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Fostering Mathematical Thinking and Creativity: The Percent Problem[1]

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Open-ended problems can foster deeper understanding of mathematical ideas, generating creative thinking and communication in students. High-order thinking tasks such as open-ended problems involve more ambiguity and higher level of personal risks for students than they are normally exposed to in routine problems. To explore the classroom-based factors that could support or inhibit such higher-order processes, this paper also describes two cases of Singapore primary school teachers who have successfully or unsuccessfully implemented an open-ended problem in their mathematics lessons.

Keywords: teaching mathematics problem solving, open-ended problems, creativity
MESC Classification: D52
MSC2010 Classification: 97D50

1. INTRODUCTION

There has always been a growing concern in mathematics education about teaching methods that focus largely on standard textbook questions and solving problems through drilling. These methods tend to encourage only development of procedural knowledge and skills without encouraging students to think more deeply about the mathematics that they are learning. Such product-oriented instructions do not bring about desirable learning outcomes like critical and creative thinking in problem solving which is the core of the Singapore mathematics curriculum framework. As to the teaching methods in our mathematics classrooms, the prevalent practice has been observed as using whole class

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teaching, textbooks and regular testing (Chang, Kaur, Koay & Lee, 2001). According to the study, students did a lot of practice sums, mostly one-method and one-answer kind, to consolidate and reinforce the mathematics concepts or procedures taught by the teacher through an expository method. Hence to have students who can think and use mathematics creatively in situations that are unfamiliar would require a change of instructional strategy that is less dependent on the typical worksheets or textbook exercises. Advocates of process-based curriculum are now arguing for more problem-solving tasks that are open-ended. In a recent survey by Foong (2008) of teachers on their conception of *mathematical creativity,* the results painted a very positive picture for Singapore mathematics education where majority of teachers conceived of mathematical creativity as embedded in the activities of mathematics lessons that have problem solving as the focus. Most of them saw mathematical creativity as an event where students solved higher-order thinking tasks in different non-conventional, unique or in ways that were not taught before. The main purpose of this paper is to describe two cases of Singapore primary school teachers who have implemented an open-ended problem in their mathematics lessons and to explain the classroom-based factors that could have supported or inhibited students' thinking and creativity.

Mathematical Thinking and Creativity

In mathematics learning which includes problem solving, there have always been attempts to clarify what "thinking" means when we want to help students think clearly or productively. Thinking has been of interest to the human race partly because we can engage in the process like solving problems. One must think in order to solve problems, but it is difficult to define it. In this regard thinking is like understanding and is a mental activity that is essential to learning most outcomes. According to Carpenter and Lehrer (1999), understanding is not an all-or-none phenomenon. It is a generative process where students acquire knowledge, they can apply that knowledge to learn new topics and solve new and unfamiliar problems. In the same way for thinking to emerge, classrooms need to provide students with tasks that can activate the following forms of interrelated mental activities:

(a) develop appropriate relationships,
(b) extend and apply their mathematical knowledge,
(c) reflect about their own experiences,
(d) articulate what they know, and
(e) make mathematical knowledge their own.

These forms of mental activities underlay the different types of thinking, of which some common terms are used to differentiate among them. They are *productive, reflective,* *critical and creative* thinking. *Productive* thinking involves developing appropriate relationship while structuring and re-structuring of one's perceptual view of a new situation, extend and apply one's mathematical knowledge to the unfamiliar problem. *Reflective* thinking or *metacognition,* involves monitoring of one's processes when in a state of doubt to want to settle and resolve a problematic or new situation. *Critical* thinking involving articulating what one knows to judge logical relationship between meanings of words and statements of a problem situation. Seeking a new understanding or not generally accepted solution would lead to *creative* thinking, as one takes ownership of the process.

There is a view emerging with increasing acceptance that mathematics is an exciting and dynamic science (de Lange, 1993) which focuses on active generative processes engaged by the learners as they do mathematics. These are characterized by activities such as making sense of mathematical ideas; exploring and looking for patterns and underlying relationships; conjecturing, reasoning and justifying in flexible and creative ways. One cannot count on the standard mathematical textbook questions used by teachers to support this new view of mathematics education. Students must encounter rich mathematical tasks where they have opportunity to show their deep thinking in mathematics and give creative productions.

Short Open-ended problems

Foong (2002) has been advocating the use of short open-ended mathematical questions that Singapore teachers can use to complement the standard type of questions commonly found in textbook exercises. These mostly closed-ended textbook questions are often structured to pre-taught procedures such as those in one-step to multiple-step words problems that require the use of a specific heuristic called model drawing in the local syllabus (Foong, 2007). Most times closed-ended questions require students to give a specific and predetermined answer in the form of a single number or figure. Hence most textbook problems normally do not allow students to reveal their thinking processes and deep conceptual understanding as well as the open-ended questions. Open-ended questions require students to explain concepts and solution processes using various modes such as diagrams, symbols and words. At the same time they allow students to demonstrate their own ways of approaching and solving problems.

Studies (Becker & Shimada, 1997; Caroll, 1999) have found that when open-ended problems were used more regularly, students developed skills of reasoning and communication in various modes of representation. The characteristic features of an open-ended question are: it should have many possible answers and, or it can be solved in different ways. Open-ended questions are often not set to look complex and complicated but simple in structure that are accessible to all pupils especially in a mixed-ability classroom. The problem solution should offer pupils room for decision making and natural mathematical way of thinking. Students' responses to open-ended questions can give teachers greater insight into how their students think and what they really have understood about mathematics. When implemented appropriately with class discussion and sharing of different approaches, open-ended problems can help develop student confidence and ownership of knowledge.

There are four types of open-ended questions that teachers can create from textbook sums (example of each type is shown in the Appendix):

1. Problem for students to solve that has missing data and/ or hidden assumptions
2. Problem for students to explain a concept, procedure and/or an error, i.e. who is correct and why?
3. Problem Posing-asking students to create the question or tell a story given a mathematical statement or model
4. Problem for students to create a situation or generate examples that would satisfy certain conditions

THE PERCENT PROBLEM

We studied the classroom experiences of two primary school teachers who had tried the Percent Problem (figure 1) in their mathematics lessons. This problem was adapted from the works of Stein, Smith, Henningsen & Silver (2000).

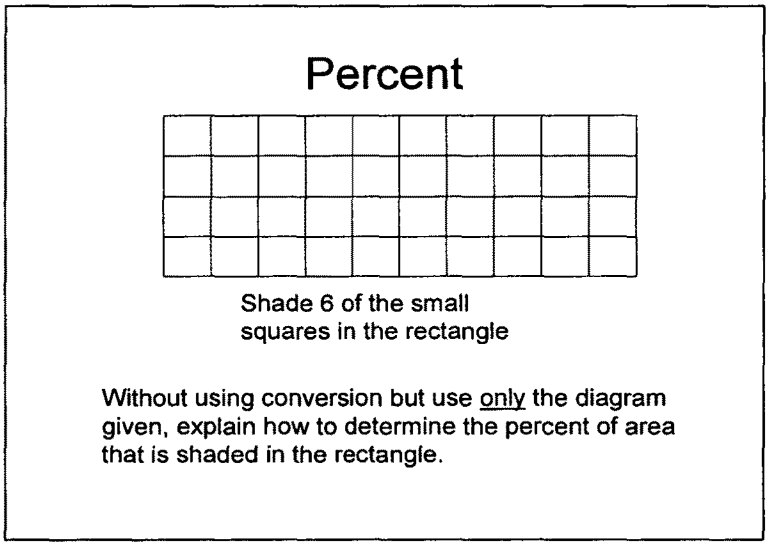


Figure 1. Percent Problem

This open-ended activity aimed to elicit students' high level cognitive processes and mathematical creativity. The task was made complex by demanding that students use only the diagram and not the traditional conversion procedures to solve it. It had been designed specifically for the following high-level cognitive outcomes:

- Students to apply non-algorithmic thinking when the given base of a percent is not a 10×10 grid.
- Students are not allowed to use the traditional conversion procedures.
- Students to construct novel ways of configuring the squares.
- Students to use visualization as a tool to analyse the diagram.
- Students to apply proportional reasoning base on conceptual understanding of fraction, decimal and percent.
- Students to communicate their reasoning.
- Students to display creativity in using possible strategies and solutions.

Two teachers from different schools used this problem with their P6 classes of 38 and 40 students. P6 level was chosen because the problem integrated nicely into the *percent* topic of the syllabus. In the two classes, the students worked in small groups. Two different results, one was implemented successfully while the other was disappointing in terms of students' creative productions. We compared features of the teachers' setup of the task, classroom norms, teachers' instructional dispositions, students' learning dispositions, students' engagement on task and outcomes, for factors that had supported or inhibited high level thinking and creativity in the two lessons.

TWO CASES

For this study the data were drawn from the case reports written by the teachers themselves. This was part of a project undertaken by the author involving in-service primary mathematics teachers who wanted to trial open-ended problem solving lessons with their students. The teachers had to report on how they set up and implemented the task; made observation of pupils at work, took notes of pupils' behaviors and cognitive processes; interviewed some for their attitudes towards such tasks and their participations in group work. They had also to analyse pupils' work; evaluate and reflect on their lesson outcomes.

The Case of Miss Tan: A Successful Lesson

Miss Tan had just completed working with her students on the concept of percentage and the conversion algorithms between fraction, decimal and percent using the 10×10 square grids as visual aids. She decided to use the open-ended *percent problem* as an enrichment to develop students' problem-solving heuristics. Through this task she also hoped that her students could see that a percentage can be represented in more than one form and can be calculated using more than one method. She had set aside every Friday of the week to be a 'PROBLEMATIC DAY'. Her students loved this day as they looked forward to doing group work where they can interact and make noise. So her pupils were mentally prepared for doing challenging task on this day.

Setting up the task: Miss Tan used the theme "*We are math investigators*" for the students to work co-operatively in groups. She told them that as math investigators they were required to solve problems and explore as many approaches as possible. They were also required to explain, compare and justify their solutions. She introduced them to the roles and expectations of 'math investigators'. The students were given 5 minutes to work individually then 15–20 minutes as a group. She told them that she would select samples of their groups' solutions to share with whole class and allow other groups to comment on their friend's work. For a start they were told:

> "Think first for a few minutes before beginning to discuss with each other. Please show and explain your solutions using drawings and/or words to justify your answers. "

In the beginning the students were at a lost when the problem stated that they could only use the diagram because they were so used to do conversion as a quick procedure for calculating the percent once they could figure out the fraction to be 6/40. Furthermore this was the first time the students were working on a grid that was not 10×10, one that they had automatically and incorrectly assumed as base of 100%.

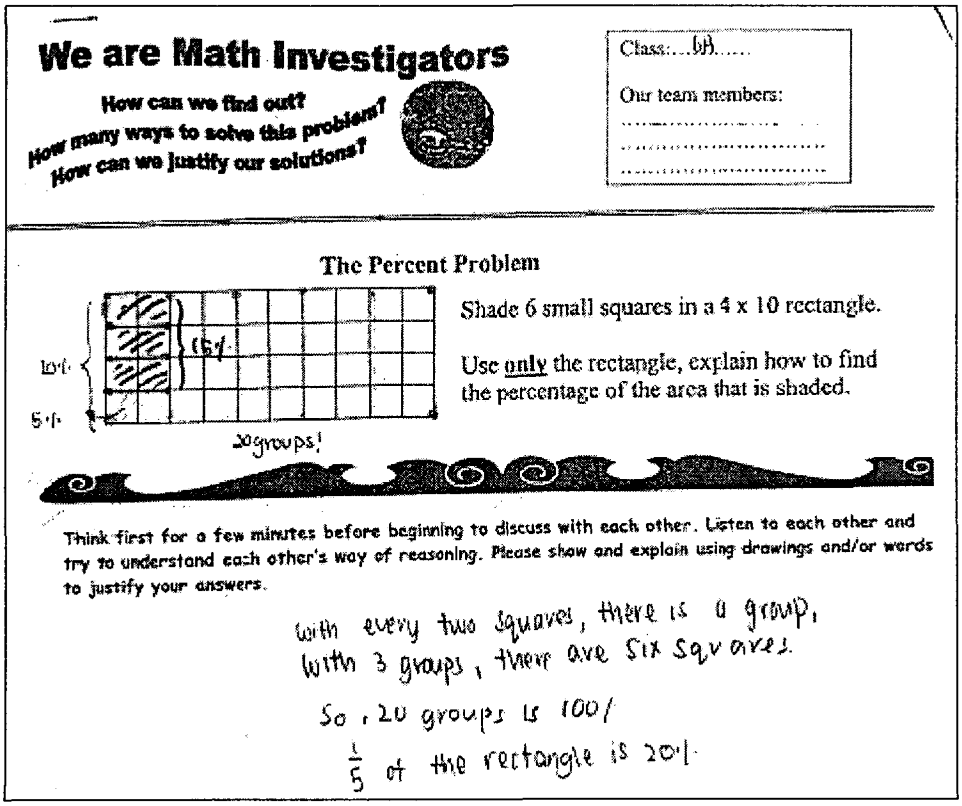
Students-at-Work

During individual work, some students started to calculate the answer using the conversion algorithm. Miss Tan had to reinforce her instructions and gave appropriate hints. Many of them were not sure if they were doing the right thing. It became easier for the students when they started to work in groups. They easily shaded six squares in different ways. Some students were heard saying it was 6% but others realized that the total number of squares was not 100 but 40, as they counted 4 *rows of 10's or 10 rows of 4's*. Miss Tan was able to assist students who were unsure without giving away too much or reducing the cognitive level of the task. While students worked in groups, she discouraged them from dismissing any of their member's ideas without first assessing it critically. They had to ask the "solver" to explain in detail and checked the solution before moving on. It was observed that a lot of learning and interactions took place because the students were used to discussing about mathematics problems every Friday. They were focused and listened to each other and gave their suggestions. They started to share what they had found out and wrote down the explanation. There was peer teaching and learning. It benefited the one who taught as it probably reinforce his understanding and for his friends they would probably understand him better, as they shared the same 'lingo'. When the groups presented their work to the class, students were surprised that there was more than one way to solve the problem.

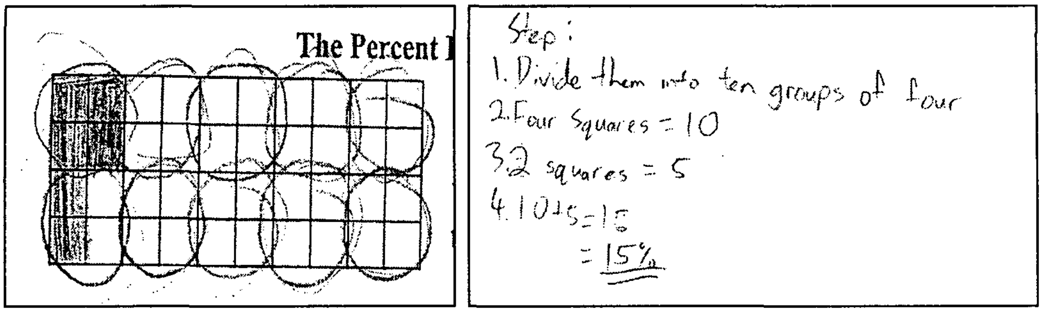
Analysis of Students' Work

Twelve out of 13 groups were able to explain the concept correctly. There were broadly four possible solution strategies using reasoning from visual diagram. They used different configurations to partition 4×10 rectangle based on part-whole relationships of the six squares out of 40. Most partitioned the 4×10 rectangle into 10 columns of 4 squares and each column of 4 squares is 10%, then 2 squares was 5%, deriving the answer 15% for 6 squares. Some divided the 40 squares into 20 groups of 2 squares of 5 %, 6 squares was 3× 5=15%. The following are some samples of the students' solutions:

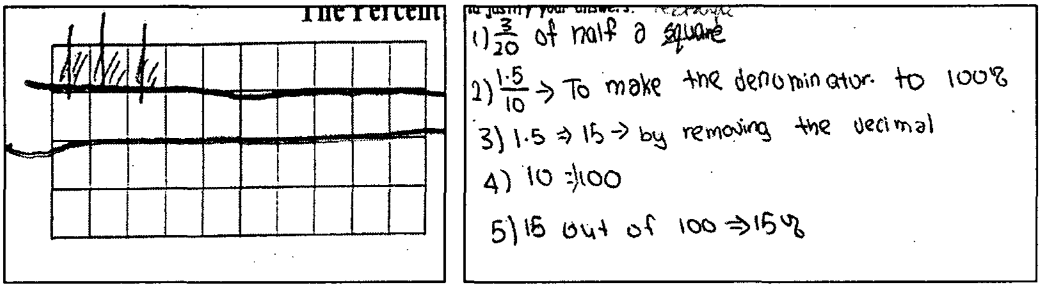
Sample A



Sample B

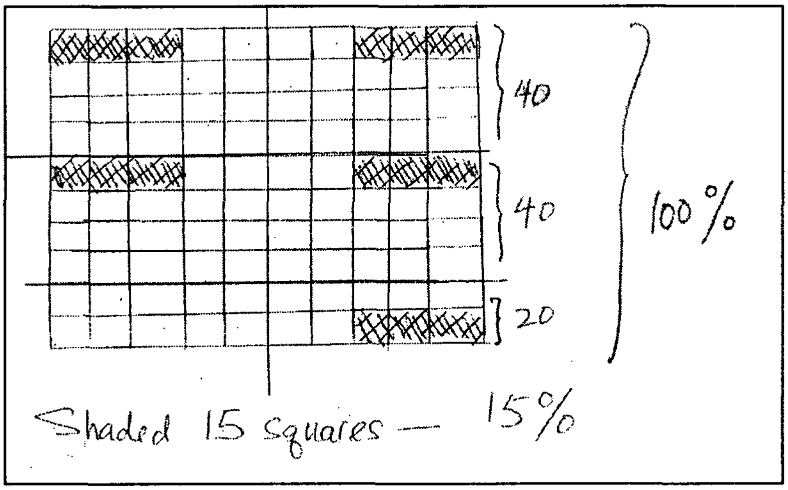


Sample C



There was one group's solution (Sample D) that was rather ingenious as compared to the rest. By extending the 4×10 rectangle to one and a half times give a total $(6+6+3)=15$ shaded squares out of 100, as shown:

Sample D



To Miss Tan's surprise the group that did not give a correct working consisted of very good students in mathematics, with two of them from China. She attributed the possible reasons for their inability to engage the task were they did not understand the instruction (*they said they did not know what I wanted them to do*) and that they had a fixed notion of how 'percent' problems should be, so they did not know how to respond when given a different type of more open-ended question.

Miss Tan's Reflection:

> I was surprised that they were able to come out with more than two solutions. I realised I should give my pupils the opportunity to explain their workings and how they derived their answers. I believe this is one important form of learning. After conducting the activity, I realise that sometimes, if not, in most time, I as a teacher forgets to inject fun in learning. I am too caught up in 'chasing' the curriculum, 'finishing' the syllabus and all. Mathematics lessons become so mundane and BORING! It is all about worksheets and more worksheets. Yet, in doing so, we are effectively putting blinders on the pupils such that they are capable only of answering exam style questions. By a mere twist of the question, making it open-ended, children are caught off-guard and are unable to answer the question, as there is "too little information" given. In giving open-ended problems, pupils are forced to remove their blinders and look further a field for more answers. The use of open-ended problems should be used more frequently in the classroom as it encourages pupils to think out of the box, to be more curious and be aware of the possibilities of more than one solution.

The Case of Miss Lim: An Unsuccessful Lesson

Although Miss Lim was keen to expose her students to open-ended problem solving she was very mindful of the time she would need to carry out such a lesson within the curriculum time that was allotted. So she decided to carry out this activity during her supplementary lesson. She had initially thought that her students would not have any problems with the question and would be able to present their solutions neatly and correctly but she realized she was wrong after the lesson the evaluation.

Setting up the task: Before she presented the open-ended percent task to her students, Miss Lim did a revision on the percentage lesson she had with them previously. "To my horror", in her own words, she found that some of the students had forgotten how to change fraction to percent. She did a quick revision with them using the 10 by 10 grid to explain concept of equivalent fractions and its relation to percentage for the next 30 minutes! To consolidate she gave them a task to shade 3/5 of the 10×10 grid and find how many squares to be shaded.

Although the intention was for students to shade the 10×10 grid in various ways to find 3/5 of the 100 squares, she was happy when students were able to use equivalent fractions to change 3/5 to base of 100 to find the percent and then shade the required squares. After this she issued out the percent problem worksheet to the class. She "warned" them to read the questions carefully and stressed to them that they had to use the diagram for their solution. She laid out the ground rules for working in pairs and the requirement of the open-ended task.

Students-at-Work: During the pairing session she noticed that students chose to work with the person that he or she was most comfortable with. This she thought was because she had reminded them not to be too critical of their friend's ideas and learned to compromise. One boy was rejected by all and had to work alone because he was known to be domineering and would get angry when his ideas were not accepted by the group. As she walked around she noticed that the first thing the students did was to convert the fraction into percentage by calculation. They even forgot to shade the six squares. She reminded them that they had to use the diagram to find the answer and not to use conversion.

There was disagreement from the students and some were at a lost. She kept telling them to think out of the box when solving the problem but she did not provide any hint to the problem. The students were stumped as how to proceed and not sure what the teacher wanted in this question. Even though she let them struggled for some time to look at the problem from another perspective she was not sure how to get them to do so. Many students were becoming uneasy with the ambiguity. Some students were heard saying "why need so many solutions—why? Just take 6/40 × 100% and the answer is 15%. No need to try other method... ." It was another 30 minutes into the lesson, the students were told to wrap up their discussion and write out their solutions.

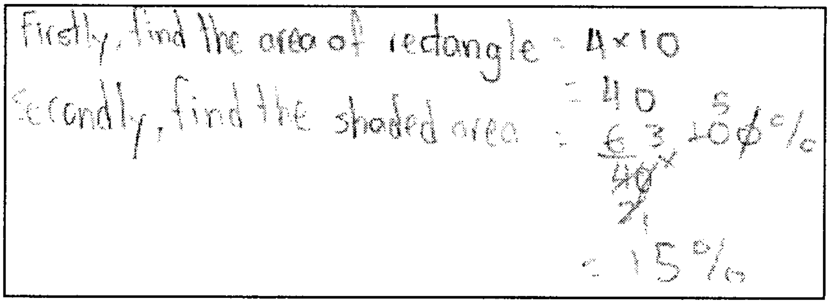
Some pairs did explore with the diagram while many were still stuck at the conversion method and some pairs were still unable to come out with any solution as they were hampered by "how to explain?" their solutions. They found difficulty expressing their thoughts in words.

Analysis of Students' Work

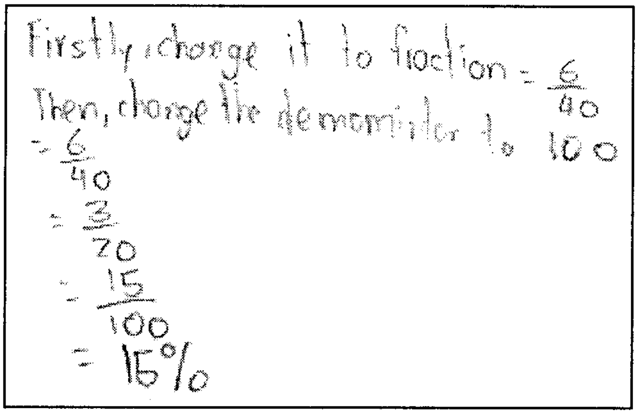
Majority of the students who completed the problem, solved it by the conversion algorithm despite Miss Lim's encouragement to them to use only the diagram. This could be due to the influence of the prior 30 minutes revision on using equivalent fractions to find percentage while setting up the actual task.

The students were so trained to use the conversion algorithm as an easy procedure to find quick answers that they did not see the relevance of the 10×10 grid diagram as a model for percent, even more so when it was a 4×10 grid. Below are two samples of the pair work which are representative of students reducing the problem-solving complexity of the task to low level cognitive procedures:

Sample E


Firstly, find the area of rectangle = 4×10
Secondly, find the shaded area = 40
= 6/40 × 100 %
= 15 %

Sample F


Firstly, change it to fraction = 6/40
Then, change the denominator to 100
= 6/40
= 3/20
= 15/100
= 15%

Miss Lim's Reflection:

> On the whole, it was quite interesting to watch them sort out their thoughts in words. They pondered and pondered. Their first reaction was to solve the problem by calculation. I supposed they are so trained that given a problem; they must work out the answer. They are more interested in the end result and not the processes. They failed to see that there may be other solutions that are unusual but correct. I feel I need to expose my students to more open-ended questions so as to nurture their creativity……. But on the other hand, I feel that time factor also play a part. That is why I carried out the activity during my supplementary lesson. Time in the classroom is very limited."

CLASSROOM-BASED FACTORS THAT SUPPORT AND INHIBIT THINKING AND CREATIVITY

Research has shown that the mere presence of high level tasks in the classroom will not automatically result in students' engagement in high-order thinking. Henningsen and Stein (1997) suggest that although attention to the nature of mathematical instructional tasks is important, attention to the classroom processes surrounding the tasks are equally important. One must create the ambient classroom environment to balance classroom management needs and the academic demands.

In their study, they found that high level tasks were more susceptible to various factors that could cause a decline in students' engagement to less demanding thought processes. In this study, the same *percent problem* was used by both Miss Tan and Miss Lim with their primary six students but ended with different students' outcomes. The task was designed to demand higher level thinking and reasoning requiring some amount of cognitive effort from the students. As the task was complex, the students were required to use only the diagram in novel ways through visualization and proportional reasoning to solve it. However, if the students used only the conversion algorithm and ignored the diagram to get the answer then they reduced the level of the task to just procedural skills.

As we compared the lessons between the two teachers, we could see that Miss Tan set-up the problem appropriately to harness an investigative approach of the students using the theme we are "math investigators." In planning the theme she was able to provide the appropriate amount of time for the whole process. She could maintain the integrity of the task through scaffolding by clarifying students' doubts and pressed them on to use the diagram as alternative solutions.

She was able to sustain pressure on pupils to provide meaning, explanation and justification to demonstrate their understanding of the mathematics underlying the task. On the other hand, Miss Lim's set-up of the task by revising the algorithms, converting fractions to percent had some how high-jacked her intention to have her students engaged in high cognitive level open-ended problem solving. It pointed to a lack of planning and time management to allow her students to grapple with the important mathematical ideas embedded in the task.

Even though in the implementation she tried to raise the level of students' engagement, they were too comfortable with already an answer to the question. The students did not see the point of the open-ended question when the teacher's focus in the very beginning of the lesson was a procedure to find an answer even though she attempted to use the 10x10 grid for developing conceptual understanding of percent. Her students were not held accountable for the required high cognitive processes as she gave in to them with their traditional conversion method. As seen in the samples of students' productions, Miss Tan's students were able to produce novel solutions using the diagram for conceptual connection between fraction and percentage while Miss Lim's students were stuck mostly in the conversion procedures.

CONCLUSION

In both case, Miss Tan and Miss Lim support what research has found. By merely giving students high level tasks in the classroom, students' will not automatically engaged in high level thinking, reasoning and sense making. A very important phase in carrying out open-ended high level tasks in the classroom is the set-up phase where the teacher announces the task in such a way as to encourage students to use more than one strategy for multiple representations and to supply explanations and justifications.

This was very evident in the case of Miss Tan, who had successfully implemented the lesson. In the case of Miss Lim, her students' engagement with the task declined to lower levels of processing. In this study, the task selected was appropriate to the mathematical prerequisite knowledge of the students although this was the first time they came across such an open-ended situation. Time was not an issue as the task was not complicated to the extent that demanded much student exploration. For the desired outcomes to be achieved, these two classroom experiences have implications for the role of the teacher in implementing open-ended tasks for students are expected to actively engage in high level mathematical reasoning and creative thinking.

Teachers must have a paradigm shift towards a more process-based approach where getting a correct answer to a problem is not the main criteria. Teachers often have a misguided idea that if they can come up with a set of procedures that students can follow to solve mathematics problems, then students will be well equipped to solve problems. It might not help them solve open-ended problems which often do not have a precise answer or a specific method of solution. If teachers carry this misconception into a process-oriented problem-solving classroom they will be reducing a high level problem to applying procedural skills. Importantly in implementing open-ended tasks, teachers must know the mathematical ideas embedded in the task and connections that might evolve, in order to create an encompassing classroom environment that allows students to take risk and ask appropriate questions. In order not to inhibit students' creativity in mathematical thinking, teachers must proactively and consistently support students' cognitive activity without reducing the high level demands of the task. They should not give in to students' anxiety and take over to show them how to get the answer.

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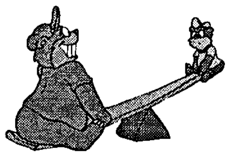
APPENDIX

EXAMPLES OF SHORT OPEN-ENDED PROBLEMS

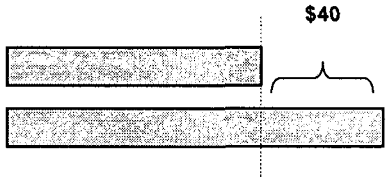
Missing data

Barney the 400kg bear wants to play with some children on a specially made see-saw. How many ***P4*** children must he play with if he wants the see-saw to balance?



Problem Posing

- Write a word problem sum that can be solved with the following model:



Explain a fraction concept

- Little Julie could not understand why the following fractions were arranged in the following order from the smallest to the biggest.

$$\frac{2}{9} \quad \frac{2}{7} \quad \frac{2}{5} \quad \frac{2}{3}$$

- How would you explain this sum to Julie. You may use words, pictures and numbers to help you...

Generate examples

I am a six-sided figure that is flexible. At times, I have only one axis of symmetry and at other times I can have as many as seven axes of symmetry. Can you draw all my possible shapes?