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HALF LIGHTLIKE SUBMANIFOLDS WITH TOTALLY UMBILICAL SCREEN DISTRIBUTIONS

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ABSTRACT. We study the geometry of half lightlike submanifold M of a semi-Riemannian space form $\bar{M}(c)$ subject to the conditions : (a) the screen distribution on M is totally umbilic in M and the coscreen distribution on M is conformal Killing on $\bar{M}$ or (b) the screen distribution is totally geodesic in M and M is irrotational.

1. INTRODUCTION

It is well known that the radical distribution $Rad(TM) = TM \cap TM^{\perp}$ of the half lightlike submanifolds (M, g) of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is a vector subbundle of the tangent bundle TM and the normal bundle $TM^{\perp}$, of rank 1. Then there exist two complementary non-degenerate distributions $S(TM)$ and $S(TM^{\perp})$ of $Rad(TM)$ in TM and $TM^{\perp}$ respectively, which called the *screen* and *coscreen distributions* on M, such that

$$TM = Rad(TM) \oplus_{orth} S(TM), \quad TM^{\perp} = Rad(TM) \oplus_{orth} S(TM^{\perp}), \tag{1.1}$$

where the symbol $\oplus_{orth}$ denotes the orthogonal direct sum. We denote such a half lightlike submanifold by $(M, g, S(TM))$. Denote by $F(M)$ the algebra of smooth functions on M and by $\Gamma(E)$ the $F(M)$ module of smooth sections of a vector bundle E over M. We use the same notation for any other vector bundle. Then there exist vector fields $\xi \in \Gamma(Rad(TM))$ and $u \in \Gamma(S(TM^{\perp}))$ such that

$$\bar{g}(\xi, v) = 0, \qquad \bar{g}(u, u) = \epsilon = \pm 1,$$

for any $v \in \Gamma(TM^{\perp})$. Consider the orthogonal complementary distribution $S(TM)^{\perp}$ to $S(TM)$ in $T\bar{M}$. Certainly ξ and u belong to $\Gamma(S(TM)^{\perp})$. Thus we have

$$S(TM)^{\perp} = S(TM^{\perp}) \oplus_{orth} S(TM^{\perp})^{\perp},$$

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where $S(TM^{\perp})^{\perp}$ is the orthogonal complementary to $S(TM^{\perp})$ in $S(TM)^{\perp}$. For any null section ξ of $Rad(TM)$ on a coordinate neighborhood $\mathcal{U} \subset M$, there exists a uniquely defined vector field $N \in \Gamma(ltr(TM))$ [1] satisfying

$$(1.2) \qquad \bar{g}(\xi, N) = 1, \quad \bar{g}(N, N) = \bar{g}(N, X) = \bar{g}(N, u) = 0,$$

for any $X \in \Gamma(S(TM))$. We call $ltr(TM)$, N and $tr(TM) = S(TM^{\perp}) \oplus_{orth} ltr(TM)$ the *lightlike transversal vector bundle, lightlike transversal vector field* and *transversal vector bundle* of M with respect to $S(TM)$ respectively. Then the tangent space $T\bar{M}$ of the ambient manifold $\bar{M}$ is decomposed as follows:

$$(1.3) \qquad \begin{aligned} T\bar{M} &= TM \oplus tr(TM) = \{Rad(TM) \oplus tr(TM)\} \oplus_{orth} S(TM) \\ &= \{Rad(TM) \oplus ltr(TM)\} \oplus_{orth} S(TM^{\perp}) \oplus_{orth} S(TM). \end{aligned}$$

The purpose of this paper is to study the geometry of half lightlike submanifolds M of semi-Riemannian space form $\bar{M}(c)$ with constant curvature c subject to the constraints (a) $S(TM)$ is totally umbilic in M and $S(TM^{\perp})$ is conformal Killing on $\bar{M}$ or (b) $S(TM)$ is totally geodesic in M and M is irrotational. In next section 2, we have a classification theorem for half lightlike submanifolds M of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, such that $S(TM)$ is totally umbilic and $S(TM^{\perp})$ is conformal Killing. This theorem shows that the lightlike and radical second fundamental forms B and C of such a half lightlike submanifold satisfy $B = 0$ or $C = 0$. In section 3, we study the geometry of irrotational half lightlike submanifolds M of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, $\epsilon c > 0$ such that $S(TM)$ is totally geodesic. Recall the following structure equations:

Let $\bar{\nabla}$ be the Levi-Civita connection of $\bar{M}$ and let P be the projection morphism of $\Gamma(TM)$ on $\Gamma(S(TM))$ with respect to the decomposition (1.1). Then the local Gauss and Weingartan formulas are given by

$$(1.4) \qquad \bar{\nabla}_X Y = \nabla_X Y + B(X, Y)N + D(X, Y)u,$$

$$(1.5) \qquad \bar{\nabla}_X N = -A_N X + \tau(X)N + \rho(X)u,$$

$$(1.6) \qquad \bar{\nabla}_X u = -A_u X + \phi(X)N,$$

$$(1.7) \qquad \nabla_X PY = \nabla^*_X PY + C(X, PY)\xi,$$

$$(1.8) \qquad \nabla_X \xi = -A^*_\xi X - \tau(X)\xi,$$

for any $X, Y \in \Gamma(TM)$, where ∇ and ∇^* are induced linear connections on TM and $S(TM)$ respectively, the bilinear forms B and D on TM are called the *local lightlike* and *screen second fundamental forms* of M respectively, C is called the *local* *radical second fundamental form* on $S(TM)$. A_N, A^*_ξ and A_u are linear operators on $\Gamma(TM)$ and τ, ρ and ϕ are 1-forms on TM.

Since $\bar{\nabla}$ is torsion-free, so is ∇ and both B and D are symmetric. From the facts $B(X, Y) = \bar{g}(\bar{\nabla}_X Y, \xi)$ and $D(X, Y) = \epsilon\bar{g}(\bar{\nabla}_X Y, u)$ for any $X, Y \in \Gamma(TM)$, we show that B and D are independent of the choice of a screen distribution and

$$B(X, \xi) = 0, \qquad D(X, \xi) = -\epsilon\phi(X), \tag{1.9}$$

for any $X \in \Gamma(TM)$. The induced connection ∇ on M is not metric and satisfies

$$(\nabla_X g)(Y, Z) = B(X, Y)\,\eta(Z) + B(X, Z)\,\eta(Y), \tag{1.10}$$

for all $X, Y, Z \in \Gamma(TM)$, where η is a 1-form on TM such that

$$\eta(X) = \bar{g}(X, N), \ \forall X \in \Gamma(TM). \tag{1.11}$$

But the connection ∇^* on M^* is metric. The above three local second fundamental forms of M and M^* are related to their shape operators by

$$B(X, Y) = g(A^*_\xi X, Y), \qquad \bar{g}(A^*_\xi X, N) = 0, \tag{1.12}$$

$$C(X, PY) = g(A_N X, PY), \quad \bar{g}(A_N X, N) = 0, \tag{1.13}$$

$$\epsilon D(X, PY) = g(A_u X, PY), \quad \bar{g}(A_u X, N) = \epsilon\rho(X), \tag{1.14}$$

$$\epsilon D(X, Y) = g(A_u X, Y) - \phi(X)\eta(Y). \tag{1.15}$$

From (1.12), A^*_ξ is $S(TM)$-valued and self-adjoint on $\Gamma(TM)$ such that

$$A^*_\xi \xi = 0. \tag{1.16}$$

We denote by $\bar{R}$, R and R^* the curvature tensors of the Levi-Civita connection $\bar{\nabla}$ on $\bar{M}$, the induced connection ∇ on M and the induced connection ∇^* on $S(TM)$ respectively. Using the Gauss-Weingarten equations for M and $S(TM)$, for any $X, Y, Z, W \in \Gamma(TM)$, we obtain the Gauss-Codazzi equations for M and $S(TM)$:

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,PW) &= g(R(X,Y)Z,PW) \\ &\quad + B(X,Z)C(Y,PW) - B(Y,Z)C(X,PW) \\ &\quad + \epsilon\{D(X,Z)D(Y,PW) - D(Y,Z)D(X,PW)\}, \end{aligned} \tag{1.17}$$

$$\begin{aligned} \bar{g}(\bar{R}(X,Y)Z,\xi) &= (\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) \\ &\quad + B(Y,Z)\tau(X) - B(X,Z)\tau(Y) \\ &\quad + D(Y,Z)\phi(X) - D(X,Z)\phi(Y), \end{aligned} \tag{1.18}$$

$$\bar{g}(\bar{R}(X,Y)Z,N) = \bar{g}(R(X,Y)Z,N) + \epsilon\{D(X,Z)\rho(Y) - D(Y,Z)\rho(X)\}, \tag{1.19}$$

$$\bar{g}(\bar{R}(X,Y)Z,u) = \epsilon\{(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) + B(Y,Z)\rho(X) - B(X,Z)\rho(Y)\}, \tag{1.20}$$

$$\bar{g}(R(X,Y)PZ,PW) = g(R^*(X,Y)PZ,PW) + C(X,PZ)B(Y,PW) - C(Y,PZ)B(X,PW), \tag{1.21}$$

$$g(R(X,Y)PZ,N) = (\nabla_X C)(Y,PZ) - (\nabla_Y C)(X,PZ) + C(X,PZ)\tau(Y) - C(Y,PZ)\tau(X). \tag{1.22}$$

2. Conformal Killing Coscreen Distributions

Definition 1. We say that (each integral leaf of) $S(TM)$ is *totally umbilic* [2] in M if, on any coordinate neighborhood $\mathcal{U} \subset M$, there is a smooth function γ such that

$$C(X,PY) = \gamma g(X,Y),\ \forall X,\ Y \in \Gamma(TM). \tag{2.1}$$

In case $\gamma = 0$ on $\mathcal{U}$, we say that (each integral leaf of) $S(TM)$ is *totally geodesic.*

In general, $S(TM)$ is not necessarily integrable. The following result gives equivalent conditions for the integrability of $S(TM)$:

Theorem 2.1 ([1]). *Let $(M,g,S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. Then the following assertions are equivalent:*

(1) *$S(TM)$ is integrable.*
(2) *C is symmetric on $\Gamma(S(TM))$.*
(3) *A_N is self-adjoint on $\Gamma(S(TM))$ with respect to g.*

Note 1. If $S(TM)$ is totally umbilic in M, then C is symmetric on $\Gamma(S(TM))$. Thus, by Theorem 2.1, $S(TM)$ is integrable and M is locally a product manifold $L \times M^*$, where L is a null curve and M^* is a leaf of $S(TM)$ [2].

Let $\bar{M}(c)$ be a semi-Riemannian space form and let $S(TM)$ be totally umbilic in M. Using (1.10), (1.19), (1.22) and (2.1), for any X, Y, $Z \in \Gamma(TM)$, we obtain

$$\begin{aligned}&\gamma\{B(Y,PZ)\eta(X) - B(X,PZ)\eta(Y)\} + \epsilon\{D(Y,PZ)\rho(X) - D(X,PZ)\rho(Y)\}\\ &= \{X[\gamma] - \gamma\tau(X) - c\eta(X)\}g(Y,PZ) - \{Y[\gamma] - \gamma\tau(Y) - c\eta(Y)\}g(X,PZ).\end{aligned}$$

Replacing Y by ξ in this equation and using (1.9), for all $X,\ Y \in \Gamma(TM)$, we have

$$\gamma B(X,Y) + \epsilon D(X,PY)\rho(\xi) + \phi(PY)\rho(X) = \{\xi[\gamma] - \gamma\tau(\xi) - c\}g(X,Y). \tag{2.2}$$

Definition 2. A vector field X on $(\bar{M},\bar{g})$ is said to be *conformal Killing*[8] if there exists a smooth function α such that $\bar{\mathcal{L}}_X\bar{g} = -2\alpha\bar{g}$, where $\bar{\mathcal{L}}_X$ denotes the Lie derivative with respect to X. In particular, if α is a constant, then X is called a *homothetic Killing*. A distribution $\mathcal{G}$ on $\bar{M}$ is said to be *conformal* (or *homothetic*) *Killing* if each vector field belonging to $\mathcal{G}$ is a conformal (or homothetic) Killing.

Theorem 2.2. *Let $(M,g,S(TM))$ be a half lightlike submanifold of a semi-Riemannian manifold $(\bar{M},\bar{g})$. If the coscreen distribution $S(TM^{\perp})$ is a conformal Killing on $\bar{M}$, then there exists a smooth function δ such that*

$$D(X,Y) = \epsilon\delta g(X,Y), \quad \forall X,\ Y \in \Gamma(TM). \tag{2.3}$$

Proof. By straightforward calculations and use (1.6) and (1.15), we have

$$(\bar{\mathcal{L}}_u\bar{g})(X,Y) = \bar{g}(\bar{\nabla}_X u, Y) + \bar{g}(X, \bar{\nabla}_Y u), \ u \in \Gamma(S(TM^{\perp})),$$
$$\bar{g}(\bar{\nabla}_X u, Y) = -g(A_u X, Y) + \phi(X)\eta(Y) = -\epsilon D(X,Y),$$

for any $X,\ Y \in \Gamma(TM)$. Thus $(\bar{\mathcal{L}}_u\bar{g})(X,Y) = -2\epsilon D(X,Y)$ for any $X,\ Y \in \Gamma(TM)$. Therefore we show that if $S(TM^{\perp})$ is a conformal Killing distribution, then there exists a smooth function δ such that $D(X,Y) = \epsilon\delta g(X,Y)$ for any $X,\ Y \in \Gamma(TM)$. □

Let $\bar{M}(c)$ be a semi-Riemannian space form such that $S(TM)$ is totally umbilic in M and $S(TM^{\perp})$ is conformal Killing on $\bar{M}$. Then, using (1.9) and (2.3), we show the 1-form ϕ vanishes identically, i.e., $\phi = 0$. Thus (1.18) and (1.20) reduce to

$$(\nabla_X B)(Y,Z) - (\nabla_Y B)(X,Z) = B(X,Z)\tau(Y) - B(Y,Z)\tau(X), \tag{2.4}$$

$$(\nabla_X D)(Y,Z) - (\nabla_Y D)(X,Z) = B(X,Z)\rho(Y) - B(Y,Z)\rho(X), \tag{2.5}$$

for any $X,\ Y,\ Z \in \Gamma(TM)$. Using (1.9), (2.2) and (2.3), we obtain

$$\gamma B(X,Y) = \{\xi[\gamma] - \gamma\tau(\xi) - \delta\rho(\xi) - c\}g(X,Y), \ \forall X,\ Y \in \Gamma(TM). \tag{2.6}$$

For the rest of this paper, by a *totally umbilical distribution* and a *conformal Killing distribution* we shall mean a *totally umbilical distribution in M* and a *conformal Killing distribution on $\bar{M}$* unless otherwise specified.

Theorem 2.3. *Let $(M,g,S(TM))$ be a half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c),\bar{g})$, $m > 2$, such that $S(TM)$ is totally umbilic and $S(TM^{\perp})$ is conformal Killing. Then $C = 0$ or $B = 0$. Moreover we show that*

(1) *$C = 0$, on any $\mathcal{U} \subset M$, implies $S(TM)$ is a totally geodesical distribution,*

(2) *$B = 0$, on any $\mathcal{U} \subset M$, implies M is totally umbilical immersed in $\bar{M}(c)$ and the induced connection ∇ on M is a metric connection.*

Proof. Assume that $C \neq 0$, that is, $\gamma \neq 0$. Then, from (2.6), we have

$$B(X,Y) = \beta g(X,Y), \ \forall X, Y \in \Gamma(TM), \tag{2.7}$$

where $\beta = \gamma^{-1}(\xi[\gamma] - \gamma\tau(\xi) - \delta\rho(\xi) - c)$. Since $S(TM)$ is totally umbilic, by Note 1, M is locally a product manifold $L \times M^*$ where L is a null curve and M^* is a leaf of $S(TM)$. From the equations (1.17), (1.21), (2.1), (2.3) and (2.7), we have

$$R^*(X,Y)Z = (c + 2\beta\gamma + \epsilon\delta^2)\{g(Y,Z)X - g(X,Z)Y\},$$

for any X, Y, $Z \in \Gamma(S(TM))$, where R^* is the curvature tensor of M^*. Let Ric^* be the symmetric Ricci tensor of M^*. From the last equation, we have

$$Ric^*(X,Y) = (c + 2\beta\gamma + \epsilon\delta^2)(m-1)g(X,Y),$$

for any X, $Y \in \Gamma(S(TM))$. Thus the leaf M^* of $S(TM)$ is an Einstein semi-Riemannian manifold of constant curvature $(c+2\beta\gamma+\epsilon\delta^2)$ due to $m > 2$. From the equation (2.6), we have $\xi[\gamma] = \beta\gamma + \gamma\tau(\xi) + \delta\rho(\xi) + c$. Differentiating (2.7) and (2.3) and then, using (1.10), (2.4) and (2.5), we have

$$\{X[\beta] + \beta\tau(X) - \beta^2\eta(X)\}g(Y,Z) = \{Y[\beta] + \beta\tau(Y) - \beta^2\eta(Y)\}g(X,Z), \tag{2.8}$$

$$\{X[\delta] + \epsilon\beta\rho(X) - \beta\delta\eta(X)\}g(Y,Z) = \{Y[\delta] + \epsilon\beta\rho(Y) - \beta\delta\eta(Y)\}g(X,Z), \tag{2.9}$$

for any X, Y, $Z \in \Gamma(S(TM))$. Replacing X by ξ in these two equations, we have $\xi[\beta] = \beta^2 - \beta\tau(\xi)$ and $\xi[\delta] = \beta\delta - \epsilon\beta\rho(\xi)$ respectively. Since $(c + 2\beta\gamma + \epsilon\delta^2)$ is a constant, we get $\xi[c + 2\beta\gamma + \epsilon\delta^2] = 2\beta(c + 2\beta\gamma + \epsilon\delta^2) = 0$. Therefore $\beta = 0$ or $c + 2\beta\gamma + \epsilon\delta^2 = 0$. If $c + 2\beta\gamma + \epsilon\delta^2 = 0$, then M^* is a semi-Euclidean space and the second fundamental form C of M^* satisfies $C = 0$. It is a contradiction to $C \neq 0$. Thus we have $\beta = 0$. Consequently, we get $B = 0$ by (2.7). In this case, from (2.3) and (2.7), we show that $h(X,Y) = \mathcal{H}g(X,Y)$ for all $X, Y \in \Gamma(TM)$, where $h(X,Y) = B(X,Y)N + D(X,Y)u = D(X,Y)u$ is the second fundamental form of M and $\mathcal{H} = \beta N + \epsilon\delta u = \epsilon\delta u$ is the curvature vector field on M. Thus M is totally umbilic in $\bar{M}$. Also, from (1.10), we see that $(\nabla_X g)(Y,Z) = 0$ for all $X, Y, Z \in \Gamma(TM)$, that is, the induced connection ∇ on M is a metric one. □

The induced Ricci type tensor $R^{(0,2)}$ of M is defined by

$$R^{(0,2)}(X,Y) = trace\{Z \to R(Z,X)Y\}, \tag{2.10}$$

for any X, $Y \in \Gamma(TM)$. Consider the induced quasi-orthonormal frame field $\{\xi; W_a\}$ on M such that $Rad(TM) = Span\{\xi\}$ and $S(TM) = Span\{W_a\}$. Using this quasi-orthonormal frame field and the equation (2.10), we obtain

$$R^{(0,2)}(X,Y) = \sum_{a=1}^{m} \epsilon_a\, g(R(W_a, X)Y, W_a) + \bar{g}(R(\xi, X)Y, N), \tag{2.11}$$

for any X, $Y \in \Gamma(TM)$ and $\epsilon_a = g(W_a, W_a)$ is the sign of W_β. In general, the induced Ricci type tensor $R^{(0,2)}$ is not symmetric [3, 5]. Therefore $R^{(0,2)}$ has no geometric or physical meaning similar to the Ricci curvature of the non-degenerate submanifolds and it is just a tensor quantity. Hence we need the following definition: A tensor field $R^{(0,2)}$ of half lightlike submanifolds M is called its *induced Ricci tensor* of M if it is symmetric. A symmetric $R^{(0,2)}$ tensor will be denoted by *Ric*.

Theorem 2.4. *Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, such that $S(TM)$ is totally umbilic and $S(TM^{\perp})$ is conformal Killing. Then M admits an induced symmetric Ricci tensor. Moreover, both M and the leaf M^* of $S(TM)$ are Einstein manifolds and the coscreen $S(TM^{\perp})$ is a homothetic Killing distribution.*

Proof. Using (1.17), (1.19), (2.11) and the fact $\beta\gamma = 0$ by Theorem 2.3, we have

$$R^{(0,2)}(X,Y) = \{c + \delta\rho(\xi) + (m-1)(c + \epsilon\delta^2)\}g(X,Y), \tag{2.12}$$

for any X, $Y \in \Gamma(TM)$. Thus $R^{(0,2)}$ is a symmetric Ricci tensor *Ric* and M is an Einstein manifold. Also, from (1.17) and (1.21), we have

$$R^*(X,Y)Z = (c + \epsilon\delta^2)\{g(Y,Z)X - g(X,Z)Y\}, \tag{2.13}$$

$$Ric^*(X,Y) = (m-1)(c + \epsilon\delta^2)g(X,Y), \tag{2.14}$$

for any X, Y, $Z \in \Gamma(S(TM))$. From (2.14), we show that M^* is also an Einstein manifold. Since $m > 2$, the function $(c+\epsilon\delta^2)$ is a constant. Therefore, the conformal factor δ is a constant, i.e., $S(TM^{\perp})$ is a homothetic Killing distribution. □

Combining Note 1 and Theorem 2.3 and 2.4, we have the following theorem:

Theorem 2.5. *Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, such that $S(TM)$ is totally umbilic and $S(TM^{\perp})$ is conformal Killing with conformal factor δ. Then M is an Einstein manifold and locally a product manifold $L \times M^*$, where L is a null curve in M and M^* is an Einstein semi-Riemannian space form of constant curvature $(c + \epsilon\delta^2)$. Furthermore, the coscreen $S(TM^{\perp})$ is a homothetic Killing distribution.*

Recall the following notion of *null sectional curvature* [2, 3, 4]. Let $x \in M$ and let ξ be a null vector of T_xM. A plane H of T_xM is called a *null plane* directed by ξ if it contains ξ, $g_x(\xi, W) = 0$ for any $W \in H$ and there exists $W_o \in H$ such that $g_x(W_o, W_o) \neq 0$. Then, the null sectional curvature of H, with respect to the null vector ξ and the induced connection ∇ of M, is defined as a real number

$$K_\xi(H) = \frac{g_x(R(W, \xi)\xi, W)}{g_x(W, W)},$$

where $W \neq 0$ is any vector in H independent with ξ. It is easy to see that $K_\xi(H)$ is independent of W but depends in a quadratic fashion on ξ.

Theorem 2.6. *Let $(M, g, S(TM))$ be a half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, such that $S(TM)$ is totally umbilic and $S(TM^\perp)$ is conformal Killing. Then every null plane H of T_xM directed by ξ has everywhere zero null sectional curvatures.*

Proof. From (1.17), (2.3) and the fact that $\beta\gamma = 0$ by Theorem 2.3, we show that

$$g(R(X,Y)Z, PW) = (c + \epsilon\delta^2)\{g(Y,Z)g(X,PW) - g(X,Z)g(Y,PW)\},$$

for any X, Y, Z, $W \in \Gamma(TM)$. Thus $K_\xi(H) = \frac{g_x(R(W,\xi)\xi, W)}{g_x(W,W)} = 0$ for any null plane H of T_xM directed by ξ. □

3. TOTALLY GEODESIC SCREEN DISTRIBUTIONS

Definition 3. A half lightlike submanifold $(M, g, S(TM))$ of a semi-Riemannian manifold $(\bar{M}, \bar{g})$ is said to be *irrotational*[7] if $\bar{\nabla}_X\xi \in \Gamma(TM)$ for any $X \in \Gamma(TM)$.

Note 2. For an irrotational M, in general, since $B(X, \xi) = 0$ due to the first equation of (1.9), we have $D(X, \xi) = 0 = \phi(X)$ for all $X \in \Gamma(TM)$.

Theorem 3.1. *Let $(M, g, S(TM))$ be an irrotational half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$; $\epsilon c > 0$, such that $S(TM)$ is totally geodesic. Then M admits an induced symmetric Ricci tensor. Moreover, M is a totally umbilical Einstein manifold with $B = 0$ and the induced connection ∇ on M is a metric connection.*

Proof. Since M is an irrotational submanifold of $\bar{M}(c)$ and $S(TM)$ is totally geodesic, we have $\gamma = 0$ and $\phi = 0$. From (2.2), we have

$$D(X, Y)\rho(\xi) = -\epsilon c g(X, Y), \ \forall X, Y \in \Gamma(TM). \tag{3.1}$$

Since $c \neq 0$, we show that $\rho(\xi) \neq 0$ and $D \neq 0$. Thus (3.1) reduces to

$$D(X,Y) = \epsilon\delta g(X,Y), \ \forall X, Y \in \Gamma(TM). \tag{3.2}$$

where $\delta = -c\rho(\xi)^{-1} \neq 0$. Differentiating (3.2) and using (1.10) and (2.5), we have

$$X[\delta]g(Y,Z) - Y[\delta]g(X,Z) = \{\delta\eta(X) - \epsilon\rho(X)\}B(Y,Z) \\ - \{\delta\eta(Y) - \epsilon\rho(Y)\}B(X,Z), \quad \forall X, Y, Z \in \Gamma(TM).$$

Replacing X by ξ in this equation and using (1.9), we obtain

$$\xi[\delta]g(X,Y) = (\delta - \epsilon\rho(\xi))B(X,Y), \ \forall X, Y \in \Gamma(TM). \tag{3.3}$$

As $S(TM)$ is totally geodesic, by Note 1, M is locally a product manifold $L \times M^*$ where L is a null curve and M^* is a leaf of $S(TM)$. Using (1.17), (1.19), (2.11) and the fact that $C = 0$ and $c + \delta\rho(\xi) = 0$ by (3.1), we have

$$R^{(0,2)}(X,Y) = (c + \epsilon\delta^2)(m-1)g(X,Y),$$

for any $X, Y \in \Gamma(TM)$. Thus $R^{(0,2)}$ is a symmetric Ricci tensor Ric and M is an Einstein manifold. Also, from (1.17), (1.21), (2.1) and (3.2), we have

$$R^*(X,Y)Z = (c + \epsilon\delta^2)\{g(Y,Z)X - g(X,Z)Y\},$$

for any $X, Y, Z \in \Gamma(S(TM))$. From this equation, we have

$$Ric^*(X,Y) = (c + \epsilon\delta^2)(m-1)g(X,Y),$$

for any $X, Y \in \Gamma(S(TM))$. Thus M^* is also an Einstein manifold of constant curvature $(c + \epsilon\delta^2)$ due to $m > 2$. Therefore δ is a constant. From (3.3), we have $(\delta - \epsilon\rho(\xi))B(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$. Also, from (3.1), we have $c + \delta\rho(\xi) = 0$. Since $\delta \neq 0$, we get $(\epsilon c + \delta^2)B(X,Y) = 0$ for all $X, Y \in \Gamma(TM)$. Since $\epsilon c > 0$ and $\delta \neq 0$, we show that $(\epsilon c + \delta^2) > 0$. Therefore $B = 0$. In this case, from (3.2), we show that $h(X,Y) = \mathcal{H}g(X,Y)$ for all $X, Y \in \Gamma(TM)$, where $h(X,Y) = D(X,Y)u$ and $\mathcal{H} = \epsilon\delta u$. Thus M is totally umbilic in $\bar{M}$. Also, from the equation (1.10), we show that $(\nabla_X g)(Y,Z) = 0$ for all $X, Y, Z \in \Gamma(TM)$. Thus the induced connection ∇ on M is a metric one. □

From Theorem 3.1, we have the following theorem:

Theorem 3.2. *Let $(M, g, S(TM))$ be an irrotational half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$; $\epsilon c > 0$, such that $S(TM)$ is totally geodesic. Then M is a totally umbilical Einstein manifold and locally a* *product manifold $L \times M^*$, where L is a null curve in M and M^* is an Einstein semi-Riemannian space form of constant curvature $(c + \epsilon\delta^2)$.*

Theorem 3.3. *Let $(M, g, S(TM))$ be an irrotational half lightlike submanifold of a semi-Riemannian space form $(\bar{M}^{m+3}(c), \bar{g})$, $m > 2$, such that $S(TM)$ is totally geodesic. Then every null plane H of T_xM directed by ξ has everywhere zero null sectional curvatures.*

Proof. From (1.9), (1.17), (1.19), (3.2) and the fact $C = 0$, we show that

$$g(R(\xi, X)Y, PW) = 0; \qquad g(R(\xi, X)Y, N) = (c + \delta\rho(\xi))g(X, Y) = 0.$$

Thus we have $R(\xi, X)Y = 0$. Therefore, $K_\xi(H) = \frac{g_x(R(\xi,W)W,\xi)}{g_x(W,W)} = 0$ for any null plane H of T_xM directed by ξ. □

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