* $\bar{M}(c)$ *such that* $S(TM)$ *is proper totally umbilical.*
(3) *The second fundamental form tensor* h *is degenerate on* M.
(4) *The vector field* U *is conjugate to any vector field on* M *with respect to* h. *In particular,* U *is an asymptotic vector field.*

Proof. From the two equations of (4.2) with $\gamma = 0$, we obtain

$$h(X, U) = 0, \quad \forall X \in \Gamma(TM). \tag{4.5}$$

Therefore h is degenerate on M and we get (3). By (4.5), we have (4).

Theorem 4.3. *Let* $(M, g, S(TM))$ *be a real half lightlike submanifold of an indefinite complex space form* $\bar{M}(c)$ *such that* $S(TM)$ *is totally umbilical in* M. *Then the curvatures* R *and* R^* *are related by*

$$R(X, Y)Z = R^*(PX, PY)PZ, \quad \forall X, Y, Z \in \Gamma(TM). \tag{4.6}$$

Proof. From (1.9) with $\phi = 0$ and (4.3) with $\gamma = c = 0$, we have

$$D(Y, Z)\rho(X) = D(X, Z)\rho(Y), \quad \forall X, Y, Z \in \Gamma(TM). \tag{4.7}$$

From this and (1.19), we obtain $\bar{g}(R(X, Y)Z, N) = 0$. Thus we see that the equation (4.6) of this theorem is equivalent with the following equation:

$$g(R(X, Y)Z, PW) = g(R^*(PX, PY)PZ, PW), \tag{4.8}$$

for all $X, Y, Z, W \in \Gamma(TM)$. Due to (1.17) with $\gamma = c = 0$, we show that $g(R(X, Y)\xi, Z) = 0$. Thus we see that (4.8) is true for $Z = \xi$. Using (1.9), (1.17) and (1.21) satisfy $\gamma = c = 0$, we derive (4.8). □

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X, Y) = trace\{Z \to R(Z, X)Y\}, \tag{4.9}$$

for any $X, Y \in \Gamma(TM)$. Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$. Using this quasi-orthonormal frame field and the equation (4.9), we obtain

$$R^{(0,2)}(X, Y) = \sum_{a=1}^{m} \epsilon_a \, g(R(W_a, X)Y, W_a) + \bar{g}(R(\xi, X)Y, N), \tag{4.10}$$

for any $X, Y \in \Gamma(TM)$ and $\epsilon_a = g(W_a, W_a)$ is the sign of W_β. In general, the induced Ricci type tensor $R^{(0,2)}$, defined by the method of the geometry of the non-degenerate submanifolds [9], is not symmetric [2, 3, 5]. A tensor field $R^{(0,2)}$ of half lightlike submanifolds M is called its *induced Ricci tensor* of M if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by *Ric*.

Theorem 4.4. *Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in M. Then the Ricci type tensor $R^{(0,2)}$ is a symmetric Ricci tensor. Moreover, if M is an Einstein manifold, then M is Ricci flat.*

Proof. By Theorem 2.3, since M is an irrotational real half lightlike submanifold of $\bar{M}(c)$, then, using (1.17) and (1.19), the equation (4.10) reduces to

$$R^{(0,2)}(X,Y) = D(X,Y)trA_L - \epsilon g(A_L X, A_L Y), \tag{4.11}$$

where trA_L is the trace of A_L. Thus $R^{(0,2)}$ is a symmetric Ricci tensor *Ric*. Let M be an Einstein manifold, that is, $R^{(0,2)} = \kappa g$ for a constant κ. Replacing Y by U in (4.11) and using the fact that $D(X,U) = g(A_L U, X) = 0$ for any $X \in \Gamma(TM)$, we obtain $\kappa g(X,U) = 0$ for all $X \in \Gamma(TM)$. Replacing X by V in this equation, we have $\kappa = 0$. Thus M is Ricci flat. □

Definition 5. A vector field X on $\bar{M}$ is said to be *conformal Killing* [5] if there exists a smooth function α such that $\bar{\mathcal{L}}_X \bar{g} = -2\alpha\bar{g}$, where $\bar{\mathcal{L}}_X$ denotes the Lie derivative with respect to X. In particular, if $\alpha = 0$, then X is called a *Killing*. A distribution $\mathcal{G}$ on $\bar{M}$ is said to be *conformal Killing* (or *Killing*) if each vector field belonging to $\mathcal{G}$ is a conformal Killing (or Killing).

Theorem 4.5. *Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in M. If $S(TM^\perp)$ is a conformal Killing distribution, then we have $D = 0$.*

Proof. Using the equations (1.6) and (1.15), we have

$$\begin{aligned}(\bar{\mathcal{L}}_L \bar{g})(X,Y) &= \bar{g}(\bar{\nabla}_X L, Y) + \bar{g}(X, \bar{\nabla}_Y L), \ \forall X,\ Y \in \Gamma(TM),\\ \bar{g}(\bar{\nabla}_X L, Y) &= -g(A_L X, Y) + \phi(X)\eta(Y) \ = \ -\epsilon D(X,Y).\end{aligned}$$

Thus $(\bar{\mathcal{L}}_L \bar{g})(X,Y) = -2\epsilon D(X,Y)$ for any $X,\ Y \in \Gamma(TM)$. We show that if $S(TM^\perp)$ is a conformal Killing distribution, then there exists a smooth function δ such that $D(X,Y) = \epsilon\delta g(X,Y)$ for all $X,\ Y \in \Gamma(TM)$. Using this and the second equation of (4.2) with $\gamma = 0$, we have $0 = D(X,U) = \epsilon\delta g(X,U)$ for any $X \in \Gamma(TM)$. Replace X by V in this equation, we obtain $\delta = 0$. □

Theorem 4.6. *Let $(M, g, S(TM))$ be a real half lightlike submanifold of an indefinite complex space form $\bar{M}(c)$ such that $S(TM)$ is totally umbilical in M. If $S(TM^{\perp})$ is a conformal Killing, then M and each leaf M^* of $S(TM)$ are spaces of curvature 0. Moreover, M is locally a product manifold $L_\xi \times M^*$, where L_ξ is a null curve and M^* is a semi-Euclidean space.*

Proof. Using (1.17), (1.19), (3.3) and (4.7) with $\gamma = c = 0$, we have

$$R(X,Y)Z = R^*(PX,PY)PZ = D(Y,Z)A_LX - D(X,Z)A_LY,$$

for any $X, Y, Z \in \Gamma(TM)$. Thus, if $S(TM^{\perp})$ is a conformal Killing, then we have $R(X,Y)Z = R^*(PX,PY)PZ = 0$ due to $D = 0$. Thus M and M^* are semi-Euclidean spaces of constant curvature 0. By Note 2, we have our assertion. □

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