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COMMON FIXED POINTS OF ASYMPTOTICALLY NONEXPANSIVE MAPPINGS BY ONE-STEP ITERATION PROCESS IN CONVEX METRIC SPACES

Mujahid Abbas, Safeer Hussain Khan and Jong Kyu Kim

Abstract. We study one-step iteration process to approximate common fixed points of two nonexpansive mappings and prove some convergence theorems in convex metric spaces. Using the so-called condition (A'), the convergence of iteratively defined sequences in a uniformly convex metric space is also obtained.

1. Introduction

Let us recall some definitions:

Definition 1.1. [15] Let (X, d) be a metric space. A mapping $W : X \times X \times [0,1] \longrightarrow X$ is said to be a convex structure on X if for each $(x, y, \lambda) \in X \times X \times [0,1]$ and $u \in X$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y).$$

A metric space X together with the convex structure W is called a convex metric space.

Definition 1.2. Let X be a convex metric space. A nonempty subset F of X is said to be convex if $W(x, y, \lambda) \in F$ whenever $(x, y, \lambda) \in F \times F \times [0,1]$.

Takahashi [15] has shown that open spheres $B(x, r) = \{y \in X : d(y, x) < r\}$ and closed spheres $B[x, r] = \{y \in X : d(y, x) \leq r\}$ are convex. All normed spaces and their convex subsets are convex metric spaces. But there are many examples of convex metric spaces which are not embedded in any normed space (see [15]).

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Definition 1.3. [1] A convex metric space X is said to be uniformly convex if for any $x, y, a \in X$,

$$[d(a, W(x, y, \frac{1}{2}))]^2 \leq \frac{1}{2}\{1 - \delta(\frac{d(x,y)}{\max\{d(a,x), d(a,y)\}})\}([d(a,x)]^2 + [d(a,y)]^2)$$

where the function δ is strictly increasing function on the set of strictly positive numbers and $\delta(0) = 0$.

Definition 1.4. Let X be a convex metric space, C a nonempty convex subset of X and $T : C \to C$ a mapping. T is called an asymptotically nonexpansive mapping if there is a sequence $\{k_n\} \subset [1, \infty)$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for all $x, y \in C$ and for all $n \in \mathbb{N}$ where $\sum_{k=1}^{\infty} (k_n - 1) < \infty$.

A point $x \in C$ is a fixed point of a mapping T provided $Tx = x$. Let C be a nonempty closed bounded convex subset of a uniformly convex complete metric space X and $T : C \to C$ be an asymptotically nonexpansive mapping, then T has a fixed point [3] (see also, [4]) . Different iteration processes have been used to approximate fixed points of nonexpansive mappings. Mann iteration process [12] and Ishikawa iteration process [9] are two well known iteratively defined processes which are generally used to solve the fixed point problems of different mappings. An iteration process used to approximate common fixed points of two mappings was introduced by [6]. Note that the notion of approximating common fixed points of mappings has a direct link with the minimization problem (see, for example, [17]). The study of convergence of iterative process in a convex metric space is a recent development (see for example, [3], [5], [7] and [14]). Recently, Wang and Liu [21] obtained the convergence of a sequence generated through an Ishikawa type iteration process with errors to a common point of two uniformly quasi-Lipshitzian mappings in convex metric spaces. The aim of this paper is to use a simple iterative process to study the convergence problem of a sequence thus obtained to common fixed points of two asymptotically nonexpansive mapping in a complete convex metric space.

2. Preparatory Lemmas

In this section, we prove some lemmas for development of our convergence results. In the sequel, we write $F = F(S) \cap F(T)$ for the set of all common fixed points of the mappings S and T. For $S, T : C \to C$, our iteration process reads as follows:

$$\begin{cases} x_1 = x \in C, \\ x_{n+1} = W(S^n x_n, T^n x_n, \frac{1}{2}), \quad \forall n \in \mathbb{N}. \end{cases} \tag{2.1}$$

Now, we state the following useful lemma.

Lemma 2.1. [11] Let $\{\delta_n\}, \{\beta_n\}$, and $\{\gamma_n\}$ be three sequences of nonnegative numbers such that

$$\beta_n \geq 1 \text{ and } \delta_{n+1} \leq \beta_n\delta_n + \gamma_n \text{ for all } n \in \mathbb{N}.$$

If $\sum_{n=1}^{\infty} \gamma_n < \infty$ and $\sum_{n=1}^{\infty} (\beta_n - 1) < \infty$, then $\lim_{n\to\infty} \delta_n$ exists.

Next, we prove the following lemma which is a generalization of Lemma 4 of [16].

Lemma 2.2. Let X be a uniformly convex metric space. If $\{x_n\}$ and $\{y_n\}$ are sequences in X such that for some $z \in X$,

$$\limsup_{n\to\infty} d(x_n, z) \leq r, \ \limsup_{n\to\infty} d(y_n, z) \leq r$$

and

$$\lim_{n\to\infty} d(z, W(x_n, y_n, \frac{1}{2})) = r$$

for some $r \geq 0$. Then $\lim_{n\to\infty} d(x_n, y_n) = 0$.

Proof. For $r = 0$, The proof is obvious. Now take $r > 0$. Assume on contrary that $\lim_{n\to\infty} d(x_n, y_n) \neq 0$ which implies that there exist subsequences $\{x_m\}$ and $\{y_m\}$ of $\{x_n\}$ and $\{y_n\}$ respectively and $\varepsilon > 0$ such that $d(x_m, y_m) \geq \varepsilon$ for all $m \in \mathbb{N}$. Since $\limsup_{m\to\infty} d(x_m, z) \leq r$ and $\limsup_{m\to\infty} d(y_m, z) \leq r$, there exists $m_0 \in \mathbb{N}$ such that $d(x_m, z) \leq r + \varepsilon$ and $d(y_m, z) \leq r + \varepsilon$ for $m \geq m_0$. Also, $\lim_{m\to\infty} d(z, W(x_m, y_m, \frac{1}{2})) = r$. Note that

$$\begin{aligned} d(z, W(x_m, y_m, \frac{1}{2})) &\leq \frac{1}{2}d(z, x_m) + \frac{1}{2}d(z, y_m) \\ &\leq \max\{d(z, x_m), d(z, y_m)\} \end{aligned}$$

Therefore, either $\liminf([d(z, x_m)]^2) > 0$ or $\liminf([d(z, y_m)]^2) > 0$. Since X is uniformly convex, therefore

$$\begin{aligned} [d(z, W(x_m, y_m, \frac{1}{2}))]^2 &\leq \frac{1}{2}\{1 - \delta(\frac{d(x_m, y_m)}{\max\{d(z, x_m), d(z, y_m)\}})\} \\ &\quad \times([d(a, x)]^2 + [d(a, y)]^2) \\ &\leq \frac{1}{2}([d(z, x_m)]^2 + [d(z, y_m)]^2) \\ &\quad -\frac{1}{2}\delta(\frac{\varepsilon}{r + \varepsilon})([d(z, x_m)]^2 + [d(z, y_m)]^2) \end{aligned}$$

Now taking limsup on both the sides, we get

$$\begin{aligned} r^2 &= \limsup_{m\to\infty}[d(z, W(x_m, y_m, \frac{1}{2}))]^2 \\ &\le \frac{1}{2}\limsup_{m\to\infty}([d(z,x_m)]^2 + [d(z,y_m)]^2) - \frac{1}{2}\delta(\frac{\varepsilon}{r+\varepsilon}) \\ &\quad \times \liminf([d(z,x_m)]^2 + [d(z,y_m)]^2) \\ &\le \frac{1}{2}\limsup_{m\to\infty}([d(z,x_m)]^2 + [d(z,y_m)]^2) \\ &\quad -\frac{1}{2}\delta(\frac{\varepsilon}{r+\varepsilon})(\liminf([d(z,x_m)]^2 + \liminf[d(z,y_m)]^2)) \\ &< \frac{1}{2}(r^2 + r^2) \ = r^2 \end{aligned}$$

which is a contradiction. The proof is complete. □

Lemma 2.3. Let C be a nonempty closed convex subset of a convex metric space X and $S, T : C \to C$ be two asymptotically nonexpansive mappings. Let $\{x_n\}$ be the sequence defined in (2.1). If $F \neq \phi$, then $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F$.

Proof. Let $x^* \in F$. Then we have

$$\begin{aligned} d(x_{n+1}, x^*) &= d(W(S^n x_n, T^n x_n, \frac{1}{2}), x^*) \\ &\le \frac{1}{2}d(S^n x_n, S^n x^*) + \frac{1}{2}d(T^n x_n, T^n x^*) \\ &\le \frac{1}{2}k_n d(x_n, x^*) + \frac{1}{2}k_n d(x_n, x^*) \\ &= k_n d(x_n, x^*). \end{aligned}$$

Thus, by Lemma 2.1, $\lim_{n\to\infty} d(x_n, x^*)$ exists for each $x^* \in F$. □

Lemma 2.4. Let X be a uniformly convex metric space and C be a nonempty closed convex subset of X. Let $S, T : C \to C$ be asymptotically nonexpansive mappings and $\{x_n\}$ be the sequence as defined in (2.1) satisfying $d(x_n, S^n x_n) \le d(S^n x_n, T^n x_n), n \in \mathbb{N}$. If $F \neq \emptyset$, then

$$\lim_{n\to\infty} d(Sx_n, x_n) = 0 = \lim_{n\to\infty} d(Tx_n, x_n).$$

Proof. By Lemma 2.3, $\lim_{n\to\infty} d(x_n, x^*)$ exists. Suppose that there exists $c \ge 0$ such that $\lim_{n\to\infty} d(x_n, x^*) = c$. Then $d(S^n x_n, x^*) \le k_n d(x_n, x^*)$ implies that

$$\limsup_{n\to\infty} d(S^n x_n, x^*) \le c.$$

Similarly,

$$\limsup_{n\to\infty} d(T^n x_n, x^*) \le c.$$

Further, $\lim_{n\to\infty} d(x_{n+1}, x^*) = c$ gives that,

$$\lim_{n\to\infty} d(W(S^n x_n, T^n x_n, \frac{1}{2}), x^*) = c.$$

Applying Lemma 2.2, we have

$$\lim_{n\to\infty} d(S^n x_n, T^n x_n) = 0. \tag{2.2}$$

But then by the condition $d(x_n, S^n x_n) \le d(S^n x_n, T^n x_n)$,

$$\limsup_{n\to\infty} d(x_n, S^n x_n) \le 0.$$

That is,

$$\lim_{n\to\infty} d(x_n, S^n x_n) = 0. \tag{2.3}$$

Also then

$$d(x_n, T^n x_n) \le d(x_n, S^n x_n) + d(S^n x_n, T^n x_n)$$

implies that

$$\lim_{n\to\infty} d(x_n, T^n x_n) = 0. \tag{2.4}$$

Now by definition, $d(x_{n+1}, T^n x_n) \le \frac{1}{2} d(S^n x_n, T^n x_n)$ so that

$$\lim_{n\to\infty} d(x_{n+1}, T^n x_n) = 0. \tag{2.5}$$

Then

$$d(x_{n+1}, S^n x_n) \le d(x_{n+1}, T^n x_n) + d(T^n x_n, S^n x_n)$$

implies

$$\lim_{n\to\infty} d(x_{n+1}, S^n x_n) = 0. \tag{2.6}$$

Similarly, by

$$d(x_{n+1}, x_n) \le d(x_{n+1}, T^n x_n) + d(x_n, T^n x_n),$$

we have

$$\lim_{n\to\infty} d(x_{n+1}, x_n) = 0. \tag{2.7}$$

Next,

$$\begin{aligned} d(x_{n+1}, Sx_{n+1}) &\le d(x_{n+1}, S^{n+1} x_{n+1}) + d(S^{n+1} x_{n+1}, S^{n+1} x_n) \\ &\quad + d(S^{n+1} x_n, S x_{n+1}) \\ &\le d(x_{n+1}, S^{n+1} x_{n+1}) + k_{n+1} d(x_{n+1}, x_n) \\ &\quad + k_1 d(S^n x_n, x_{n+1}) \end{aligned}$$

yields

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0. \tag{2.8}$$

Moreover,

$$\begin{aligned} d(Sx_{n+1},Tx_{n+1}) &\leq d(Sx_{n+1},S^{n+1}x_{n+1})+d(S^{n+1}x_{n+1},T^{n+1}x_{n+1}) \\ &\quad +d(T^{n+1}x_{n+1},T^{n+1}x_n)+d(T^{n+1}x_n,Tx_{n+1}) \\ &\leq k_1 d(x_{n+1},S^n x_{n+1})+d(S^{n+1}x_{n+1},T^{n+1}x_{n+1}) \\ &\quad +k_{n+1}d(x_{n+1},x_n)+k_1 d(T^n x_n,x_{n+1}) \\ &\leq k_1\left(d(x_{n+1},S^n x_n)+d(S^n x_n,S^n x_{n+1})\right) \\ &\quad +d(S^{n+1}x_{n+1},T^{n+1}x_{n+1})+k_{n+1}d(x_{n+1},x_n) \\ &\quad +k_1 d(T^n x_n,x_{n+1}) \\ &\leq k_1\left(d(x_{n+1},S^n x_n)+k_n d(x_n,x_{n+1})\right) \\ &\quad +d(S^{n+1}x_{n+1},T^{n+1}x_{n+1})+k_{n+1}d(x_{n+1},x_n) \\ &\quad +k_1 d(T^n x_n,x_{n+1}) \end{aligned}$$

gives by $(2.2),(2.5),(2.6)$ and (2.7) that

$$\lim_{n\to\infty} d(Sx_n,Tx_n)=0. \tag{2.9}$$

In turn, by (2.8) and (2.9) we get

$$\lim_{n\to\infty} d(x_n,Tx_n)=0.$$

This completes the proof of the lemma. □

3. Convergence Theorems

We approximate common fixed points of the mappings S and T through convergence of the sequence $\{x_n\}$ defined in (2.1).

The first convergence result in an arbitrary convex metric space goes as follows:

Theorem 3.1. Let C be a nonempty compact and convex subset of a uniformly convex metric space X and S,T and $\{x_n\}$ be as in Lemma 2.4. If $F\neq\emptyset$, then there is a subsequence of $\{x_n\}$ which converges to a common fixed point of S and T.

Proof. Since $\lim\limits_{n\to\infty} d(Sx_n,x_n)=0=\lim\limits_{n\to\infty} d(Tx_n,x_n)$ and C is compact, we have subsequence $\{x_{n_j}\}$ of $\{x_n\}$ with $x_{n_j}\to q$ in C. Continuity of T and S imply $Tx_{n_j}\to Tq$ and $Sx_{n_j}\to Sq$, as $n_j\to\infty$. Thus, $d(Sq,q)=0=d(Tq,q)$. Therefore, $Tq=Sq=q$. □

Theorem 3.2. Let C be a nonempty closed convex subset of a complete convex metric space X, $\{x_n\}$, S and T be as in Lemma 2.4. If $F\neq\emptyset$, then $\{x_n\}$ converges to a common fixed point of S and T if and only if $\liminf_{n\to\infty} d(x_n,F)=0$, where $d(x,F)=\inf\{d(x,p):p\in F\}$.

Proof. Necessity is obvious. Conversely, suppose that $\liminf_{n\to\infty} d(x_n, F) = 0$. As in Lemma 2.3, we have

$$d(x_{n+1}, p) \leq k_n d(x_n, p).$$

This implies

$$d(x_{n+1}, F) \leq k_n d(x_n, F),$$

so that $\lim_{n\to\infty} d(x_n, F)$ exists. Then by the hypothesis,

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

Next, we show that $\{x_n\}$ is a Cauchy sequence in C. Let $\varepsilon > 0$ be arbitrarily chosen. Put $k_n = 1+u_n$. Then $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ implies that $\sum_{n=1}^{\infty} u_n < \infty$.

Now from $d(x_{n+1}, x^*) \leq k_n d(x_n, x^*)$ for all $x^* \in F$ combined with $1+x \leq e^x$ for all $x \geq 0$, we have

$$\begin{aligned} d(x_{n+m}, x^*) &\leq k_{n+m-1} d(x_{n+m-1}, x^*) \\ &= (1 + u_{n+m-1})\, d(x_{n+m-1}, x^*) \\ &\leq e^{u_{n+m-1}} d(x_{n+m-1}, x^*) \\ &\vdots \\ &\leq \left(e^{\sum_{i=n}^{n+m-1} u_i}\right) d(x_n, x^*) \\ &\leq \left(e^{\sum_{i=1}^{\infty} u_i}\right) d(x_n, x^*) \end{aligned}$$

for all $x^* \in F$ and for all $m, n \in \mathbb{N}$.

Since $\sum_{n=1}^{\infty} u_n < \infty$, there is a positive real number M such that $e^{\sum_{i=1}^{\infty} u_i} = M$. Thus

$$d(x_{n+m}, x^*) \leq M d(x_n, x^*)$$

for all $x^* \in F$ and for all $m, n \in \mathbb{N}$.

Next, from $\lim_{n\to\infty} d(x_{n_0}, F) = 0$, there exists a positive integer n_0 such that

$$d(x_{n_0}, F) < \frac{\varepsilon}{M+1}.$$

This says that there exists a $p \in F$ such that

$$d(x_{n_0}, p) < \frac{\varepsilon}{M+1}.$$

Thus for all $m \in \mathbb{N}$,

$$\begin{aligned} d(x_{n_0+m}, x_{n_0}) &\leq d(x_{n_0+m}, p) + d(x_{n_0}, p) \\ &\leq Md(x_{n_0}, p) + d(x_{n_0}, p) \\ &< (M+1)\left(\frac{\varepsilon}{M+1}\right) \\ &= \varepsilon. \end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a complete convex metric space X and so it must converge to a point q in C. Now, $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since F is closed, so we have $q \in F$. □

Khan and Fukhar-ud-din [8] introduced the so-called Condition (A') and gave a bit improved version of it in [8]. We give its metric analogue as follows:

Definition 3.3. Two mappings $S, T : C \to C$ are said to satisfy the Condition (A') if there exists a nondecreasing function $f : [0, \infty) \to [0, \infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$ such that either $d(x, Tx) \geq f(d(x, F))$ or $d(x, Sx) \geq f(d(x, F))$ for all $x \in C$.

We use the Condition (A') to study convergence of $\{x_n\}$ defined in (2.1).

Theorem 3.4. Let X be a uniformly complete convex metric space, C and $\{x_n\}$ be as in Lemma 2.4. Let $S, T : C \to C$ be two asymptotically nonexpansive mappings satisfying the Condition (A'). If $F \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of S and T.

Proof. By Lemma 2.3, $\lim_{n\to\infty} d(x_n, x^*)$ exists for all $x^* \in F$. Let this limit be c, where $c \geq 0$. If $c = 0$, there is nothing to prove. Suppose that $c > 0$. Now, $d(x_{n+1}, x^*) \leq k_n d(x_n, x^*)$ gives that

$$\inf_{x^*\in F} d(x_{n+1}, x^*) \leq k_n \inf_{x^*\in F} d(x_n, x^*),$$

which means that

$$d(x_{n+1}, F) \leq k_n d(x_n, F)$$

and so $\lim_{n\to\infty} d(x_n, F)$ exists. By using the Condition (A'), either

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Tx_n) = 0$$

or

$$\lim_{n\to\infty} f(d(x_n, F)) \leq \lim_{n\to\infty} d(x_n, Sx_n) = 0.$$

In both the cases, we have

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since f is a nondecreasing function and $f(0) = 0$, it follows that

$$\lim_{n\to\infty} d(x_n, F) = 0.$$

The rest of the proof follows the pattern of the above theorem and is therefore omitted. □

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Mujahid Abbas
Department of Mathematics
Lahore University of Management Sciences
54792- Lahore, Pakistan
E-mail address: mujahid@lums.edu.pk

Safeer Hussain Khan
Department of Mathematics, Statistics and Physics
Qatar University
Doha 2713, Qatar.
E-mail address: safeer@qu.edu.qa; safeerhussain5@yahoo.com

Jong Kyu Kim
Department of Mathematics Education
Kyungnam University
Masan, Kyungnam, 631-701, Korea
E-mail address: jongkyuk@kyungnam.ac.kr