East Asian Mathematical Journal
Vol. 26 (2010), No. 5, pp. 627–639

STRONG CONVERGENCE THEOREMS FOR NONEXPANSIVE MAPPINGS AND INVERSE-STRONGLY-MONOTONE MAPPINGS IN A BANACH SPACE

Ying Liu

Abstract. In this paper, we introduce a new iterative sequence for finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of the variational inequality for an inverse-strongly-monotone mapping in a Banach space. Then we show that the sequence converges strongly to a common element of two sets. Using this result, we consider the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse-strongly-monotone mapping, the fixed point problem and the classical variational inequality problem. Our results improve and extend the corresponding results announced by many others.

1. Introduction

Let E be a real Banach space with norm $\|\cdot\|$, let E^* denote the dual of E and let $\langle x, f\rangle$ denote the value of $f \in E^*$ at $x \in E$. Suppose that C is a nonempty, closed convex subset of E and A is a monotone operator of C into E^*. Then we study the problem of finding a point $u \in C$ such that

$$\langle v-u, Au\rangle \geq 0 \quad \forall v \in C. \tag{1.1}$$

This problem is called the *variational inequality problem* [8]. The set of solutions of the variational inequality problem is denoted by $VI(C, A)$. Such a problem is connected with the convex minimization problem, the complementarity problem, the problem of finding a point $u \in E$ satisfying $0 = Au$ and so on. An operator A of C into E^* is said to be *inverse-strongly-monotone* [4,7,9] if there exists a positive real number α such that

$$\langle x-y, Ax-Ay\rangle \geq \alpha\|Ax-Ay\|^2$$

Received August 16, 2010; Revised October 25, 2010; Accepted November 20, 2010.
2000 *Mathematics Subject Classification.* 47H09, 47H05, 47J25.
Key words and phrases. Nonexpansive mapping,generalized projection, inverse-strongly-monotone mapping, variational inequality, p-uniformly convex.
This work was financially supported by the Natural Science Youth Foundation of Hebei Education Commission (No. 2010110) and the Natural Science Youth Foundation of Hebei Province (No. A2010000191).

for all $x, y \in C$. In such a case, A is said to be $\alpha-$inverse-strongly-monotone. If A is an $\alpha-$inverse-strongly-monotone mapping of C into E^*, then it is obvious that A is $\frac{1}{\alpha}-$Lipschitz continuous.

A mapping T of C into E is called nonexpansive if

$$\|Tx - Ty\| \leq \|x - y\|, \ \ \forall x, y \in C.$$

We denoted by $F(T)$ the set of fixed points of T. In 2005, Iiduka and Takahashi[5] proved strong convergence theorems for finding a common element of the set of solution of the variational inequality problem for an inverse-strongly-monotone mapping and the set of fixed points of a nonexpansive mapping in a Hilbert space. In 2008, Matsushita and Takahashi[10]proved a strong convergence theorem for a nonexpansive mapping T in a Banach space by using the following hybrid method:

$$\begin{cases} x_0 = x \in C, \\ C_n = \bar{co}\{z \in C : \|z - Tz\| \leq t_n\|x_n - Tx_n\|\}, \\ D_n = \{z \in C : \langle x_n - z, J(x - x_n)\rangle \geq 0\}, \\ x_{n+1} = P_{C_n \bigcap D_n} x, n = 0, 1, 2, \cdots, \end{cases} \tag{1.2}$$

where $P_{C_n \bigcap D_n}$ is the metric projection from C into $C_n \bigcap D_n$, $\bar{co}D$ denotes the convex closure of the set D and $\{t_n\}$ is a sequence in $(0, 1)$ with $t_n \to 0$ as $n \to \infty$. Then, they proved that the sequence $\{x_n\}$ converges strongly to $P_{F(T)}x$. Recently, Iiduka and Takahashi[6] proved a weak convergence theorem for finding a solution of the variational inequality problem for an operator A that satisfies the following conditions in a 2-uniformly convex and uniformly smooth Banach space E :

(A1) A is $\alpha-$inverse-strongly-monotone;

(A2) $VI(C, A) \neq \emptyset$;

(A3) $\|Ay\| \leq \|Ay - Au\|$ for all $y \in C$ and $u \in VI(C, A)$.

Inspired and motivated by these facts, our purpose in this paper is to obtain a strong convergence theorem for finding a common element of the set of solutions of a variational inequality problem and the set of fixed points of a nonexpansive mapping in a Banach space by using the hybrid method. Our results generalize the results of [5] from Hilbert spaces to Banach spaces. Furthermore, our results also generalize the result of [6] from weak convergence to strong convergence.

2. Preliminaries

Throughout this paper, we denote by N and R the sets of positive integers and real numbers, respectively. When $\{x_n\}$ is a sequence in E, we denote strong convergence of $\{x_n\}$ to $x \in E$ by $x_n \to x$ and weak convergence by $x_n \rightharpoonup x$.

A multi-valued operator $S : E \to 2^{E^*}$ with domain $D(S) = \{z \in E : Sz \neq \emptyset\}$ and range $R(S) = \bigcup\{Sz \in E^* : z \in D(S)\}$ is said to be monotone if $\langle x_1 - x_2, y_1 - y_2\rangle \geq 0$ for each $x_i \in D(S)$ and $y_i \in Sx_i$, $i = 1, 2$. A monotone operator S is said to be maximal if its graph $G(S) = \{(x, y) : y \in Sx\}$ is not properly contained in the graph of any other monotone operator.

Let $U = \{x \in E : \|x\| = 1\}$. A Banach space E is said to be strictly convex if for any $x, y \in U$ and $x \neq y$ implies $\|\frac{x+y}{2}\| < 1$. It is also said to be uniformly convex if for each $\epsilon \in (0, 2]$, there exists $\delta > 0$ such that for any $x, y \in U$, $\|x - y\| \geq \epsilon$ implies $\|\frac{x+y}{2}\| \leq 1 - \delta$. It is known that a uniformly convex Banach space is reflexive and strictly convex. And we define a function $\delta : [0, 2] \to [0, 1]$ called the modulus of convexity of E as follows:

$$\delta(\epsilon) = inf\{1 - \|\frac{x + y}{2}\| : x, y \in U, \|x - y\| \geq \epsilon\}.$$

Then E is uniformly convex if and only if $\delta(\epsilon) > 0$ for all $\epsilon \in (0, 2]$. Let p be a fixed real number with $p \geq 2$. A Banach space E is said to be $p-$*uniformly convex* if there exists a constant $c > 0$ such that $\delta(\epsilon) \geq c\epsilon^p$ for all $\epsilon \in [0, 2]$. For example, see [3] and [13] for more details. We know the following fundamental characterization [3,6] of $p-$uniformly convex Banach spaces:

Lemma 2.1. ([3]) *Let p be a real number with $p \geq 2$ and let E be a Banach space. Then E is $p-$*uniformly convex *if and only if there exists a constant $0 < c \leq 1$ such that*

$$\frac{1}{2}(\|x + y\|^p + \|x - y\|^p) \geq \|x\|^p + c^p\|y\|^p \tag{2.1}$$

for all $x, y \in E$.

The best constant $1/c$ in Lemma 2.1 is called the $p-$*uniformly convexity constant* of E[3]. Putting $x = \frac{(u+v)}{2}$ and $y = \frac{(u-v)}{2}$ in (2.1), we readily conclude that, for all $u, v \in E$,

$$\frac{1}{2}(\|u\|^p + \|v\|^p) \geq \|\frac{u + v}{2}\|^p + c^p\|\frac{u - v}{2}\|^p. \tag{2.2}$$

A Banach space E is said to be smooth if the limit

$$\lim_{n\to\infty} \frac{\|x + ty\| - \|x\|}{t} \tag{2.3}$$

exists for all $x, y \in U$. It is also said to be uniformly smooth if the limit (2.3) is attained uniformly for $x, y \in U$. One should note that no Banach space is $p-$uniformly convex for $1 < p < 2$; see [13] for more details. It is well known that Hilbert and the Lebesgue $L^q(1 < q \leq 2)$ spaces are $2-$uniformly convex, uniformly smooth.

On the other hand, with each $p > 1$, the (generalized) duality mapping J_p from E into 2^{E^*} is defined by

$$J_p(x) := \{v \in E^* : \langle x, v\rangle = \|x\|^p, \|v\| = \|x\|^{p-1}\}, \quad \forall x \in E.$$

In particular, $J = J_2$ is called the *normalized duality mapping.* If E is a Hilbert space, then $J = I$, where I is the identity mapping. The duality mapping J has the following properties:

(i) if E is smooth, then J is single-valued;
(ii) if E is strictly convex, then J is one-to-one;
(iii) if E is reflexive, then J is surjective.
(iv) if E is uniformly smooth, then J is uniformly norm-to-norm continuous on each bounded subset of E.

Lemma 2.2. ([6]) *Let p be a given real number with $p \geq 2$ and let E be a $p-$uniformly convex Banach space. Then, for all $x, y \in E, j_x \in J_p x$ and $j_y \in J_p y$,*

$$\langle x - y, j_x - j_y \rangle \geq \frac{c^p}{2^{p-2}p} \|x - y\|^p,$$

where J_p is the generalized duality mapping of E and $1/c$ is the $p-$uniformly convexity constant of E.

A Banach space E is said to have the K-K property if a sequence $\{x_n\}$ of E satisfying that $x_n \rightharpoonup x \in E$ and $\|x_n\| \to \|x\|$, then $x_n \to x$. It is known that if E is uniformly convex, then E has the K-K property. Let E be a smooth Banach space. The function $\phi : E \times E \to R$ is defined by

$$\phi(y, x) = \|y\|^2 - 2\langle y, Jx \rangle + \|x\|^2$$

for all $x, y \in E$. It is obvious from the definition of the function ϕ that

$$(\|y\| - \|x\|)^2 \leq \phi(y, x) \leq (\|y\| + \|x\|)^2 \quad \forall x, y \in E. \tag{2.4}$$

Remark 1. From Remark 2.1 of [11], we know that $\phi(x, y) = 0$ if and only if $x = y$.

Lemma 2.3. ([11]) *Let E be a uniformly convex and smooth Banach space and let $\{y_n\}, \{z_n\}$ be two sequences of E. If $\phi(y_n, z_n) \to 0$, and either $\{y_n\}$, or $\{z_n\}$ is bounded, then $y_n - z_n \to 0$.*

Let C be a nonempty closed convex subset of E. Suppose that E is reflexive, strictly convex and smooth. Then, for any $x \in E$, there exists a unique element $x_0 \in C$ such that

$$\phi(x_0, x) = \min_{y \in C} \phi(y, x).$$

The mapping $\Pi_C : E \to C$ defined by $\Pi_C x = x_0$ is called the *generalized projection* [2,6,11]. In a Hilbert space, $\Pi_C = P_C$(metric projection). The following are well-known results.

Remark 2. From Remark 1, it is easy to see that $\Pi_E = I$.

Lemma 2.4. ([2, 6, 11]) *Let C be a nonempty closed convex subset of a smooth Banach space E and $x \in E$. Then, $x_0 = \Pi_C x$ if and only if*

$$\langle x_0 - y, Jx - Jx_0 \rangle \geq 0$$

for all $y \in C$.

Lemma 2.5. ([2, 6, 11]) *Let E be a reflexive, strictly convex and smooth Banach space, let C be a nonempty closed convex subset of E and let $x \in E$. Then*

$$\phi(y, \Pi_C x) + \phi(\Pi_C x, x) \le \phi(y, x)$$

for all $y \in C$.

Lemma 2.6. ([10]) *Let C be a closed convex subset of a uniformly convex Banach space. Then for each $r > 0$, there exists a strictly increasing convex continuous function $\gamma : [0, \infty) \to [0, \infty)$ such that $\gamma(0) = 0$ and*

$$\gamma(\|T(\sum_{j=0}^{n} \lambda_j x_j) - \sum_{j=0}^{n} \lambda_j T x_j\|) \le \max_{0 \le j < k \le n} (\|x_j - x_k\| - \|Tx_j - Tx_k\|)$$

for all $n \in N$, $\{\lambda_i\}_{i=0}^{n} \in \Delta^n$, $\{x_i\}_{i=0}^{n} \subset C \bigcap B_r$ and $T \in Lip(C, 1)$, where $\Delta^n = \{\{\lambda_0, \lambda_1, \lambda_2, \cdots, \lambda_n\} : 0 \le \lambda_i (0 \le i \le n)$ and $\sum_{i=0}^{n} \lambda_i = 1\}$, $B_r = \{z \in E : \|z\| \le r\}$ and $Lip(C, 1)$ is the set of all nonexpansive mappings from C into E.

Let E be a reflexive, strictly convex, smooth Banach space and let J be the duality mapping from E into E^*. Then J^{-1} is also single-valued, one-to-one, surjective, and it is the duality mapping from E^* into E. We make use of the following mapping V studied in Alber [1]:

$$V(x, x^*) = \|x\|^2 - 2\langle x, x^*\rangle + \|x^*\|^2 \tag{2.5}$$

for all $x \in E$ and $x^* \in E^*$. In other words, $V(x, x^*) = \phi(x, J^{-1}(x^*))$ for all $x \in E$ and $x^* \in E^*$. For each $x \in E$, the mapping g defined by $g(x^*) = V(x, x^*)$ for all $x^* \in E^*$ is a continuous, convex function from E^* into R. We know the following lemma [1]:

Lemma 2.7. ([1]) *Let E be a reflexive, strictly convex, smooth Banach space and let V be as in (2.5). Then*

$$V(x, x^*) + 2\langle J^{-1}(x^*) - x, y^*\rangle \le V(x, x^* + y^*)$$

for all $x \in E$ and $x^*, y^* \in E^*$.

An operator A of C into E^* is said to be hemicontinuous if for all $x, y \in C$, the mapping f of $[0, 1]$ into E^* defined by $f(t) = A(tx + (1-t)y)$ is continuous with respect to the weak* topology of E^*. We denote by $N_C(v)$ the normal cone for C at a point $v \in C$, that is,

$$N_C(v) = \{x^* \in E^* : \langle v - y, x^*\rangle \ge 0 \text{ for all } y \in C\}.$$

We know the following theorem [12]:

Theorem 2.8. (See Rockafellar [12]) *Let C be a nonempty, closed convex subset of a Banach space E and let A be a monotone, hemicontinuous operator of C into E^*. Let $T \subset E \times E^*$ be an operator defined as follows:*

$$Tv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \bar{\in} C. \end{cases}$$

Then T is maximal monotone and $T^{-1}0 = VI(C, A)$.

Lemma 2.9. ([6]) *Let C be a nonempty, closed convex subset of a Banach space E and let A be a monotone, hemicontinuous operator of C into E^*. Then*

$$VI(C, A) = \{u \in C : \langle v - u, Av\rangle \geq 0 \text{ for all } v \in C\}.$$

It is obvious from Lemma 2.9 that the set $VI(C, A)$ is a closed convex subset of C.

3. Main results

Theorem 3.1. *Let E be a 2-uniformly convex, uniformly smooth Banach space. Let C be a nonempty, closed convex subset of E. Assume that A is an operator of C into E^* that satisfies the conditions $(A1) - (A3)$. Assume that T is a nonexpansive mapping from C into itself such that $F = F(T) \bigcap VI(C, A) \neq \emptyset$. The sequence $\{x_n\}$ is defined by*

$$\begin{cases} x_0 \in C \quad \text{chosen} \quad \text{arbitrarily}, \\ y_n = J^{-1}(\beta_n Jx_n + (1 - \beta_n)J\Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n))), \\ \bar{C_n} = \bar{co}\{z \in C : \|z - Tz\| \leq t_n\|x_n - Tx_n\|\}, \\ \tilde{C_n} = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ C_n = \bar{C_n} \bigcap \tilde{C_n}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0, \end{cases} \tag{3.1}$$

where $\{\beta_n\}$ and $\{t_n\}$ satisfy: $0 \leq \beta_n < 1$, and $\limsup_{n\to\infty} \beta_n < 1$, $\{t_n\} \subset (0, 1)$ and $\lim_{n\to\infty} t_n = 0$. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges strongly to $\Pi_F x_0$, where $\frac{1}{c}$ is the $2-$uniformly convexity constant of E.

Proof. From the definition of C_n and Q_n, it is obvious that $C_n \bigcap Q_n$ is closed and convex for each $n \in N \bigcup\{0\}$. Next, we show that $F \subset C_n \bigcap Q_n$ for all $n \in N \bigcup\{0\}$. Put $u_n = J^{-1}(Jx_n - \lambda_n Ax_n)$ for every $n \in N \bigcup\{0\}$. Let $p \in F$. It holds from Lemmas 2.5 and 2.7 that

$$\begin{aligned} \phi(p, \Pi_C u_n) &\leq \phi(p, u_n) \\ &= V(p, Jx_n - \lambda_n Ax_n) \\ &\leq V(p, (Jx_n - \lambda_n Ax_n) + \lambda_n Ax_n) \\ &\quad - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - p, \lambda_n Ax_n\rangle \\ &= V(p, Jx_n) - 2\lambda_n\langle u_n - p, Ax_n\rangle \\ &= \phi(p, x_n) - 2\lambda_n\langle x_n - p, Ax_n\rangle + 2\langle u_n - x_n, -\lambda_n Ax_n\rangle \end{aligned} \tag{3.2}$$

for every $n \in N \bigcup \{0\}$. From the condition(A1) and $p \in VI(C, A)$, we have

$$\begin{aligned} -2\lambda_n\langle x_n - p, Ax_n\rangle &= -2\lambda_n\langle x_n - p, Ax_n - Ap\rangle - 2\lambda_n\langle x_n - p, Ap\rangle \\ &\leq -2\lambda_n\alpha\|Ax_n - Ap\|^2 \end{aligned} \tag{3.3}$$

for every $n \in N \bigcup \{0\}$. By Lemma 2.2 and the condition(A3), we also have

$$\begin{aligned} 2\langle u_n - x_n, -\lambda_n Ax_n\rangle &= 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}Jx_n, -\lambda_n Ax_n\rangle \\ &\leq 2\|J^{-1}(Jx_n - \lambda_n Ax_n) - J^{-1}(Jx_n)\|\|\lambda_n Ax_n\| \\ &\leq \frac{4}{c^2}\|Jx_n - \lambda_n Ax_n - Jx_n\|\|\lambda_n Ax_n\| \\ &= \frac{4}{c^2}\lambda_n^2\|Ax_n\|^2 \\ &\leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2. \end{aligned} \tag{3.4}$$

Therefore, from (3.3), (3.4) and (3.2), we have

$$\phi(p, \Pi_C u_n) \leq \phi(p, x_n) + 2a(\frac{2}{c^2}b - \alpha)\|Ax_n - Ap\|^2. \tag{3.5}$$

Then, by the convexity of $\|\cdot\|^2$ and (3.5), we have

$$\begin{aligned} \phi(p, y_n) &= \|p\|^2 - 2\langle p, \beta_n Jx_n + (1 - \beta_n)J\Pi_C u_n\rangle \\ &\quad + \|\beta_n Jx_n + (1 - \beta_n)J\Pi_C u_n\|^2 \\ &\leq \|p\|^2 - 2\beta_n\langle p, Jx_n\rangle - 2(1 - \beta_n)\langle p, J\Pi_C u_n\rangle + \beta_n\|x_n\|^2 \\ &\quad + (1 - \beta_n)\|\Pi_C u_n\|^2 \\ &= \beta_n\phi(p, x_n) + (1 - \beta_n)\phi(p, \Pi_C u_n) \\ &\leq \phi(p, x_n) + (1 - \beta_n)2a(\frac{2}{c^2}b - \alpha)\|Ax_n - Ap\|^2 \\ &\leq \phi(p, x_n). \end{aligned} \tag{3.6}$$

Thus, we have $p \in \tilde{C_n}$. It is obvious that $p \in \bar{C_n}$. Therefore we obtain $F \subset C_n$ for each $n \in N \bigcup \{0\}$. Using the same argument presented in the proof of [11, Theorem 3.1;pp.261-262] we have $F \subset C_n \bigcap Q_n$ for each $n \in N \bigcup \{0\}$. This implies that $\{x_n\}$ is well defined. It follows from the definition of Q_n and lemma 2.4 that $x_n = \Pi_{Q_n} x_0$. Using $x_n = \Pi_{Q_n} x_0$ and lemma 2.5, we have

$$\phi(x_n, x_0) \leq \phi(p, x_0) - \phi(p, x_n) \leq \phi(p, x_0)$$

for each $p \in F \subset Q_n$ for each $n \in N \bigcup \{0\}$. Therefore, $\phi(x_n, x_0)$ is bounded. Moreover, from (2.4), we have that $\{x_n\}$ is bounded.

Since $x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0 \in Q_n$, and $x_n = \Pi_{Q_n} x_0$, we have $\phi(x_n, x_0) \leq \phi(x_{n+1}, x_0)$ for each $n \in N \bigcup \{0\}$. Therefore, $\{\phi(x_n, x_0)\}$ is nondecreasing. So there exists the limit of $\phi(x_n, x_0)$. From the lemma 2.5, we have

$$\phi(x_{n+1}, x_n) \leq \phi(x_{n+1}, x_0) - \phi(x_n, x_0)$$

for each $n \in N\bigcup\{0\}$. This implies that $\lim_{n\to\infty} \phi(x_{n+1}, x_n) = 0$. Since $x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0 \in C_n \subset \tilde{C_n}$, from the definition of $\tilde{C_n}$, we also have

$$\phi(x_{n+1}, y_n) \leq \phi(x_{n+1}, x_n)$$

for each $n \in N\bigcup\{0\}$. Tending $n \to \infty$, we have $\lim_{n\to\infty} \phi(x_{n+1}, y_n) = 0$. Using lemma 2.3, we obtain

$$\lim_{n\to\infty} \|x_{n+1} - y_n\| = \lim_{n\to\infty} \|x_{n+1} - x_n\| = 0. \tag{3.7}$$

From $\|x_n - y_n\| \leq \|x_n - x_{n+1}\| + \|x_{n+1} - y_n\|$, we have

$$\|x_n - y_n\| \to 0, \ \ (n \to \infty). \tag{3.8}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\lim_{n\to\infty} \|Jx_{n+1} - Jy_n\| = \lim_{n\to\infty} \|Jx_{n+1} - Jx_n\| = \lim_{n\to\infty} \|Jx_n - Jy_n\| = 0. \tag{3.9}$$

Therefore, for each $p \in F$, we have

$$\begin{aligned} \phi(p, x_n) - \phi(p, y_n) &= 2\langle p, Jy_n - Jx_n\rangle + \|x_n\|^2 - \|y_n\|^2, \\ &\leq 2\|p\|\|Jy_n - Jx_n\| + (\|x_n\| - \|y_n\|)(\|x_n\| + \|y_n\|) \\ &\to 0. \end{aligned} \tag{3.10}$$

From (3.6), we have

$$-(1-\beta_n)2a(\frac{2}{c^2}b - \alpha)\|Ax_n - Ap\|^2 \leq \phi(p, x_n) - \phi(p, y_n).$$

By (3.10) and $\limsup_{n\to\infty} \beta_n < 1$, we have

$$\|Ax_n - Ap\| \to 0, \qquad \text{as } n \to \infty. \tag{3.11}$$

From lemmas 2.5 and 2.7, and (3.4), for each $n \in N\bigcup\{0\}$, we have

$$\begin{aligned} \phi(x_n, \Pi_C u_n) \leq \phi(x_n, u_n) &= \phi(x_n, J^{-1}(Jx_n - \lambda_n Ax_n)) \\ &= V(x_n, Jx_n - \lambda_n Ax_n) \\ &\leq V(x_n, Jx_n - \lambda_n Ax_n + \lambda_n Ax_n) \\ &\quad - 2\langle J^{-1}(Jx_n - \lambda_n Ax_n) - x_n, \lambda_n Ax_n\rangle \\ &= \phi(x_n, x_n) + 2\langle u_n - x_n, -\lambda_n Ax_n\rangle \\ &= 2\langle u_n - x_n, -\lambda_n Ax_n\rangle \\ &\leq \frac{4}{c^2}\lambda_n^2\|Ax_n - Ap\|^2. \end{aligned}$$

By (3.11), we get

$$\phi(x_n, \Pi_C u_n) \to 0, \qquad \text{as } n \to \infty. \tag{3.12}$$

Applying lemma 2.3, we obtain from (3.12) that

$$\|x_n - \Pi_C u_n\| \to 0, \qquad \text{as } n \to \infty. \tag{3.13}$$

Since J is uniformly norm-to-norm continuous on bounded sets, we have

$$\|J\Pi_C u_n - Jx_n\| \to 0, \quad \text{as } n \to \infty. \tag{3.14}$$

On the other hand, since $x_{n+1} \in C_n \subset \bar{C_n}$ and $t_n > 0$, there exist $m \in N$, $\{\lambda_i\} \in \Delta^m$ and $\{z_i\}_{i=0}^m \subset C$ such that

$$\|x_{n+1} - \sum_{i=0}^m \lambda_i z_i\| < t_n \text{ and } \|z_i - Tz_i\| \leq t_n \|x_n - Tx_n\| \text{ for all } i \in \{0, 1, \cdots, m\}. \tag{3.15}$$

Put $r_0 = 2 \sup_{n \geq 0} \|x_n - u\|$, where $u = \Pi_F x_0$. It follows from Lemma 2.6 and (3.15) that

$$\begin{aligned}\|x_{n+1} - Tx_{n+1}\| &\leq \|x_{n+1} - \sum_{i=0}^m \lambda_i z_i\| + \|\sum_{i=0}^m \lambda_i (z_i - Tz_i)\| \\ &\quad + \|\sum_{i=0}^m \lambda_i Tz_i - T(\sum_{i=0}^m \lambda_i z_i)\| + \|T(\sum_{i=0}^m \lambda_i z_i) - Tx_{n+1}\| \\ &\leq (2 + r_0)t_n + \gamma^{-1}\big(\max_{0 \leq i < j \leq m} (\|z_i - z_j\| - \|Tz_i - Tz_j\|)\big) \\ &\leq (2 + r_0)t_n + \gamma^{-1}\big(\max_{0 \leq i < j \leq m} (\|z_i - Tz_i\| + \|z_j - Tz_j\|)\big) \\ &\leq (2 + r_0)t_n + \gamma^{-1}(2r_0 t_n).\end{aligned}$$

This gives us that $\|x_{n+1} - Tx_{n+1}\| \to 0$ as $n \to \infty$. Since T is nonexpansive, T is demiclosed. So, we have that if $\{x_{n_i}\}$ is a subsequence of $\{x_n\}$ such that $x_{n_i} \rightharpoonup \hat{x}$, then $\hat{x} \in F(T)$.

We next prove $\hat{x} \in VI(C, A)$. from (3.13), we have $\Pi_C u_n \rightharpoonup \hat{x}$. Let $S \subset E \times E^*$ be an operator as follows:

$$Sv = \begin{cases} Av + N_C(v), & v \in C, \\ \emptyset, & v \bar{\in} C. \end{cases}$$

By Theorem 2.8, S is maximal monotone and $S^{-1}0 = VI(C, A)$. Let $(v, w) \in G(S)$. Since $w \in Sv = Av + N_C(v)$, we have $w - Av \in N_C(v)$. From $\Pi_C u_n \in C$, we get

$$\langle v - \Pi_C u_n, w - Av \rangle \geq 0. \tag{3.16}$$

On the other hand, from lemma 2.4, we have $\langle v - \Pi_C u_n, J\Pi_C u_n - Ju_n \rangle \geq 0$ and hence

$$\langle v - \Pi_C u_n, \frac{Jx_n - J\Pi_C u_n}{\lambda_n} - Ax_n \rangle \leq 0. \tag{3.17}$$

Then it holds from (3.16) and (3.17) that

$$\begin{aligned}
\langle v - \Pi_C u_n, w\rangle &\geq \langle v - \Pi_C u_n, Av\rangle \\
&\geq \langle v - \Pi_C u_n, Av\rangle + \langle v - \Pi_C u_n, \frac{Jx_n - J\Pi_C u_n}{\lambda_n} - Ax_n\rangle \\
&= \langle v - \Pi_C u_n, Av - Ax_n\rangle + \langle v - \Pi_C u_n, \frac{Jx_n - J\Pi_C u_n}{\lambda_n}\rangle \\
&= \langle v - \Pi_C u_n, Av - A\Pi_C u_n\rangle + \langle v - \Pi_C u_n, A\Pi_C u_n - Ax_n\rangle \\
&\quad + \langle v - \Pi_C u_n, \frac{Jx_n - J\Pi_C u_n}{\lambda_n}\rangle \\
&\geq -\|v - \Pi_C u_n\| \frac{\|\Pi_C u_n - x_n\|}{\alpha} - \|v - \Pi_C u_n\| \frac{\|J\Pi_C u_n - Jx_n\|}{a} \\
&\geq -M\left(\frac{\|\Pi_C u_n - x_n\|}{\alpha} + \frac{\|J\Pi_C u_n - Jx_n\|}{a}\right),
\end{aligned}$$

for every $n \in N \bigcup \{0\}$, where $M = \sup\{\|v - \Pi_C u_n\| : n \in N \bigcup \{0\}\}$. Taking $n = n_i$, from (3.13) and (3.14), we have $\langle v - \hat{x}, w\rangle \geq 0$ as $i \to \infty$. By the maximality of S, we obtain $\hat{x} \in S^{-1}0$ and hence $\hat{x} \in VI(C, A)$. Therefore, $\hat{x} \in F$.

Finally, we show that $x_n \to \Pi_F x_0$. Let $\tilde{x} = \Pi_F x_0$. For any $n \in N$, from $x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0$ and $\tilde{x} \in F \subset C_n \bigcap Q_n$, we have $\phi(x_{n+1}, x_0) \leq \phi(\tilde{x}, x_0)$. On the other hand, from weakly lower semicontinuity of the norm, we have

$$\begin{aligned}
\phi(\hat{x}, x_0) &= \|\hat{x}\|^2 - 2\langle \hat{x}, Jx_0\rangle + \|x_0\|^2 \\
&\leq \liminf_{i\to\infty} (\|x_{n_i}\|^2 - 2\langle x_{n_i}, Jx_0\rangle + \|x_0\|^2) \\
&= \liminf_{i\to\infty} \phi(x_{n_i}, x_0) \\
&\leq \limsup_{i\to\infty} \phi(x_{n_i}, x_0) \\
&\leq \phi(\tilde{x}, x_0).
\end{aligned}$$

From the definition of $\Pi_F x_0$, we obtain $\hat{x} = \tilde{x}$ and hence, $\lim_{i\to\infty} \phi(x_{n_i}, x_0) = \phi(\hat{x}, x_0)$. So, we have $\lim_{i\to\infty} \|x_{n_i}\| = \|\hat{x}\|$. Using the K-K property of E, we obtain $x_{n_i} \to \Pi_F x_0$. Since x_{n_i} is an arbitrary convergent subsequence of $\{x_n\}$, we can conclude that $\{x_n\}$ converges strongly to $\Pi_F x_0$. □

Corollary 3.2. *Let E be a 2-uniformly convex, uniformly smooth Banach space. Let A be an $\alpha-$inverse-strongly monotone operator of E into itself and T be a nonexpansive mapping of E into itself such that $F(T) \bigcap A^{-1}0 \neq \emptyset$.*

Suppose that the sequence $\{x_n\}$ *is defined by*

$$\begin{cases} x_0 \in E \quad chosen \quad arbitrarily, \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)(Jx_n - \lambda_n Ax_n)), \\ \bar{C_n} = \bar{co}\{z \in C : \|z - Tz\| \le t_n \|x_n - Tx_n\|\}, \\ \tilde{C_n} = \{z \in C : \phi(z, y_n) \le \phi(z, x_n)\}, \\ C_n = \bar{C_n} \bigcap \tilde{C_n}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n \rangle \ge 0\}, \\ x_{n+1} = \Pi_{C_n \bigcap Q_n} x_0, \end{cases} \tag{3.1}$$

where $\{\beta_n\}$ *and* $\{t_n\}$ *satisfy:* $0 \le \beta_n < 1$*, and* $\limsup_{n\to\infty} \beta_n < 1$*,* $\{t_n\} \subset (0,1)$ *and* $\lim_{n\to\infty} t_n = 0$*. If* $\{\lambda_n\}$ *is chosen so that* $\lambda_n \in [a,b]$ *for some* a, b *with* $0 < a < b < c^2\alpha/2$*, then the sequence* $\{x_n\}$ *converges strongly to* $\Pi_{F(T)\bigcap A^{-1}0} x_0$*, where* $\frac{1}{c}$ *is the* $2-$*uniformly convexity constant of* E*.*

Proof. In Theorem 3.1, we put $C = E$. By $\Pi_E = I$, we have $y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)(Jx_n - \lambda_n Ax_n)) = J^{-1}(\beta_n Jx_n + (1-\beta_n)J\Pi_E(J^{-1}(Jx_n - \lambda_n Ax_n)))$ for every $n = 0, 1, 2...$ From Remark 2.2 and Lemma 2.4, We also have $VI(E, A) = A^{-1}0$ and $\|Ay\| = \|Ay - 0\| = \|Ay - Au\|$ for all $y \in E$ and $u \in A^{-1}0$. So, by using Theorem 3.1, $\{x_n\}$ converges strongly to $\Pi_{F(T)\bigcap A^{-1}0} x_0$. □

Remark 3. In Theorem 4.2 of [5], Iiduka and Takahashi proved the following conclusion:

Let H be a real Hilbert space. Let A be an $\alpha-$inverse-strongly monotone operator of H into itself and T be a nonexpansive mapping of H into itself such that $F(T)\bigcap A^{-1}0 \neq \emptyset$. Suppose $x_1 = x \in H$ and $\{x_n\}$ is given by

$$x_{n+1} = \alpha_n x + (1-\alpha_n)T(x_n - \lambda_n Ax_n)$$

for every $n = 1, 2, ...$, where $\{\alpha_n\}$ is a sequence in $[0, 1)$ and $\{\lambda_n\}$ is a sequence in $[0, 2\alpha]$. If $\{\alpha_n\}$ and $\{\lambda_n\}$ are chosen so that $\lambda_n \in [a, b]$ for some a, b with $0 < a < b < 2\alpha$,

$$\lim_{n\to\infty} \alpha_n = 0, \sum_{n=1}^{\infty} \alpha_n = \infty, \sum_{n=1}^{\infty} |\alpha_{n+1} - \alpha_n| < \infty \text{ and } \sum_{n=1}^{\infty} |\lambda_{n+1} - \lambda_n| < \infty,$$

then $\{x_n\}$ converges strongly to $P_{F(T)\bigcap A^{-1}0} x$.

Therefore, it's obvious that Corollary 3.1 generalize the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of zeros of an inverse-strongly-monotone mapping from Hilbert spaces to 2-uniformly convex, uniformly smooth Banach spaces without assuming any additional conditions on operators A and T. Furthermore, these conditions that $0 \le \beta_n < 1$, $\limsup_{n\to\infty} \beta_n < 1$, $\{t_n\} \subset (0, 1)$ and $\lim_{n\to\infty} t_n = 0$ on control sequences $\{\beta_n\}$, $\{t_n\}$ are easier to implement than these conditions that

$\lim_{n\to\infty}\alpha_n = 0$, $\sum_{n=1}^{\infty}\alpha_n = \infty$, $\sum_{n=1}^{\infty}|\alpha_{n+1}-\alpha_n| < \infty$ and $\sum_{n=1}^{\infty}|\lambda_{n+1}-\lambda_n| < \infty$ assumed by Theorem 4.2 of [5]. In addition, in Corollary 3.1, we obtain that the strong convergence point of $\{x_n\}$ is $\Pi_{F(T)\bigcap A^{-1}0}x_0$. If $E = H$, then $\Pi_{F(T)\bigcap A^{-1}0}x_0 = P_{F(T)\bigcap A^{-1}0}x_0$. Hence, this is the same as the convergent result of Theorem 4.2 of [5].

Corollary 3.3. *Let E be a 2-uniformly convex, uniformly smooth Banach space. Let C be a nonempty, closed convex subset of E. Assume that T is a nonexpansive mapping from C into itself such that $F(T) \neq \emptyset$. Suppose that the sequence $\{x_n\}$ is defined by*

$$\begin{cases} x_0 \in C \quad chosen \quad arbitrarily, \\ \bar{C_n} = \bar{co}\{z \in C : \|z - Tz\| \leq t_n\|x_n - Tx_n\|\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{\bar{C_n}\bigcap Q_n}x_0, \end{cases} \tag{3.1}$$

where $\{t_n\}$ satisfies: $\{t_n\} \subset (0,1)$ and $\lim_{n\to\infty} t_n = 0$. Then the sequence $\{x_n\}$ converges strongly to $\Pi_{F(T)}x_0$.

Proof. Taking $A \equiv 0$ in Theorem 3.1, we have $y_n = x_n$, $VI(C,A) = C$, $\tilde{C_n} = C$ and $C_n = \bar{C_n}$. Then, it is easy to obtain the desired result. □

Corollary 3.4. *Let E be a 2-uniformly convex, uniformly smooth Banach space. Let C be a nonempty, closed convex subset of E. Assume that A is an operator of C into E^* that satisfies the conditions $(A1)-(A3)$. The sequence $\{x_n\}$ is defined by*

$$\begin{cases} x_0 \in C \quad chosen \quad arbitrarily, \\ y_n = J^{-1}(\beta_n Jx_n + (1-\beta_n)J\Pi_C(J^{-1}(Jx_n - \lambda_n Ax_n))), \\ C_n = \{z \in C : \phi(z, y_n) \leq \phi(z, x_n)\}, \\ Q_n = \{z \in C : \langle x_n - z, Jx_0 - Jx_n\rangle \geq 0\}, \\ x_{n+1} = \Pi_{C_n\bigcap Q_n}x_0, \end{cases} \tag{3.1}$$

where $\{\beta_n\}$ satisfies: $0 \leq \beta_n < 1$, and $\limsup_{n\to\infty}\beta_n < 1$. If $\{\lambda_n\}$ is chosen so that $\lambda_n \in [a,b]$ for some a,b with $0 < a < b < c^2\alpha/2$, then the sequence $\{x_n\}$ converges strongly to $\Pi_{VI(C,A)}x_0$, where $\frac{1}{c}$ is the $2-$uniformly convexity constant of E.

Proof. Taking $T = I$(the identity mapping) in Theorem 3.1, we have $\bar{C_n} = C$ and $C_n = \tilde{C_n}$. Then, it is easy to obtain the desired result. □

Remark 4. Corollary 3.3 generalize theorem 3.1 of [6] from weak convergence to strong convergence.

References

[1] Ya. I. Alber, Metric and generalized projection operators in Banach spaces: Properties and applications, in: A.G. Kartsatos (Ed.), Theory and Applications of Nonlinear Operators of Accretive and Monotone Type, in: Lecture Notes Pure Appl. Math., vol. 178, Dekker, New York, 1996, pp. 15–50.

[2] Ya. I. Alber and S. Reich, *An iterative method for solving a class of nonlinear operator equations in Banach spaces*, Panamer. Math. J. **4** (1994), 39–54.

[3] K. Ball, E. A. Carlen and E. H. Lieb, *Sharp uniform convexity and smoothness inequalities for trace norms*, Invent. Math. **115** (1994), 463–482.

[4] F. E. Browder and W. V. Petryshyn, *Construction of fixed points of nonlinear mappings in Hilbert space*, J. Math. Anal. Appl. **20** (1967), 197–228.

[5] H. Iiduka and W. Takahashi, *Strong convergence theorems for nonexpansive mappings and inverse-strongly monotone mappings*, Nonlinear Analysis **61** (2005), 341–350.

[6] H. Iiduka and W. Takahashi, *Weak convergence of a projection algorithm for variational inequalities in a Banach space*, J. Math. Anal. Appl. **339** (2008), 668–679.

[7] H. Iiduka, W. Takahashi and M. Toyoda, *Approximation of solutions of variational inequalities for monotone mappings*, Panamer. Math. J. **14** (2004), 49–61.

[8] J. L. Lions and G. Stampacchia, *Variational inequalities*, Comm. Pure Appl. Math. **20** (1967), 493–517.

[9] F. Liu and M. Z. Nashed, *Regularization of nonlinear ill-posed variational inequalities and convergence rates*, Set-Valued Anal. **6** (1998), 313–344.

[10] S. Matsushita and W. Takahashi, *Approximating fixed points of nonexpansive mappings in a Banach space by metric projections*, Applied Mathematics and Computation **196** (2008), 422–425.

[11] S. Matsushita and W. Takahashi, *A strong convergence theorem for relatively nonexpansive mappings in a Banach space*, Journal of Approximation Theory **134** (2005), 257–266.

[12] R. T. Rockafellar, *On the maximality of sums of nonlinear monotone operators*, Trans. Amer. Math. Soc. **149** (1970), 75–88.

[13] Y. Takahashi, K. Hashimoto and M. Kato, *On sharp uniform convexity, smoothness, and strong type, cotype inequalities*, J. Nonlinear Convex Anal. **3** (2002), 267–281.

Ying Liu
College of Mathematics and Computer
Hebei University
No. 180, Wusi eastRoad, Baoding, Hebei, China, 071002
E-mail address: ly_cyh2007@yahoo.com.cn