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A SYSTEM OF NONLINEAR VARIATIONAL INCLUSIONS WITH GENERAL H-MONOTONE OPERATORS IN BANACH SPACES

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Abstract. A system of nonlinear variational inclusions involving general H-monotone operators in Banach spaces is introduced. Using the resolvent operator technique, we suggest an iterative algorithm for finding approximate solutions to the system of nonlinear variational inclusions, and establish the existence of solutions and convergence of the iterative algorithm for the system of nonlinear variational inclusions.

1. Introduction

Variational inequality theory, which was introduced by Stampacchia [7] in 1964, has emerged as an useful and interesting branch of pure and applied sciences with a wide range of applications in mathematical programming, optimization theory, engineering, elasticity theory and transportation equilibrium etc.

In recent years, variational inequalities have been extended and generalized in different directions, and one of the most important generalizations is called the variational inclusion. Fang and Huang [3] introduced and studied a system of variational inclusions involving H-monotone operators. Moreover, Verma [8] and Fang et al. [4] introduced a system of variational inclusions involving A-monotone operators and (H, η)-monotone operators, respectively. Fang and Huang [1] introduced a new class of generalized accretive operator named H-accretive operators in Banach spaces. As a promotion of these results, Xia and Huang [9] introduced a new system of variational inclusions involving general H-monotone operators in Banach spaces.

Motivated and inspired by the research work in [1-4,6-10], we introduce and study a new system of nonlinear variational inclusions with general H-monotone operators in Banach spaces, which contains the variational inequalities and variational inclusions in [4,6] as special cases. By using the resolvent

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operator technique for the general H-monotone operator, we construct an iterative algorithm of the system of nonlinear variational inclusions and prove the existence of solutions and convergence of the iterative algorithm for the system of nonlinear variational inclusions. The result in this paper extends and improves Theorem 3.4 in [9].

2. Preliminaries

Assume that $(B, \|\cdot\|)$ is a Banach space. Let $CB(B)$ denote the families of all nonempty closed bounded subsets of B and $D^*(\cdot,\cdot)$ denote the Hausdorff metric on $CB(B)$ defined by

$$D^*(A,B) = \max\left\{\sup_{a\in A} d(a,B), \sup_{b\in B} d(A,b)\right\}, \quad \forall A,B \in CB(B),$$

where $d(a,B) = \inf_{b\in B}\|a-b\|$ and $d(A,b) = \inf_{a\in A}\|a-b\|$.

Definition 2.1. ([9]) Let B be a Banach space with the dual space B^* and $P : B \to B^*$ and $g : B \to B$ be two mappings.

(1) P is said to be *monotone* if

$$\langle P(x) - P(y), x-y\rangle \ge 0, \quad \forall x, y \in B;$$

(2) P is said to be *strictly monotone* if P is monotone and

$$\langle P(x) - P(y), x-y\rangle = 0 \quad \text{if and only if} \quad x = y;$$

(3) P is said to be *α-strongly monotone* if there exists $\alpha > 0$ satisfying

$$\langle P(x) - P(y), x-y\rangle \ge \alpha\|x-y\|^2, \quad \forall x, y \in B;$$

(4) P is said to be *β-Lipschitz continuous* if there exists $\beta > 0$ satisfying

$$\|P(x) - P(y)\| \le \beta\|x-y\|, \quad \forall x, y \in B;$$

(5) g is said to be *η-strongly accretive* if there exists $\eta > 0$ satisfying

$$\langle g(x) - g(y), j(x-y)\rangle \ge \eta\|x-y\|^2, \quad \forall x, y \in B,$$

where $j(x-y) \in J(x-y)$ and $J : B \to 2^{B^*}$ is the normalized duality mapping defined by

$$J(x) = \{f \in B^* : \langle f, x\rangle = \|f\|\cdot\|x\|, \|f\| = \|x\|\}, \quad \forall x \in B.$$

Definition 2.2. ([6,9]) Let B be a Banach space with the dual space B^* and $T : B \to 2^{B^*}$ and $A : B \to CB(B)$ be set-valued mappings.

(1) T is said to be *μ-strongly monotone* if there exists $\mu > 0$ satisfying

$$\langle u-v, x-y\rangle \ge \mu\|x-y\|^2, \quad \forall x, y \in B,\ u \in Tx,\ v \in Ty;$$

(2) A is said to be *D^*-Lipschitz* if there exists a constant $\xi > 0$ such that

$$D^*(A(x), A(y)) \le \xi\|x-y\|, \quad \forall x, y \in B.$$

Definition 2.3. For $i \in \{1, 2\}$, let $(B_i, \|\cdot\|_i)$ be a Banach space with the dual space B_i^*. A mapping $F : B_1 \times B_2 \times B_1 \times B_2 \to B_1^*$ is said to be *mixed-Lipschitz continuous* if there exist $\delta > 0$, $\epsilon > 0$, $\varepsilon > 0$ and $\zeta > 0$ such that

$$\begin{aligned}&\|F(x_1, y_1, u_1, v_1) - F(x_2, y_2, u_2, v_2)\|_1\\&\leq \delta\|x_1 - x_2\|_1 + \epsilon\|y_1 - y_2\|_2 + \varepsilon\|u_1 - u_2\|_1 + \zeta\|v_1 - v_2\|_2\end{aligned}$$

for all $x_1, x_2, u_1, u_2 \in B_1$ and $y_1, y_2, v_1, v_2 \in B_2$.

Similarly we can define the mixed-Lipschitz continuity of a mapping $G : B_1 \times B_2 \times B_1 \times B_2 \to B_2^*$.

Definition 2.4. ([9]) Let B be a Banach space with the dual space B^* and $H : B \to B^*$ be a mapping. A set-valued mapping $M : B \to 2^{B^*}$ is said to be *general H-monotone* if M is monotone and $(H + \lambda M)(B) = B^*$ holds for every $\lambda > 0$.

Definition 2.5. ([9]) Let B be a reflexive Banach space with the dual space B^*, $H : B \to B^*$ be a strictly monotone mapping and $M : B \to 2^{B^*}$ be a general H-monotone mapping. A *resolvent operator* (or *proximal mapping*) $R^H_{M,\lambda}$ is defined by

$$R^H_{M,\lambda}(x^*) = (H + \lambda M)^{-1}(x^*), \quad \forall x^* \in B^*,$$

where $\lambda > 0$ is a constant.

Lemma 2.1. ([13]) *Assume that B is a reflexive Banach space with the dual space B^*. Let $H : B \to B^*$ be a mapping and $M : B \to 2^{B^*}$ be a general H-monotone mapping.*

(a) *If $H : B \to B^*$ is a strongly monotone mapping with constant $\gamma > 0$, then the resolvent operator $R^H_{M,\lambda} : B^* \to B$ is Lipschitz continuous with constant $\frac{1}{\gamma}$;*

(b) *If $H : B \to B^*$ is a strictly monotone mapping and $M : B \to 2^{B^*}$ is a strongly monotone mapping with constant $\beta > 0$, then the resolvent operator $R^H_{M,\lambda} : B^* \to B$ is Lipschitz continuous with constant $\frac{1}{\lambda\beta}$.*

Lemma 2.2. ([9]) *Let B be a uniformly smooth Banach space and J be the normalized duality mapping from B into B^*. Then*

(a) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, J(x + y)\rangle$, $\forall x, y \in B$;

(b) $\langle x-y, J(x)-J(y)\rangle \leq 2d^2\rho_B\big(\frac{4}{d}\|x-y\|\big)$, *where* $d = \big(\frac{1}{2}(\|x\|^2 + \|y\|^2)\big)^{\frac{1}{2}}$, $\forall x$, $y \in B$.

3. A system of nonlinear variational inclusions and an iterative algorithm

Let $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$ be two Banach spaces with the topological dual spaces ${B_1}^*$ and ${B_2}^*$, respectively, $H_1 : B_1 \to B_1^*$, $H_2 : B_2 \to B_2^*$, $g_1 : B_1 \to B_1$, $g_2 : B_2 \to B_2$, $F : B_1 \times B_2 \times B_1 \times B_2 \to B_1^*$, $G : B_1 \times B_2 \times B_1 \times B_2 \to B_2^*$ be six mappings and $A, C : B_1 \to CB(B_1)$, $B, D : B_2 \to CB(B_2)$ be four set-valued mappings, $M : B_1 \to 2^{{B_1}^*}$ be a general H_1-monotone mapping and $N : B_2 \to 2^{{B_2}^*}$ be a general H_2-monotone mapping. We consider the following problem of finding (x, y, u, v, w, z) such that $(x, y) \in B_1 \times B_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$ satisfying

$$\begin{cases} 0 \in F(x, y, u, v) + M(g_1(x)), \\ 0 \in G(x, y, w, z) + N(g_2(y)). \end{cases} \tag{3.1}$$

The problem (3.1) is called a system of nonlinear variational inclusions.

Some special cases of the problem (3.1) are as follows:

(A) If B_1 and B_2 are two Hilbert spaces, $F(x, y, u, v) = F_1(x, y) + P(u, v)$, $G(x, y, u, v) = G_1(x, y) + Q(u, v)$ for all $x, u \in B_1$, $y, v \in B_2$, where $F_1, P : B_1 \times B_2 \to B_1$, $G_1, Q : B_1 \times B_2 \to B_2$ are mappings, then the problem (3.1) reduces to the below system of variational inclusions with general H-monotone operators [6], which is to find (x, y, u, v, w, z) with $(x, y) \in B_1 \times B_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$ satisfying

$$\begin{cases} 0 \in F_1(x, y) + P(u, v) + M(g_1(x)), \\ 0 \in G_1(x, y) + Q(w, z) + N(g_2(y)). \end{cases} \tag{3.2}$$

(B) If B_1 and B_2 are two Hilbert spaces, $g_1 \equiv I_1$, $g_2 \equiv I_2$, $F(x, y, u, v) = F_1(x, y)$, $G(x, y, u, v) = G_1(x, y)$ for all $x, u \in B_1$, $y, v \in B_2$, where $F_1 : B_1 \times B_2 \to B_1$, $G_1 : B_1 \times B_2 \to B_2$ are mappings, then the problem (3.1) reduces to the system of variational inclusions [4], which is to find $(x, y) \in B_1 \times B_2$ satisfying

$$\begin{cases} 0 \in F_1(x, y) + M(x), \\ 0 \in G_1(x, y) + N(y). \end{cases} \tag{3.3}$$

Lemma 3.1. *Let $(B_1, \|\cdot\|_1)$ and $(B_2, \|\cdot\|_2)$ be two Banach spaces with the topological dual spaces ${B_1}^*$ and ${B_2}^*$, respectively. Let $H_1 : B_1 \to {B_1}^*$ be a strongly monotone mapping and $H_2 : B_2 \to {B_2}^*$ be a strictly monotone mapping, $g_1 : B_1 \to B_1$, $g_2 : B_2 \to B_2$, $F : B_1 \times B_2 \times B_1 \times B_2 \to B_1^*$ and $G : B_1 \times B_2 \times B_1 \times B_2 \to B_2^*$ be four mappings and $A, C : B_1 \to CB(B_1)$, $B, D : B_2 \to CB(B_2)$ be four set-valued mappings, $M : B_1 \to 2^{{B_1}^*}$ be a general H_1-monotone mapping and $N : B_2 \to 2^{{B_2}^*}$ be a general H_2-monotone mapping. Then (x, y, u, v, w, z) with $(x, y) \in B_1 \times B_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$ is a solution of the problem* (3.1) *if and only if*

$$g_1(x) = R^{H_1}_{M,\lambda}\big(H_1(g_1(x)) - \lambda F(x, y, u, v)\big),$$

$$g_2(x) = R^{H_2}_{N,\rho}\big(H_2(g_2(y)) - \rho G(x, y, w, z)\big),$$

where $R^{H_1}_{M,\lambda} = (H_1 + \lambda M)^{-1}$, $R^{H_2}_{N,\rho} = (H_2 + \rho N)^{-1}$, $\lambda > 0$ and $\rho > 0$ are constants.

Based on Lemma 3.1 and Nadler's result [5], we suggest the following

Algorithm 3.1. *For any given $x_0 \in B_1$, $y_0 \in B_2$, compute the iterative sequences $\{x_n\}_{n\geq 0}$, $\{y_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$ by, $\forall n \geq 0$,*

$$x_{n+1} = x_n - g_1(x_n) + R^{H_1}_{M,\lambda}\big(H_1(g_1(x_n)) - \lambda F(x_n, y_n, u_n, v_n)\big), \tag{3.4}$$

$$y_{n+1} = y_n - g_2(y_n) + R^{H_2}_{N,\rho}\big(H_2(g_2(y_n)) - \rho G(x_n, y_n, w_n, z_n)\big), \tag{3.5}$$

$$\begin{aligned}
&\exists\, u_n \in A(x_n),\ \|u_{n+1} - u_n\|_1 \leq \left(1 + \frac{1}{n+1}\right) D_1^*(A(x_{n+1}), A(x_n)),\\
&\exists\, v_n \in B(y_n),\ \|v_{n+1} - v_n\|_2 \leq \left(1 + \frac{1}{n+1}\right) D_2^*(B(y_{n+1}), B(y_n)),\\
&\exists\, w_n \in C(x_n),\ \|w_{n+1} - w_n\|_1 \leq \left(1 + \frac{1}{n+1}\right) D_1^*(C(x_{n+1}), C(x_n)),\\
&\exists\, z_n \in D(y_n),\ \|z_{n+1} - z_n\|_2 \leq \left(1 + \frac{1}{n+1}\right) D_2^*(D(y_{n+1}), D(y_n)).
\end{aligned} \tag{3.6}$$

4. Existence of solutions for the problem (3.1) and convergence of Algorithm 3.1

In this section, we prove the existence of solutions for the problem (3.1) and convergence of the iterative sequences generated by Algorithm 3.1.

Theorem 4.1. *For $i \in \{1, 2\}$, let $(B_i, \|\cdot\|_i)$ be a uniformly smooth Banach space with the dual space B_i^* and $\rho_{B_i}(t) \leq C_i t^2$ for all $t \geq 0$, where $C_i > 0$ is a constant. Let $H_1 : B_1 \to B_1^*$ be γ-strongly monotone and s_1-Lipschitz continuous, $H_2 : B_2 \to B_2^*$ be strictly monotone and s_2-Lipschitz continuous, $g_1 : B_1 \to B_1$ be k_1-strongly accretive and l_1-Lipschitz continuous, $g_2 : B_2 \to B_2$ be k_2-strongly accretive and l_2-Lipschitz continuous, respectively. Let $A, C : B_1 \to CB(B_1)$ be D_1^*-Lipschitz continuous with constants l_A and l_C, respectively, and $B, D : B_2 \to CB(B_2)$ be D_2^*-Lipschitz continuous with constants l_B and l_D, respectively. Let $F : B_1 \times B_2 \times B_1 \times B_2 \to B_1^*$ and $G : B_1 \times B_2 \times B_1 \times B_2 \to B_2^*$ be mixed-Lipschitz continuous with constants a_1, b_1, c_1, d_1 and a_2, b_2, c_2, d_2, respectively. Assume that $M : B_1 \to 2^{B_1^*}$ is a general H_1-monotone and $N : B_2 \to 2^{B_2^*}$ is a general H_2-monotone and β-strongly monotone. If there exist constants $\lambda > 0$ and $\rho > 0$ such that*

$$\begin{aligned}
\max\Bigg\{ &\left(1 - 2k_1 + 64C_1 {l_1}^2\right)^{\frac{1}{2}} + \frac{s_1 l_1 + \lambda a_1 + \lambda c_1 l_A}{\gamma} + \frac{a_2 + c_2 l_C}{\beta},\\
&\left(1 - 2k_2 + 64C_2 {l_2}^2\right)^{\frac{1}{2}} + \frac{s_2 l_2 + \rho b_2 + \rho d_2 l_D}{\rho\beta} + \frac{\lambda b_1 + \lambda d_1 l_B}{\gamma} \Bigg\}\\
&< 1,
\end{aligned} \tag{4.1}$$

then the problem (3.1) has a solution (x, y, u, v, w, z) with $(x, y) \in B_1 \times B_2$, $u \in A(x)$, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$ and the iterative sequences $\{x_n\}_{n\geq 0}$,

$\{y_n\}_{n\geq 0}$, $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$ and $\{z_n\}_{n\geq 0}$ generated by Algorithm 3.1 converge to x, y, u, v, w, z, respectively.

Proof. By (3.4), Lemma 2.1 and the Lipschitz continuity of H_1 and g_1, we have

$$\begin{aligned}
&\|x_{n+1} - x_n\|_1 \\
&= \big\|x_n - g_1(x_n) + R^{H_1}_{M,\lambda}\big(H_1(g_1(x_n)) - \lambda F(x_n, y_n, u_n, v_n)\big) \\
&\quad - \big(x_{n-1} - g_1(x_{n-1}) + R^{H_1}_{M,\lambda}\big(H_1(g_1(x_{n-1})) \\
&\quad - \lambda F(x_{n-1}, y_{n-1}, u_{n-1}, v_{n-1})\big)\big)\big\|_1 \\
&\leq \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|_1 \\
&\quad + \big\|R^{H_1}_{M,\lambda}\big(H_1(g_1(x_n)) - \lambda F(x_n, y_n, u_n, v_n)\big) \\
&\quad - R^{H_1}_{M,\lambda}\big(H_1(g_1(x_{n-1})) - \lambda F(x_{n-1}, y_{n-1}, u_{n-1}, v_{n-1})\big)\big\|_1 \\
&\leq \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|_1 \\
&\quad + \frac{1}{\gamma}\big\|H_1(g_1(x_n)) - H_1(g_1(x_{n-1})) \\
&\quad - \lambda\big(F(x_n, y_n, u_n, v_n) - F(x_{n-1}, y_{n-1}, u_{n-1}, v_{n-1})\big)\big\|_1 \\
&\leq \|x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))\|_1 + \frac{s_1 l_1}{\gamma}\|x_n - x_{n-1}\|_1 \\
&\quad + \frac{\lambda}{\gamma}\|F(x_n, y_n, u_n, v_n) - F(x_{n-1}, y_{n-1}, u_{n-1}, v_{n-1})\|_1, \quad \forall n \geq 1.
\end{aligned} \tag{4.2}$$

Note that g_1 is k_1-strongly accretive and B_1 is a uniformly smooth Banach space. By Lemma 2.2, we get that

$$\begin{aligned}
&\|x_n - x_{n-1} - g_1(x_n) + g_1(x_{n-1})\|_1^2 \\
&\leq \|x_n - x_{n-1}\|_1{}^2 \\
&\quad - 2\big\langle g_1(x_n) - g_1(x_{n-1}), J_1(x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1})))\big\rangle \\
&= \|x_n - x_{n-1}\|_1{}^2 - 2\big\langle g_1(x_n) - g_1(x_{n-1}), J_1(x_n - x_{n-1})\big\rangle \\
&\quad - 2\big\langle g_1(x_n) - g_1(x_{n-1}), J_1(x_n - x_{n-1} - (g_1(x_n) - g_1(x_{n-1}))) \\
&\quad - J_1(x_n - x_{n-1})\big\rangle \\
&\leq \|x_n - x_{n-1}\|_1{}^2 - 2k_1\|x_n - x_{n-1}\|_1{}^2 \\
&\quad + 4d^2\rho_B\left(\frac{4}{d}\|g_1(x_n) - g_1(x_{n-1})\|_1\right) \\
&\leq (1 - 2k_1)\|x_n - x_{n-1}\|_1{}^2 + 64C_1\|g_1(x_n) - g_1(x_{n-1})\|_1{}^2 \\
&\leq (1 - 2k_1 + 64C_1 l_1{}^2)\|x_n - x_{n-1}\|_1{}^2, \quad \forall n \geq 1,
\end{aligned} \tag{4.3}$$

where $J_1 : B_1 \to {B_1}^*$ is the normalized duality mapping. By the mixed Lipschitz continuity of F, the D_1^*-Lipschitz continuity of A, the D_2^*-Lipschitz continuity of B, (3.6) and (4.2), we infer that

$$
\begin{aligned}
&\|F(x_n, y_n, u_n, v_n) - F(x_{n-1}, y_{n-1}, u_{n-1}, v_{n-1})\|_1 \\
&\le a_1\|x_n - x_{n-1}\|_1 + b_1\|y_n - y_{n-1}\|_2 + c_1\|u_n - u_{n-1}\|_1 \\
&\quad + d_1\|v_n - v_{n-1}\|_2 \\
&\le a_1\|x_n - x_{n-1}\|_1 + b_1\|y_n - y_{n-1}\|_2 \\
&\quad + c_1\left(1 + \frac{1}{n}\right)D_1^*(A(x_n), A(x_{n-1})) \\
&\quad + d_1\left(1 + \frac{1}{n}\right)D_2^*(B(y_n), B(y_{n-1})) \\
&\le a_1\|x_n - x_{n-1}\|_1 + b_1\|y_n - y_{n-1}\|_2 + c_1\left(1 + \frac{1}{n}\right)l_A\|x_n - x_{n-1}\|_1 \\
&\quad + d_1\left(1 + \frac{1}{n}\right)l_B\|y_n - y_{n-1}\|_2, \quad \forall n \ge 1.
\end{aligned} \tag{4.4}
$$

It follows from (4.2)-(4.4) that

$$
\begin{aligned}
&\|x_{n+1} - x_n\|_1 \\
&\le \left(\left(1 - 2k_1 + 64C_1{l_1}^2\right)^{\frac{1}{2}} + \frac{s_1 l_1 + \lambda a_1 + \lambda c_1(1 + \frac{1}{n})l_A}{\gamma}\right)\|x_n - x_{n-1}\|_1 \\
&\quad + \frac{\lambda b_1 + \lambda d_1(1 + \frac{1}{n})l_B}{\gamma}\|y_n - y_{n-1}\|_2, \quad \forall n \ge 1.
\end{aligned} \tag{4.5}
$$

Similarly we conclude that

$$
\begin{aligned}
&\|y_{n+1} - y_n\|_2 \\
&= \big\|y_n - g_2(y_n) + R_{N,\rho}^{H_2}\big(H_2(g_2(y_n)) - \rho G(x_n, y_n, w_n, z_n)\big) \\
&\qquad - \big(y_{n-1} - g_2(y_{n-1}) + R_{N,\rho}^{H_2}\big(H_2(g_2(y_{n-1})) \\
&\qquad - \rho G(x_{n-1}, y_{n-1}, w_{n-1}, z_{n-1})\big)\big)\big\|_2 \\
&\le \|y_n - y_{n-1} - (g_2(y_n) - g_2(y_{n-1}))\|_2 \\
&\quad + \big\|R_{N,\rho}^{H_2}\big(H_2(g_2(y_n)) - \rho G(x_n, y_n, w_n, z_n)\big) \\
&\qquad - R_{N,\rho}^{H_2}\big(H_2(g_2(y_{n-1})) - \rho G(x_{n-1}, y_{n-1}, w_{n-1}, z_{n-1})\big)\big\|_2 \\
&\le \|y_n - y_{n-1} - (g_2(y_n) - g_2(y_{n-1}))\|_2 \\
&\quad + \frac{1}{\rho\beta}\big\|H_2(g_2(y_n)) - H_2(g_2(y_{n-1})) \\
&\qquad - \rho\big(G(x_n, y_n, w_n, z_n) - G(x_{n-1}, y_{n-1}, w_{n-1}, z_{n-1})\big)\big\|_2
\end{aligned} \tag{4.6}
$$

$$\begin{aligned}
&\leq \|y_n - y_{n-1} - (g_2(y_n) - g_2(y_{n-1}))\|_2 + \frac{s_2 l_2}{\rho\beta}\|y_n - y_{n-1}\|_2 \\
&\quad + \frac{1}{\beta}\|G(x_n, y_n, w_n, z_n) - G(x_{n-1}, y_{n-1}, w_{n-1}, z_{n-1})\|_2 \\
&\leq \left(\left(1 - 2k_2 + 64C_2{l_2}^2\right)^{\frac{1}{2}} + \frac{s_2 l_2 + \rho b_2 + \rho d_2(1 + \frac{1}{n})l_D}{\rho\beta}\right)\|y_n - y_{n-1}\|_2 \\
&\quad + \frac{a_2 + c_2(1 + \frac{1}{n})l_C}{\beta}\|x_n - x_{n-1}\|_1, \quad \forall n \geq 1.
\end{aligned}$$

By (4.5) and (4.6), we have

$$\begin{aligned}
&\|x_{n+1} - x_n\|_1 + \|y_{n+1} - y_n\|_2 \\
&\leq \left(\left(1 - 2k_1 + 64C_1{l_1}^2\right)^{\frac{1}{2}} + \frac{s_1 l_1 + \lambda a_1 + \lambda c_1(1 + \frac{1}{n})l_A}{\gamma}\right. \\
&\qquad \left. + \frac{a_2 + c_2(1 + \frac{1}{n})l_C}{\beta}\right)\|x_n - x_{n-1}\|_1 \\
&\quad + \left(\left(1 - 2k_2 + 64C_2{l_2}^2\right)^{\frac{1}{2}} + \frac{s_2 l_2 + \rho b_2 + \rho d_2(1 + \frac{1}{n})l_D}{\rho\beta}\right. \\
&\qquad \left. + \frac{\lambda b_1 + \lambda d_1(1 + \frac{1}{n})l_B}{\gamma}\right)\|y_n - y_{n-1}\|_2 \\
&\leq \theta_n\big(\|x_n - x_{n-1}\|_1 + \|y_n - y_{n-1}\|_2\big), \quad \forall n \geq 1,
\end{aligned} \tag{4.7}$$

where

$$\begin{aligned}
\theta_n = \max\bigg\{&\left(1 - 2k_1 + 64C_1{l_1}^2\right)^{\frac{1}{2}} + \frac{s_1 l_1 + \lambda a_1 + \lambda c_1(1 + \frac{1}{n})l_A}{\gamma} \\
&+ \frac{a_2 + c_2(1 + \frac{1}{n})l_C}{\beta}, \left(1 - 2k_2 + 64C_2{l_2}^2\right)^{\frac{1}{2}} \\
&+ \frac{s_2 l_2 + \rho b_2 + \rho d_2(1 + \frac{1}{n})l_D}{\rho\beta} + \frac{\lambda b_1 + \lambda d_1(1 + \frac{1}{n})l_B}{\gamma}\bigg\}, \quad \forall n \geq 1.
\end{aligned}$$

Let

$$\begin{aligned}
\theta = \max\bigg\{&\left(1 - 2k_1 + 64C_1{l_1}^2\right)^{\frac{1}{2}} + \frac{s_1 l_1 + \lambda a_1 + \lambda c_1 l_A}{\gamma} + \frac{a_2 + c_2 l_C}{\beta}, \\
&\left(1 - 2k_2 + 64C_2{l_2}^2\right)^{\frac{1}{2}} + \frac{s_2 l_2 + \rho b_2 + \rho d_2 l_D}{\rho\beta} + \frac{\lambda b_1 + \lambda d_1 l_B}{\gamma}\bigg\}.
\end{aligned}$$

It is clear that $\theta_n \to \theta$ as $n \to \infty$. By (4.1), we know that $0 < \theta < 1$. It follows from (4.7) that $\{x_n\}_{n\geq 0}$ and $\{y_n\}_{n\geq 0}$ are both Cauchy sequences. Consequently there exist $x \in B_1$ and $y \in B_2$ such that $x_n \to x$ and $y_n \to y$ as $n \to \infty$, respectively.

Next we prove that $u_n \to u \in A(x)$, $v_n \to v \in B(y)$, $w_n \to w \in C(x)$ and $z_n \to z \in D(y)$ as $n \to \infty$. In fact, it follows from the Lipschitz continuity of A, B, C, D and (3.4)-(3.6) that $\{u_n\}_{n\geq 0}$, $\{v_n\}_{n\geq 0}$, $\{w_n\}_{n\geq 0}$, $\{z_n\}_{n\geq 0}$ are also

Cauchy sequences. Consequently, there exist $u \in B_1$, $v \in B_2$, $w \in B_1$, $z \in B_2$ such that $u_n \to u$, $v_n \to v$, $w_n \to w$, $z_n \to z$ as $n \to \infty$. Note that

$$\begin{aligned} d_1(u, A(x)) &\le \|u - u_{n+1}\|_1 + d_1(u_{n+1}, A(x)) \\ &\le \|u - u_{n+1}\|_1 + D_1^*(A(x_{n+1}), A(x)) \\ &\le \|u - u_{n+1}\|_1 + l_A\|x_n - x\|_1 \to 0 \quad \text{as } n \to \infty. \end{aligned}$$

Since $A(x)$ is closed, it follows that $u \in A(x)$. Similarly, $v \in B(y)$, $w \in C(x)$, $z \in D(y)$. By the Lipschitz continuity of $g_1, g_2, B_1, B_2, F, G, P, Q$, $R_{M,\lambda}^{H_1}$, $R_{N,\rho}^{H_2}$ and Algorithm 3.1, we know that x, y, u, v, w, z satisfy the following relations:

$$g_1(x) = R_{M,\lambda}^{H_1}\big(H_1(g_1(x)) - \lambda F(x, y, u, v)\big),$$

$$g_2(y) = R_{N,\rho}^{H_2}\big(H_2(g_2(y)) - \rho G(x, y, w, z)\big).$$

Lemma 3.1 guarantees (x, y, u, v, w, z) is a solution of the problem (3.1). This completes the proof. □

Remark 4.1. Theorem 4.1 extends and improves Theorem 3.4 in [9].

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