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Building Mathematics Competence via Multiple Choice Competitions[1]

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The paper focuses on a type of mathematics competence noted as synthetic. It is needed for success in multiple choice competition problems and it is crucial for the further development of advanced school students. The concept of synthetic competence includes a large fraction of mathematical competences, but also competences and skills related to informatics and linguistics.
The general structure of a multiple choice competition is considered as the first level of assessment of the mathematics component of the synthetic competence. As the second level under consideration is the set of complex tasks in a competition test any of which is a bouquet of ideas. Our standpoint on the didactic specifics and goals of the two levels in perspective of the gifted education is briefly presented in the article and some examples are given.

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1. KEY COMPETENCES

The "Education and Training 2010" program of the European Commission (EC) points out eight key competences[2] which are supposed to be formed at the end of the compulsory education (European Commission, 2004). The following definition evidences

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[1] This paper has been presented at the 15th International Seminar on Education of Talented Children and Creativity Development in Mathematics at Woosuk University, Samrye, Jeonbuk, Korea; February 19–20, 2010.

[2] They appear on pp. 7–8 of
http://ec.europa.eu/education/policies/2010/doc/basicframe.pdf
and also on pp. 13–18 of
http://eur-lex.europa.eu/LexUriServ/LexUriServ.do?uri=OJ:L:2006:394:0010:0018:EN:PDF

the importance of the concept.

Key competences represent a transferable, multifunctional package of knowledge, skills and attitudes that all individuals need for personal fulfillment and development, inclusion and employment.

The concept of competence refers to a combination of skills, knowledge, aptitudes and attitudes, and to include disposition to learn as well as know-how. The *key competence* focuses on three aspects that are expected to be fulfilled:

1) Personal fulfillment and development throughout life (the cultural capital): key competences must enable people to pursue individual objectives in life driven by personal interests, aspirations and the desire to continue learning throughout life;
2) Active citizenship and inclusion (the social capital): these key competences should allow everybody to participate as an active citizen in society;
3) Employability (the human capital): the capacity of each and every person to obtain a decent job in the labor market.

The concept of competence comes to replace the too restrictive concept of *basic skills* — the last one was generally referring to basic literacy and numeracy, or as survival or life skills. Excellent! But let us see what the projection of the definition of key competences on the mathematics education is.

2. MATHEMATICS COMPETENCE FROM EC PERSPECTIVE

The key-competence (the 3rd of 8 key competences) that refers to mathematics given in the overview of the key competences on page 3 in European Commission (2004) declares

> Mathematical literacy is the ability to use addition, subtraction, multiplication, division and ratios in mental and written computation to solve a range of problems in everyday situations. The emphasis is on process rather than output, on activity rather than knowledge.[3]

This definition is quite restrictive even for compulsory education. It is hard to imagine that even complete coverage of the listed abilities is enough for 'a successful life in a knowledge-based society'. One can say that it contradicts the general view on key competence given above as far as it deals with the rejected concept of 'skills'. Our belief is that in the second decade of the 21st century a kind of broader concept will come to replace the considered view point. Further we try to give an idea of what we think as a possible direction for changes and how it works in a particular mathematics competition.

[3] The definition continues with some speculation on science related competences.

3. NEED FOR EXTENSION

The compulsory education view point does not take into account the competence in mathematics that is necessary for numerous subjects in higher education. Fortunately, a more comprehensive table attached to the general definition describes a wider range of topics in which mathematics is potentially applicable (we put the quotation as appendix) and let we assign to this list of desired skills the label mathematics key competence (MKC). Further in the Framework (European Commission, 2004) is noted that math competence involves the use of mathematical modes of thought (logical and spatial thinking) and presentation (formulas, models, constructs, graphs or charts) which have universal application in explaining, and describing reality. Elaboration of these topics in secondary school curriculum gives a partial solution of the preparation of citizens for the knowledge-based society.

However, the *competence approach*, widely accepted in higher education, requires more abilities in mathematics and its applications. Our experience is related to an extension of the range of the math-key-competence for students who are advanced or gifted in mathematics. It is a widely spread practice to meet the intellectual needs in mathematics of advanced and gifted students by providing extracurricular activities. Such practice has a long tradition in Bulgaria, especially in the form of mathematics competitions, where the development of another kind of mathematics competence is stated as a goal.

4. TOPICS IN EXTRACURRICULAR MATH ACTIVITIES

Multiple choice competitions (MCC) cover a large area of mathematical topics that appear in extracurricular activities. A competition theme of 25 or more items contains as a rule problems of every important branch.

Such a variety allows students to prove competences in many particular branches of mathematics. The knowledge and skills required for a good performance in a math tournament are far beyond the everyday individual needs.

A sample distribution

The 2008 theme for the 11–12th graders of *Chernorizets Hrabar Tournament* (ChH) includes problems in:

Arithmetic – 4,
Algebra and calculus – 7,

Probability and combinatorics – 6,
Geometry – 8,
Miscellaneous – 5

The distribution given above is quite typical for the MCCs. To get a high score in such a theme students should be familiar with the particular knowledge and skills in any of the math branches. For instance, the arithmetic problems are composed on ideas of Diophantine equations, congruencies, number bases, famous theorems as Fermat's theorem, etc.

As a rule, students attending math circles achieve competence in all these topics, *i.e.*, math circles make provision for a kind of *math-tournament competence* (MTC). We are not going to define exactly what math-tournament competence is but in general it includes knowledge and abilities to apply advanced mathematical results and methods in solving competition problems, *i.e.*, a kind of development of MKC in theoretical aspect.

5. FORMULA FOR SUCCESS

Students who have math-tournament competence (corresponding to their grade) as a rule perform successfully in a MCC. The intellectual abilities of advanced students in mathematics are challenged by a regular MCC. Deep understanding of the extracurricular mathematics is a base for reaching the top of Math Olympiad and serves the early preparation of future professional mathematicians. However, it happens that having all these skills is still not enough to be a champion in the Chernorizets Hrabar Tournament (Lazarov & Kortezov, 2006). Some mixed-type problems could be attacked successfully only when several techniques work together. Thus, a need for another kind of extension of the competence appears. To clarify our idea we consider two problems from the 2006 ChH issue.

5.1. Example. Which of the numbers

A) 1   B) 1.5   C) 2   D) 2.5   E) 3

gives the best approximation to the result of the execution of the following program fragment:

```
insert n;
    x := -1; s := 0; h := 1/n
    while x < 1 do
        (p := h * √(1 - x²);   s := s + p;   x := x + h)
print s
```

upon entering $n = 100$?

Solution and comments

The variable h indicates the magnitude of change of x; the variable s aggregates the areas of the rectangles that have base h and a vertex on the semicircle with diameter the segment between -1 and 1 on the x-axe. Since n is large enough, the area of the rectangles is close enough to the area of the semicircle, *i.e.*, to

$$\frac{\pi}{2} \approx 1.5.$$

In this problem students are expected

- to spell out the algorithm
- to recognize the semicircle as an analytical-geometry object
- to guess that the total area of 100 inscribed rectangles is a good enough approximation to the area of the semicircle

The first two bullets refer to student's knowledge but the third one calls student's sense of mathematics. Finally, a synthesis of abilities united by sense of mathematics is the formula for successful performance in such a complex problem.

5.2. Example 2. *The Pirate* is a gambling automat in which a positive integer a is loaded randomly. The gambler puts 1 Euro in *The Pirate* and it randomly generates another positive integer b; both a and b do not exceed 2006. If a is closer to b than to 0, *The Pirate* wins the Euro; otherwise it gives back 3 Euro to the gambler. What is approximately the expected profit of a casino from 10000 games with *The Pirate*?

A) 1250 B) 2500 C) 3750 D) 5000 E) 7500

Solution and comments

We can present the sample space as all pairs $(a;b)$ where $a, b \in \{1, 2, 3, ..., 2006\}$, *i.e.*, the nodes of a grid that lie in a square with side 2006. The outcomes of success (casino wins) satisfy the condition

$$a > \frac{b}{2}.$$

In a coordinate system xOy these points belong to the trapezium $OABC$, where $A(2006; 0)$, $B(2006; 2006)$ $C(1003; 2006)$.

The number of nodes is approximately the area of the trapezium, so the probability that the casino will gain the euro in a single game is approximately the ratio of the areas of the trapezium and the square which is

$$\frac{3}{4} \text{ (the exact value is 0.7487).}$$

Hence the casino wins

$$1 \text{ EUR with probability } \frac{3}{4} \text{ and loses 2 EUR with probability } \frac{1}{4}.$$

The expected profit from 10000 games is about

$$10000(\frac{3}{4} - 2 \cdot \frac{1}{4}) = 2500 \,\text{EUR}.$$

In this problem students are expected in a very limited time;

- to construct a phase space
- to determine the points of success
- to evaluate the number of nodes via the area of a polygon
- to perform a rough calculation for the expectation of a random variable

This example shows that a synthesis of mathematics competences that includes the sense of mathematics is an absolutely necessary condition to solve the problem in a limited time.

6. THE MATHEMATICAL SYNTHETIC COMPETENCE

We define the concept of mathematical synthetic competence (MSC) as

The ability to use mathematical knowledge (concepts and methods) combined with a level of skills to use ideas related to computer technologies for solving problems; the ability to analyze the initial data of a problem and to transform them in an appropriate form to apply complex methods; the ability to evaluate the outcomes and to apply them.

The examples we gave indicate a significant difference between MKC, MTC and MSC. The topics from the list in section 4 check particular skills, knowledge, ability, but not the MSC as an integrated characteristic. The MSC does not include necessarily a high level of calculation techniques neither rigorous deduction abilities. It allows to discover connections, to estimate values of variables and to take decisions based on both deduction and common sense. This is what we mean when we talk further about *sense of mathematics.*

6.1. Arnold's point of view

To advocate introduction of the concept of mathematical synthetic competence let us consider the following problem which implicitly refers to the sense of mathematics.

Calculate

$$\int_1^{10} x^x dx.$$

Error should not exceed 10%.

It appeared in the booklet *Zadachi dlya detey* by Arnold (2008), where he wrote that the culture of thinking could be build by encouraging students to *speculate* about common but not easy problems from an early age (5–15 years old).

6.2. How it works

The following speculations could give an idea of what author's sense of mathematics says when he attacked the above Arnold's problem.

- First of all the accuracy of 10% is a clue that no exact methods could work, but estimations are welcome.
- Function x^x grows quite fast and this gives us grounds to estimate

$$\int_1^{10} x^x dx \quad \text{by} \quad \int_9^{10} x^x dx.$$

- x^x is between x^n and x^{n+1} when $x \in [n;\ n+1]$, hence

$$l = \int_9^{10} x^9 dx < \int_9^{10} x^x dx < \int_9^{10} x^{10} dx = u.$$

- It is plausible to take a kind of mean of l and u to estimate the desired integral, let it be their arithmetic mean

$$\int_9^{10} x^x dx \approx 3.4 \cdot 10^9.$$

Check by MATHEMATICA:

In [1]:= NIntegrate [x^x, {x, 1, 10}]
Out [1] = $3.05749 * 10^9$
Accuracy: about 12%. :-(

Open question: Is the Arnold's problem solved by the author?

7. BUILDING SYNTHETIC COMPETENCE

The mathematic key competence as it has been declared does not serve the aspects 1)–3) quoted in the first section. The mathematic tournament competence becomes more and more difficult to be built up and, when it is finally built up, only a small fraction of the students get full-scale advantage of their MTC. But students having MTC, as a rule, prefer tournaments of traditional type where they can apply explicitly deduction and mathematical knowledge.

Our standpoint is that the output of the mathematics education of advanced and gifted students should be a synthetic competence. An irreplaceable part of this competence is the sense of mathematics. Such a sense could be grown in multiple choice competitions where students are free of heavy calculations and rigorous solutions but are encouraged to combine knowledge, common sense, special techniques and skills as far as they can.

Knowledge-based society is a competitive one. Development of mathematical synthetic competence not only serves intellectual needs of a person but also raises chances for survival and success, *i.e.*, meets the three aspects quoted in section 1.

Finally, we confess that it is a tricky business to design problems of complex type to challenge students' sense of mathematics and MSC respectively. Chernorizec Hrabar Tournament (Lazarov & Kortezov, 2006) has a good practice in this area. The next step is our good practice to be turned into regular one for multiple choice competitions.

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APPENDIX

FRAMEWORK OF KEY COMPETENCES IN A KNOWLEDGE-BASED SOCIETY[4]

Column Skills

Mathematical literacy has many applications in everyday life:

- managing a household budget (equating income to expenditure, planning ahead, saving),
- shopping (comparing prices, understanding weights and measures, value for money),
- travel and leisure (relating distances to travel time; comparing currencies and prices),
- decoding and interpreting symbolic and formal mathematical language (symbols and formulae), and understanding its relations to natural language,
- handling mathematical symbols and formulae,
- representing mathematical entities, understanding and utilizing (decoding, interpreting, distinguishing between) different sorts of representations of mathematical objects, phenomena and situations, choosing and switching between representations as and when appropriate,
- following and assessing chains of arguments, put forward by others, uncovering the basic ideas in a given line of argument (especially a proof) etc,
- thinking and reasoning mathematically (mastering mathematical modes of thought),
- abstracting and generalizing when relevant to the question; modeling mathematically (i.e., analyzing and building models) - using and applying existing models to questions at hand,
- communicating in, with, and about mathematics,
- making use of aids and tools (IT included),
- knowing the kinds of questions that mathematics may offer the answer to,
- distinguishing between different kinds of mathematical statements (is something an assertion or an assumption, etc),
- understanding the scope and limitations of a given concept,
- understanding mathematical proofs,
- critical thinking,
- use and manipulate technological tools and machines as well as scientific data and insights to obtain a goal or reach a conclusion.

[4] given in European Commission (2005)

Column Attitudes

- overcoming 'fear of numbers',
- willingness to use numerical computation in order to solve problems in the course of day-to-day work and domestic life,
- respect for truth,
- willingness to look for reasons to support one's assertions,
- willingness to accept or reject the opinions of others on the basis of valid (or invalid) reasons or proofs.