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A SYSTEM OF VARIATIONAL INCLUSIONS IN BANACH SPACES

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Abstract. A system of variational inclusions with (A, η, m)-accretive operators in real q-uniformly smooth Banach spaces is introduced. Using the resolvent operator technique associated with (A, η, m)-accretive operators, we prove the existence and uniqueness of solutions for this system of variational inclusions and propose a Mann type iterative algorithm for approximating the unique solution for the system of variational inclusions.

1. Introduction

Variational inclusions have a wide range of applications in the fields of optimization, economics, transportation equilibrium and engineering sciences. For details, we refer the reader to [1-5,7-9] and the references therein.

Recently, some authors discussed several systems of variational inclusions in Hilbert and Banach spaces. Fang and Huang [2] introduced and studied a system of variational inclusions involving (H, η)-monotone operators in Hilbert spaces. Afterwards, Fang and Huang [3] considered a system of variational inclusions involving H-accretive operators in Banach spaces. Ding and Feng [1] and Peng and Zhu [9] discussed, respectively, a system of generalized mixed quasi-variational inclusions with (A, η)-accretive operators and a system of variational inclusions with P-η-accretive operators in q-uniformly smooth Banach spaces. Lately, Peng [7] investigated a system of variational inclusions with (A, η, m)-accretive operators in q-uniformly smooth Banach spaces.

Motivated and inspired by the research work in [1-5,7-9], we introduce and study a system of variational inclusions with (A, η, m)-accretive operators in real q-uniformly smooth Banach spaces. By means of the resolvent operator technique associated with (A, η, m)-accretive operators, we establish the existence and uniqueness of solutions for the system of variational inclusions and

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suggest a Mann type iterative algorithm for approximating the unique solution of the system of variational inclusions.

2. Preliminaries

Throughout this paper, we assume that $(E, \|\cdot\|)$ is a real Banach space with the dual space and the generalized dual pair denoted by E^* and $\langle\cdot,\cdot\rangle$, respectively, 2^E is the family of all the nonempty subsets of E and the *generalized duality mapping* $J_q : E \to 2^{E^*}$ is defined by

$$J_q(x) = \{f^* \in E^* : \langle x, f^*\rangle = \|f^*\| \cdot \|x\|,\ \|f^*\| = \|x\|^{q-1}\}, \quad \forall x \in E,$$

where $q > 1$ is a constant. In particular, J_2 is the usual normalized duality mapping. It is known that, in general, $J_q(x) = \|x\|^{q-2}J_2(x)$ for all $x \neq 0$ and J_q is single-valued if E^* is strictly convex.

The *modulus of smoothness* of E is the function $\rho_E : [0, \infty) \to [0, \infty)$ defined by

$$\rho_E(t) = \sup\left\{\frac{1}{2}(\|x + y\| + \|x - y\|) - 1 : \|x\| \leq 1, \|y\| \leq t\right\}.$$

A Banach space E is called *uniformly smooth* if

$$\lim_{t\to 0} \frac{\rho_E(t)}{t} = 0.$$

E is called *q-uniformly smooth* if there exists a constant $c > 0$ such that

$$\rho_E(t) \leq ct^q, \quad q > 1.$$

Note that J_q is single-valued if E is uniformly smooth.

We recall some definitions needed later.

Definition 2.1. ([4]) Let E be a real uniformly smooth Banach space and $T : E \to E$ and $\eta : E \times E \to E$ be two single-valued operators. T is said to be

(1) *η-accretive* if

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle \geq 0, \quad \forall x, y \in E;$$

(2) *strictly η-accretive* if T is η-accretive and

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle = 0 \quad \text{if and only if} \quad x = y;$$

(3) *r-strongly η-accretive* if there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), J_q(\eta(x, y))\rangle \geq r\|x - y\|^q, \quad \forall x, y \in E;$$

(4) *Lipschitz continuous* if there exists a constant $s > 0$ such that

$$\|T(x) - T(y)\| \leq s\|x - y\|, \quad \forall x, y \in E.$$

Definition 2.2. ([4]) Let E be a real uniformly smooth Banach space and $T : E \to E$ and $g : E \times E \to E$ be two single-valued operators. T is said to be

(1) (α, ξ)-*relaxed cocoercive* if there exist constants $\alpha, \xi > 0$ such that

$$\langle T(x) - T(y), J_q(x - y)\rangle \geq -\alpha\|T(x) - T(y)\|^q + \xi\|x - y\|^q, \quad \forall x, y \in E;$$

(2) (α, ξ)-*relaxed cocoercive* with respect to g if there exist constants $\alpha, \xi > 0$ such that

$$\langle T(x) - T(y), J_q(g(x) - g(y))\rangle \geq -\alpha\|T(x) - T(y)\|^q + \xi\|x - y\|^q, \quad \forall x, y \in E;$$

(3) ξ-*strongly accretive* with respect to g if there exists a constant $\xi > 0$ such that

$$\langle T(x) - T(y), J_q(g(x) - g(y))\rangle \geq \xi\|x - y\|^q, \quad \forall x, y \in E.$$

Definition 2.3. ([4,7]) Let $\eta : E \times E \to E$ be a single-valued operator and $M : E \to 2^E$ be a multi-valued operator. M is said to be *relaxed* η-*accretive* with a constant m if there exists a constant $m > 0$ such that

$$\langle u - v, J_q(\eta(x, y))\rangle \geq -m\|x - y\|^q, \quad \forall x, y \in E,\ u \in M(x),\ v \in M(y).$$

Definition 2.4. ([4,7]) Let $\eta : E \times E \to E$, $A : E \to E$ be single-valued operators and $M : E \to 2^E$ be a multi-valued operator. M is said to be (A, η, m)-*accretive* if M is relaxed η-accretive with a constant m and $(A + \rho M)(E) = E$ holds for all $\rho > 0$.

Definition 2.5. ([9]) A single-valued operator $\eta : E \times E \to E$ is said to be τ-*Lipschitz continuous* if there exists a constant $\tau > 0$ such that

$$\|\eta(u, v)\| \leq \tau\|u - v\|, \quad \forall u, v \in E.$$

Based on Theorem 3.2 and Definition 3.2 in [4], we introduce the following concept.

Definition 2.6. Let $\eta : E \times E \to E$ be a single-valued operator, $A : E \to E$ be a strictly η-accretive single-valued operator and $M, N : E \times E \to 2^E$ satisfy that for each $\omega \in E$, $M(\cdot, \omega)$ and $N(\omega, \cdot)$ are (A, η, m)-accretive. Then for each $\omega \in E$, the *resolvent operators* $R^{A,\eta}_{M(\cdot,\omega),\lambda,m}, R^{A,\eta}_{N(\omega,\cdot),\lambda,m} : E \to E$ associated with $A, \eta, m, M, N, \lambda$ are defined, respectively, by

$$R^{A,\eta}_{M(\cdot,\omega),\lambda,m}(u) = (A + \lambda M(\cdot, \omega))^{-1}(u), \quad \forall u \in E,$$

$$R^{A,\eta}_{N(\omega,\cdot),\lambda,m}(u) = (A + \lambda N(\omega, \cdot))^{-1}(u), \quad \forall u \in E.$$

Definition 2.7. A single-valued operator $F : E \times E \to E$ is said to be (l, θ)-*Lipschitz continuous* if there exist two constants $l > 0$ and $\theta > 0$ such that

$$\|F(u_1, v_1) - F(u_2, v_2)\| \leq l\|u_1 - u_2\| + \theta\|v_1 - v_2\|, \quad \forall u_1, u_2, v_1, v_2 \in E;$$

Lemma 2.1. ([10]) *Let E be a real uniformly smooth Banach space. Then E is q-uniformly smooth if and only if there exists a constants $c_q > 0$ satisfying*

$$\|x + y\|^q \leq \|x\|^q + q\langle y, J_q(x)\rangle + c_q\|y\|^q, \quad \forall x, y \in E.$$

It follows from Theorem 3.3 in [4] that

Lemma 2.2. *For each $i \in \{1,2\}$, let $\eta_i : E \times E \to E$ be a Lipschitz continuous operator with a constant τ_i and $A_i : E \to E$ be a γ_i-strongly η_i-accretive operator. Let $M : E \times E \to 2^E$ satisfy that for each $u \in E$, $M(\cdot,u)$ is (A_1,η_1,m_1)-accretive and $N : E \times E \to 2^E$ satisfy that for each $u \in E$, $N(u,\cdot)$ is (A_2,η_2,m_2)-accretive. Then for each $u \in E$, the resolvent operators $R^{A_1,\eta_1}_{M(\cdot,u),\lambda_1,m_1}$, $R^{A_2,\eta_2}_{N(u,\cdot),\lambda_2,m}$: $E \to E$ are Lipschitz continuous with constants $\frac{\tau_i^{q-1}}{\gamma_i - m_i\lambda_i}$, respectively, i.e.,*

$$\left\|R^{A_1,\eta_1}_{M(\cdot,u),\lambda_1,m_1}(x) - R^{A_1,\eta_1}_{M(\cdot,u),\lambda_1,m_1}(y)\right\| \le \frac{\tau_1^{q-1}}{\gamma_1 - m_1\lambda_1}\|x-y\|, \quad \forall x,y \in E,$$

$$\left\|R^{A_2,\eta_2}_{N(u,\cdot),\lambda_2,m_2}(x) - R^{A_2,\eta_2}_{N(u,\cdot),\lambda_2,m_2}(y)\right\| \le \frac{\tau_2^{q-1}}{\gamma_2 - m_2\lambda_2}\|x-y\|, \quad \forall x,y \in E,$$

where $\lambda_i \in \left(0, \frac{\gamma_i}{m_i}\right)$ is a constant.

Lemma 2.3. ([6]) *Let $\{\alpha_n\}_{n\ge 0}$, $\{\beta_n\}_{n\ge 0}$, $\{\gamma_n\}_{n\ge 0}$ and $\{t_n\}_{n\ge 0}$ be four nonnegative sequences satisfying the inequality*

$$\alpha_{n+1} \le (1-t_n)\alpha_n + t_n\beta_n + \gamma_n, \quad \forall n \ge 0,$$

where $\{t_n\}_{n\ge 0} \subset [0,1]$, $\sum_{n=0}^{\infty} t_n = +\infty$, $\lim_{n\to\infty}\beta_n = 0$ and $\sum_{n=0}^{\infty}\gamma_n < +\infty$. Then $\lim_{n\to\infty}\alpha_n = 0$.

3. A system of variational inclusions

In this section, we introduce a system of variational inclusions with (A,η,m)-accretive operators. In what follows, unless specified otherwise, we always suppose that E is a real q-uniformly smooth Banach space, $A_i : E \to E$, $\eta_i : E \times E \to E$ for $i \in \{1,2\}$, $f,g : E \to E$, $p,h : E \to E$, $F : E \times E \to E$ and $G : E \times E \to E$ are all single-valued operators. Let $M : E \times E \to 2^E$ satisfy that for each $u \in E, M(\cdot,u)$ is (A_1,η_1,m_1)-accretive and $N : E \times E \to 2^E$ satisfy that for each $u \in E, N(u,\cdot)$ is (A_2,η_2,m_2)-accretive. We consider the following problem of finding $(x,y) \in E \times E$ such that

$$\begin{cases} 0 \in F(x,y) + M(f(x),p(y)), \\ 0 \in G(x,y) + N(g(x),h(y)). \end{cases} \tag{3.1}$$

Lemma 3.1. *For $i \in 1,2$, let λ_i be a positive constant, E be a real q-uniformly Banach space, $f,g,h,p : E \to E$, $\eta_i, F, G : E \times E \to E$ be single-valued operators and $A_i : E \to E$ be a strictly η_i-accretive operator. If $M : E \times E \to 2^E$ satisfies that for each $u \in E, M(\cdot,u)$ is (A_1,η_1,m_1)-accretive and $N : E \times E \to 2^E$ satisfies that for each $u \in E$, $N(u,\cdot)$ is (A_2,η_2,m_2)-accretive, then $(x,y) \in$*

$E\times E$ is a solution of the problem (3.1) *if and only if $(x,y)\in E\times E$ satisfies that*

$$\begin{cases} f(x)=R^{A_1,\eta_1}_{M(\cdot,p(y)),\lambda_1,m_1}\big(A_1(f(x))-\lambda_1F(x,y)\big),\\ h(y)=R^{A_2,\eta_2}_{N(g(x),\cdot),\lambda_2,m_2}\big(A_2(h(y))-\lambda_2G(x,y)\big).\end{cases}\tag{3.2}$$

Remark 3.1. The equality (3.2) can be written as

$$\begin{cases} x=(1-\mu_1)x+\mu_1\big[x-f(x)\\ \qquad +R^{A_1,\eta_1}_{M(\cdot,p(y)),\lambda_1,m_1}\big(A_1(f(x))-\lambda_1F(x,y)\big)\big],\\ y=(1-\mu_2)y+\mu_2\big[y-h(y)\\ \qquad +R^{A_2,\eta_2}_{N(g(x),\cdot),\lambda_2,m_2}\big(A_2(h(y))-\lambda_2G(x,y)\big)\big],\end{cases}\tag{3.3}$$

where $\mu_1,\mu_2\in(0,1]$ are two parameters.

4. Existence of solution and Mann iterative approximations for a system of variational inclusions

Now we prove the existence and uniqueness of solutions for the problem (3.1) and construct the Mann type iterative algorithm for approximating the unique solution of the problem (3.1).

Theorem 4.1. *For $i\in\{1,2\}$, let E be a real q-uniformly smooth Banach space, $\eta_i:E\times E\to E$ be τ_i-Lipschitz continuous, $A_i:E\to E$ be γ_i-strongly η_i-accretive and δ_i-Lipschitz continuous, $f:E\to E$ be (t_1,r_1)-relaxed cocoercive and s_1-Lipschitz continuous, $g:E\to E$ be ξ_1-Lipschitz continuous, $p:E\to E$ be ξ_2-Lipschitz continuous, $h:E\to E$ be (t_2,r_2)-relaxed cocoercive and s_2-Lipschitz continuous, $F:E\times E\to E$ be (α_1,β_1)-relaxed cocoercive with respect to $A_1\circ f$ and (σ_1,θ_1)-Lipschitz continuous, $G:E\times E\to E$ be (α_2,β_2)-relaxed cocoercive with respect to $A_2\circ h$ and (σ_2,θ_2)-Lipschitz continuous, $M:E\times E\to 2^E$ satisfy that for each $u\in E$, $M(\cdot,u)$ is (A_1,η_1,m_1)-accretive and $N:E\times E\to 2^E$ satisfy that for each $u\in E, N(u,\cdot)$ is (A_2,η_2,m_2)-accretive. If there exist positive constants $\rho_1,\rho_2,\lambda_1,\lambda_2,\mu_1,\mu_2$ satisfying $\lambda_i<\frac{\gamma_i}{m_i},\mu_i\le 1$ for $i\in\{1,2\}$,*

$$\big\|R^{A_1,\eta_1}_{M(\cdot,y),\lambda_1,m_1}(x)-R^{A_1,\eta_1}_{M(\cdot,\overline{y}),\lambda_1,m_1}(x)\big\|\le\rho_1\|y-\overline{y}\|,\quad \forall x\in E,\ y,\overline{y}\in E,\tag{4.1}$$

$$\big\|R^{A_2,\eta_2}_{N(x,\cdot),\lambda_2,m_2}(y)-R^{A_2,\eta_2}_{N(\overline{x},\cdot),\lambda_2,m_2}(y)\big\|\le\rho_2\|x-\overline{x}\|,\quad \forall x,\overline{x}\in E,\ y\in E\tag{4.2}$$

and

$$\begin{aligned} k=\max\Big\{&1-\mu_1+\mu_1\big(1+qt_1s_1^q-qr_1+c_qs_1^q\big)^{\frac1q}\\ &+\frac{\mu_1\tau_1^{q-1}}{\gamma_1-m_1\lambda_1}\big(\delta_1^qs_1^q+q\lambda_1\alpha_1\sigma_1^q-q\lambda_1\beta_1+c_q\lambda_1^q\sigma_1^q\big)^{\frac1q}+\mu_2\rho_2\xi_1\\ &+\frac{\mu_2\tau_2^{q-1}\lambda_2\sigma_2}{\gamma_2-m_2\lambda_2},1-\mu_2+\mu_2\big(1+qt_2s_2^q-qr_2+c_qs_2^q\big)^{\frac1q}\end{aligned}\tag{4.3}$$

$$+ \frac{\mu_2 \tau_2^{q-1}}{\gamma_2 - m_2\lambda_2}\big(\delta_2^q s_2^q + q\lambda_2\alpha_2\theta_2^q - q\lambda_2\beta_2 + c_q\lambda_2^q\sigma_2^q\big)^{\frac{1}{q}} + \mu_1\rho_1\xi_2$$
$$+ \frac{\mu_1\tau_1^{q-1}\lambda_1\theta_1}{\gamma_1 - m_1\lambda_1}\Big\}$$
$$< 1,$$

then

(a) *the problem* (3.1) *admits a unique solution* $(x, y) \in E \times E$;

(b) *for any given* $(x_0, y_0) \in E\times E$*, the Mann iterative sequence* $\{(x_n, y_n)\}_{n\geq 0}$ *defined by*

$$\begin{cases} x_{n+1} = (1-a_n)x_n + a_n\big\{(1-\mu_1)x_n + \mu_1\big[x_n - f(x_n) \\ \qquad +R^{A_1,\eta_1}_{M(\cdot,p(y_n)),\lambda_1,m_1}\big(A_1\big(f(x_n)\big) - \lambda_1 F(x_n, y_n)\big)\big]\big\} + c_n, \\ y_{n+1} = (1-b_n)y_n + b_n\big\{(1-\mu_2)y_n + \mu_2\big[y_n - h(y_n) \\ \qquad +R^{A_2,\eta_2}_{N(g(x_n),\cdot),\lambda_2,m_2}\big(A_2\big(h(y_n)\big) - \lambda_2 G(x_n, y_n)\big)\big]\big\} + d_n \end{cases} \tag{4.4}$$

for each $n \geq 0$*, where* $\{a_n\}_{n\geq 0}$*,* $\{b_n\}_{n\geq 0} \subset [0, 1]$ *and* $\{c_n\}_{n\geq 0}$*,* $\{d_n\}_{n\geq 0}$ *are any bounded sequences in* E *satisfying*

$$\begin{aligned} &\sum_{n=0}^{\infty}\big[\min\{a_n, b_n\} - k\max\{a_n, b_n\}\big] = +\infty, \\ &\min\{a_n, b_n\} \geq k\max\{a_n, b_n\}, \quad \forall n \geq 0 \end{aligned} \tag{4.5}$$

and

$$\sum_{n=0}^{\infty}\big(\|c_n\| + \|d_n\|\big) < +\infty \tag{4.6}$$

converges strongly to the unique solution (x, y) *of the problem* (3.1)*.*

Proof. Define $S : E \times E \to E$ and $T : E \times E \to E$ by

$$\begin{aligned} S(u, v) = (1-\mu_1)u + \mu_1\big[&u - f(u) \\ &+ R^{A_1,\eta_1}_{M(\cdot,p(v)),\lambda_1,m_1}\big(A_1(f(u)) - \lambda_1 F(u, v)\big)\big] \end{aligned} \tag{4.7}$$

and

$$T(u, v) = (1-\mu_2)v + \mu_2\big[v - h(v) + R^{A_2,\eta_2}_{N(g(u),\cdot),\lambda_2,m_2}\big(A_2(h(v)) - \lambda_2 G(u, v)\big)\big] \tag{4.8}$$

for all $(u, v) \in E \times E$.

Put $(u_1,v_1),(u_2,v_2)\in E\times E$. It follows from (4.7) and (4.8) that

$$\begin{aligned}
&\big\|S(u_1,v_1)-S(u_2,v_2)\big\|\\
&=\big\|(1-\mu_1)(u_1-u_2)+\mu_1\big[u_1-u_2-\big(f(u_1)-f(u_2)\big)\\
&\quad+R^{A_1,\eta_1}_{M(\cdot,p(v_1)),\lambda_1,m_1}\big(A_1(f(u_1))-\lambda_1F(u_1,v_1)\big)\\
&\quad-R^{A_1,\eta_1}_{M(\cdot,p(v_2)),\lambda_1,m_1}\big(A_1(f(u_2))-\lambda_1F(u_2,v_2)\big)\big]\big\|\\
&\le(1-\mu_1)\|u_1-u_2\|+\mu_1\big\|u_1-u_2-(f(u_1)-f(u_2))\big\|\\
&\quad+\mu_1\big\|R^{A_1,\eta_1}_{M(\cdot,p(v_1)),\lambda_1,m_1}\big(A_1(f(u_1))-\lambda_1F(u_1,v_1)\big)\\
&\quad-R^{A_1,\eta_1}_{M(\cdot,p(v_1)),\lambda_1,m_1}\big(A_1(f(u_2))-\lambda_1F(u_2,v_2)\big)\big\|\\
&\quad+\mu_1\big\|R^{A_1,\eta_1}_{M(\cdot,p(v_1)),\lambda_1,m_1}\big(A_1(f(u_2))-\lambda_1F(u_2,v_2)\big)\\
&\quad-R^{A_1,\eta_1}_{M(\cdot,p(v_2)),\lambda_1,m_1}\big(A_1(f(u_2))-\lambda_1F(u_2,v_2)\big)\big\|
\end{aligned}\tag{4.9}$$

and

$$\begin{aligned}
&\big\|T(u_1,v_1)-T(u_2,v_2)\big\|\\
&=\big\|(1-\mu_2)(v_1-v_2)+\mu_2\big[v_1-v_2-\big(h(v_1)-h(v_2)\big)\\
&\quad+R^{A_2,\eta_2}_{N(g(u_1),\cdot),\lambda_2,m_2}\big(A_2(h(v_1))-\lambda_2G(u_1,v_1)\big)\\
&\quad-R^{A_2,\eta_2}_{N(g(u_2),\cdot),\lambda_2,m_2}\big(A_2(h(v_2))-\lambda_2G(u_2,v_2)\big)\big]\big\|\\
&\le(1-\mu_2)\|v_1-v_2\|+\mu_2\big\|v_1-v_2-(h(v_1)-h(v_2))\big\|\\
&\quad+\mu_2\big\|R^{A_2,\eta_2}_{N(g(u_1),\cdot),\lambda_2,m_2}\big(A_2(h(v_1))-\lambda_2G(u_1,v_1)\big)\\
&\quad-R^{A_2,\eta_2}_{N(g(u_1),\cdot),\lambda_2,m_2}\big(A_2(h(v_2))-\lambda_2G(u_2,v_2)\big)\big\|\\
&\quad+\mu_2\big\|R^{A_2,\eta_2}_{N(g(u_1),\cdot),\lambda_2,m_2}\big(A_2(h(v_2))-\lambda_2G(u_2,v_2)\big)\\
&\quad-R^{A_2,\eta_2}_{N(g(u_2),\cdot),\lambda_2,m_2}\big(A_2(h(v_2))-\lambda_2G(u_2,v_2)\big)\big\|.
\end{aligned}\tag{4.10}$$

On account of Lemmas 2.1 and 2.2 and our assumptions, we obtain that

$$\begin{aligned}
&\big\|u_1-u_2-(f(u_1)-f(u_2))\big\|^q\\
&\le\|u_1-u_2\|^q-q\langle f(u_1)-f(u_2),J_q(u_1-u_2)\rangle+c_q\|f(u_1)-f(u_2)\|^q\\
&\le\|u_1-u_2\|^q-q\big(-t_1\|f(u_1)-f(u_2)\|^q+r_1\|u_1-u_2\|^q\big)\\
&\quad+c_qs_1^q\|u_1-u_2\|^q\\
&\le\big(1+qt_1s_1^q-qr_1+c_qs_1^q\big)\|u_1-u_2\|^q,
\end{aligned}\tag{4.11}$$

$$\begin{aligned}
&\big\|v_1-v_2-(h(v_1)-h(v_2))\big\|^q\\
&\le\|v_1-v_2\|^q-q\langle h(v_1)-h(v_2),J_q(v_1-v_2)\rangle+c_q\|h(v_1)-h(v_2)\|^q\\
&\le\|v_1-v_2\|^q-q\big(-t_2\|h(v_1)-h(v_2)\|^q+r_2\|v_1-v_2\|^q\big)\\
&\quad+c_qs_2^q\|v_1-v_2\|^q\\
&\le\big(1+qt_2s_2^q-qr_2+c_qs_2^q\big)\|v_1-v_2\|^q,
\end{aligned}\tag{4.12}$$

$$\begin{aligned}
&\big\|R^{A_1,\eta_1}_{M(\cdot,p(v_1)),\lambda_1,m_1}\big(A_1(f(u_1))-\lambda_1F(u_1,v_1)\big)\\
&\quad-R^{A_1,\eta_1}_{M(\cdot,p(v_1)),\lambda_1,m_1}\big(A_1(f(u_2))-\lambda_1F(u_2,v_2)\big)\big\|\\
&\le\frac{\tau_1^{q-1}}{\gamma_1-m_1\lambda_1}\big\|A_1(f(u_1))-A_1(f(u_2))-\lambda_1\big(F(u_1,v_1)-F(u_2,v_2)\big)\big\|,
\end{aligned}\tag{4.13}$$

$$\begin{aligned}
&\big\|R^{A_2,\eta_2}_{N(g(u_1),\cdot),\lambda_2,m_2}\big(A_2(h(v_1))-\lambda_2G(u_1,v_1)\big)\\
&\quad-R^{A_2,\eta_2}_{N(g(u_1),\cdot),\lambda_2,m_2}\big(A_2(h(v_2))-\lambda_2G(u_2,v_2)\big)\big\|\\
&\le\frac{\tau_2^{q-1}}{\gamma_2-m_2\lambda_2}\big\|A_2(h(v_1))-A_2(h(v_2))-\lambda_2\big(G(u_1,v_1)-G(u_2,v_2)\big)\big\|,
\end{aligned}\tag{4.14}$$

$$\begin{aligned}
&\big\|R^{A_1,\eta_1}_{M(\cdot,p(v_1)),\lambda_1,m_1}\big(A_1(f(u_2))-\lambda_1F(u_2,v_2)\big)\\
&\quad-R^{A_1,\eta_1}_{M(\cdot,p(v_2)),\lambda_1,m_1}\big(A_1(f(u_2))-\lambda_1F(u_2,v_2)\big)\big\|\\
&\le\rho_1\|p(v_1)-p(v_2)\|\\
&\le\rho_1\xi_2\|v_1-v_2\|,
\end{aligned}\tag{4.15}$$

$$\begin{aligned}
&\big\|R^{A_2,\eta_2}_{N(g(u_1),\cdot),\lambda_2,m_2}\big(A_2(h(v_2))-\lambda_2G(u_2,v_2)\big)\\
&\quad-R^{A_2,\eta_2}_{N(g(u_2),\cdot),\lambda_2,m_2}\big(A_2(h(v_2))-\lambda_2G(u_2,v_2)\big)\big\|\\
&\le\rho_2\|g(u_1)-g(u_2)\|\\
&\le\rho_2\xi_1\|u_1-u_2\|,
\end{aligned}\tag{4.16}$$

$$\begin{aligned}
&\big\|A_1(f(u_1))-A_1(f(u_2))-\lambda_1(F(u_1,v_1)-F(u_2,v_1))\big\|^q\\
&\le\|A_1(f(u_1))-A_1(f(u_2))\|^q\\
&\quad-q\lambda_1\langle F(u_1,v_1)-F(u_2,v_1),J_q(A_1(f(u_1))-A_1(f(u_2)))\rangle\\
&\quad+c_q\lambda_1^q\|F(u_1,v_1)-F(u_2,v_1)\|^q\\
&\le\delta_1^qs_1^q\|u_1-u_2\|^q-q\lambda_1\big(-\alpha_1\|F(u_1,v_1)-F(u_2,v_1)\|^q\\
&\qquad+\beta_1\|u_1-u_2\|^q\big)+c_q\lambda_1^q\sigma_1^q\|u_1-u_2\|^q\\
&\le\big(\delta_1^qs_1^q+q\lambda_1\alpha_1\sigma_1^q-q\lambda_1\beta_1+c_q\lambda_1^q\sigma_1^q\big)\|u_1-u_2\|^q
\end{aligned}\tag{4.17}$$

and

$$\begin{aligned}
&\big\|A_2(h(v_1))-A_2(h(v_2))-\lambda_2(G(u_1,v_1)-G(u_1,v_2))\big\|^q\\
&\le\|A_2(h(v_1))-A_2(h(v_2))\|^q\\
&\quad-q\lambda_2\langle G(u_1,v_1)-G(u_1,v_2),J_q(A_1(h(v_1))-A_2(h(v_2)))\rangle\\
&\quad+c_q\lambda_2^q\|G(u_1,v_1)-G(u_1,v_2)\|^q\\
&\le\delta_2^qs_2^q\|v_1-v_2\|^q-q\lambda_2\big(-\alpha_2\|G(u_1,v_1)-G(u_1,v_2)\|^q\\
&\qquad+\beta_2\|v_1-v_2\|^q\big)+c_q\lambda_2^q\sigma_2^q\|v_1-v_2\|^q\\
&\le\big(\delta_2^qs_2^q+q\lambda_2\alpha_2\theta_2^q-q\lambda_2\beta_2+c_q\lambda_2^q\sigma_2^q\big)\|v_1-v_2\|^q.
\end{aligned}\tag{4.18}$$

It follows from (4.9)-(4.18) that

$$\begin{aligned}
&\big\|S(u_1,v_1)-S(u_2,v_2)\big\| \\
&\le (1-\mu_1)\|u_1-u_2\| + \mu_1\big(1+qt_1s_1^q-qr_1+c_qs_1^q\big)^{\frac{1}{q}}\|u_1-u_2\| \\
&\quad + \frac{\mu_1\tau_1^{q-1}}{\gamma_1-m_1\lambda_1}\big(\|A_1(f(u_1))-A_1(f(u_2)) \\
&\quad - \lambda_1(F(u_1,v_1)-F(u_2,v_2))\|\big) + \mu_1\rho_1\xi_2\|v_1-v_2\| \\
&\le \bigg[1-\mu_1+\mu_1\big(1+qt_1s_1^q-qr_1+c_qs_1^q\big)^{\frac{1}{q}} \\
&\qquad + \frac{\mu_1\tau_1^{q-1}}{\gamma_1-m_1\lambda_1}\big(\delta_1^qs_1^q+q\lambda_1\alpha_1\sigma_1^q-q\lambda_1\beta_1+c_q\lambda_1^q\sigma_1^q\big)^{\frac{1}{q}}\bigg]\|u_1-u_2\| \\
&\quad + \bigg(\mu_1\rho_1\xi_2+\frac{\mu_1\tau_1^{q-1}\lambda_1\theta_1}{\gamma_1-m_1\lambda_1}\bigg)\|v_1-v_2\|
\end{aligned} \tag{4.19}$$

and

$$\begin{aligned}
&\big\|T(u_1,v_1)-T(u_2,v_2)\big\| \\
&\le (1-\mu_2)\|v_1-v_2\| + \mu_2\big(1+qt_2s_2^q-qr_2+c_qs_2^q\big)^{\frac{1}{q}}\|v_1-v_2\| \\
&\quad + \frac{\mu_2\tau_2^{q-1}}{\gamma_2-m_2\lambda_2}\big(\|A_2(h(v_1))-A_2(h(v_2)) \\
&\qquad - \lambda_2(G(u_1,v_1)-G(u_2,v_2))\|\big) + \mu_2\rho_2\xi_1\|u_1-u_2\| \\
&\le \bigg[1-\mu_2+\mu_2\big(1+qt_2s_2^q-qr_2+c_qs_2^q\big)^{\frac{1}{q}} \\
&\qquad + \frac{\mu_2\tau_2^{q-1}}{\gamma_2-m_2\lambda_2}\big(\delta_2^qs_2^q+q\lambda_2\alpha_2\theta_2^q-q\lambda_2\beta_2+c_q\lambda_2^q\sigma_2^q\big)^{\frac{1}{q}}\bigg]\|v_1-v_2\| \\
&\quad + \bigg(\mu_2\rho_2\xi_1+\frac{\mu_2\tau_2^{q-1}\lambda_2\sigma_2}{\gamma_2-m_2\lambda_2}\bigg)\|u_1-u_2\|.
\end{aligned} \tag{4.20}$$

Now (4.19) and (4.20) jointly imply that

$$\begin{aligned}
&\big\|S(u_1,v_1)-S(u_2,v_2)\big\| + \big\|T(u_1,v_1)-T(u_2,v_2)\big\| \\
&\le \bigg[1-\mu_1+\mu_1\big(1+qt_1s_1^q-qr_1+c_qs_1^q\big)^{\frac{1}{q}} \\
&\qquad + \frac{\mu_1\tau_1^{q-1}}{\gamma_1-m_1\lambda_1}\big(\delta_1^qs_1^q+q\lambda_1\alpha_1\sigma_1^q-q\lambda_1\beta_1+c_q\lambda_1^q\sigma_1^q\big)^{\frac{1}{q}}+\mu_2\rho_2\xi_1 \\
&\qquad + \frac{\mu_2\tau_2^{q-1}\lambda_2\sigma_2}{\gamma_2-m_2\lambda_2}\bigg]\|u_1-u_2\|
\end{aligned} \tag{4.21}$$

$$
\begin{aligned}
&+\bigg[1-\mu_2+\mu_2\big(1+qt_2s_2^q-qr_2+c_qs_2^q\big)^{\frac{1}{q}}\\
&\quad+\frac{\mu_2\tau_2^{q-1}}{\gamma_2-m_2\lambda_2}\big(\delta_2^qs_2^q+q\lambda_2\alpha_2\theta_2^q-q\lambda_2\beta_2+c_q\lambda_2^q\sigma_2^q\big)^{\frac{1}{q}}+\mu_1\rho_1\xi_2\\
&\quad+\frac{\mu_1\tau_1^{q-1}\lambda_1\theta_1}{\gamma_1-m_1\lambda_1}\bigg]\|v_1-v_2\|\\
&\le k(\|u_1-u_2\|+\|v_1-v_2\|).
\end{aligned}
$$

Define $\|\cdot\|_1$ on $E\times E$ by

$$\|(u,v)\|_1=\|u\|+\|v\|,\quad \forall (u,v)\in E\times E.$$

It is easy to see that $(E\times E,\|\cdot\|_1)$ is a Banach space. Define $V: E\times E\to E\times E$ by

$$V(u,v)=(S(u,v),T(u,v)),\quad \forall (u,v)\in E\times E. \tag{4.22}$$

It follows from (4.3) and (4.21) that

$$\|V(u_1,v_1)-V(u_2,v_2)\|_1\le k\|(u_1,v_1)-(u_2,v_2)\|_1. \tag{4.23}$$

That is, $V: E\times E\to E\times E$ is a contraction operator. Thus the Banach fixed point theorem ensures that V possesses a unique fixed $(x,y)\in E\times E$, that is, (3.3) holds. In view of Lemma 3.1, (x,y) is the unique solution of the problem (3.1).

On account of (4.3)-(4.5), (4.7), (4.8), (4.22) and (4.23), we deduce that

$$
\begin{aligned}
&\|(x_{n+1},y_{n+1})-(x,y)\|_1\\
&=\|x_{n+1}-x\|+\|y_{n+1}-y\|\\
&\le(1-a_n)\|x_n-x\|+a_n\|S(x_n,y_n)-S(x,y)\|+\|c_n\|\\
&\quad+(1-b_n)\|y_n-y\|+b_n\|T(x_n,y_n)-T(x,y)\|+\|d_n\|\\
&\le\max\{1-a_n,1-b_n\}(\|x_n-x\|+\|y_n-y\|)\\
&\quad+\max\{a_n,b_n\}\big(\|S(x_n,y_n)-S(x,y)\|+\|T(x_n,y_n)-T(x,y)\|\big)\\
&\quad+\|c_n\|+\|d_n\|\\
&\le\big(1-\min\{a_n,b_n\}\big)\|(x_n,y_n)-(x,y)\|_1\\
&\quad+k\max\{a_n,b_n\}\|(x_n,y_n)-(x,y)\|_1+\|c_n\|+\|d_n\|\\
&\le\big\{1-[\min\{a_n,b_n\}-k\max\{a_n,b_n\}]\big\}\|(x_n,y_n)-(x,y)\|_1\\
&\quad+\|c_n\|+\|d_n\|,\quad\forall n\ge 0,
\end{aligned}
$$

which together with (4.3), (4.5), (4.6) and Lemma 2.3 implies that $\{(x_n,y_n)\}_{n\ge0}$ converges strongly to the unique solution (x,y) of the problem (3.1). This completes the proof. □

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