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# A PRIORI $L^2$-ERROR ESTIMATES OF THE CRANK-NICOLSON DISCONTINUOUS GALERKIN APPROXIMATIONS FOR NONLINEAR PARABOLIC EQUATIONS

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Abstract. In this paper, we analyze discontinuous Galerkin methods with penalty terms, namly symmetric interior penalty Galerkin methods, to solve nonlinear parabolic equations. We construct finite element spaces on which we develop fully discrete approximations using extrapolated Crank-Nicolson method. We adopt an appropriate elliptic-type projection, which leads to optimal $\ell^\infty(L^2)$ error estimates of discontinuous Galerkin approximations in both spatial direction and temporal direction.

## 1. Introduction

In this work we shall approximate the solution of nonlinear parabolic equations using a symmetric discontinuous Galerkin method with interior penalties for the spatial discretization and extrapolated Crank-Nicolson method for the time stepping. By implementing the extrapolated technique, we induce the linear systems which can be solved explicitly, thus obviate the order reduction phenomenon which occurs when the system involved is nonlinear.

Compared to the classical Galerkin method, the discontinuous Galerkin method is very well suited for adaptive control of error and can deliver high orders of accuracy when the exact solution is sufficiently smooth.

Discontinuous Galerkin methods with interior penalties for elliptic and parabolic equations were introduced by several authors [1, 4, 12]. They generalized Nitsche method in [5] to treat the Dirichlet boundary condition with penalty terms on the boundary of the domain. These methods referred to as interior penalty Galerkin schemes are not locally mass conservative.

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A new type of elementwise conservative discontinuous Galerkin method for diffusion problem was introduced and analyzed by Oden et al. [6]. Recently, Riviere and Wheeler [9] introduced a locally conservcative discontinuous Galerkin formulation for nonlinear parabolic equations and derived a priori $L^\infty(L^2)$ and $L^2(H^1)$ error estimates. However, the error estimate in the $L^\infty(L^2)$ norm is not optimal.

In [7], Ohm, Lee and Shin constructed semidiscrete discontinuous Galerkin approximations using interior pinalty terms and obtained the optimal $L^\infty(L^2)$ error estimate.

Rievière and Wheeler [10] construct semidiscrete approximations which converge optimally in $h$ and suboptimally in $r$ for the energy norm and suboptimally for the $L^2$ norm. They also constructed fully discrete approximations and proved the optimal convergence in the temporal direction. Sun and Wheeler in [11] analyzed three discontinuous Galerkin methods, namely, symmetric interior penalty Galerkin method, nonsymmetric interior penalty Galerkin method, and incomplete interior penalty Galerkin method to approximate the solution of reactive transport problems. They obtained error estimates in $L^2(H^1)$ which are optimal in $h$ and nearly optimal in $p$ and they developed a parabolic lift-technique for SIPG which leads to $h$-optimal and nearly $p$-optimal error estimates in $L^2(L^2)$ and negative norms.

This paper is organized as follows: In section 2, we introduce model problems and preliminaries. In section 3, we construct appropriate finite element spaces and define an auxiliary projection. In section 4, we construct the extrapolated fully discrete discontinuous Galerkin method and we prove the optimal convergence in both spacial and temporal directions in $L^2$ normed space.

## 2. Model problems and preliminaries

Consider the following nonlinear parabolic differential equation:

$$\begin{aligned} u_t - \nabla\cdot\{a(u)\nabla u\} &= f(u) \quad \text{in } \Omega\times(0,T] \\ (a(u)\nabla u)\cdot n &= 0 \quad \text{on } \partial\Omega\times(0,T] \\ u(x,0) &= u_0(x) \quad \text{on } \Omega \end{aligned} \tag{2.1}$$

where $\Omega$ is a bounded convex domain in $R^d$ with $d \geq 2$ and $n$ denotes the unit outward normal vector to $\partial\Omega$ and $u_0(x)$ is a given function defined on $\Omega$. The initial data $u_0(x)$, $f$, $a$ are assumed to be such that (1.1) admits a solution sufficiently smooth to guarantee the convergence results to be presented below.

Assume that the following conditions are satisfied.

1. For any bounded subset B of real numbers. there exist constants $\gamma$ and $\gamma^*$ such that

$$0 < \gamma \leq a(x,p) \leq \gamma^*, \quad 0 < \gamma \leq \frac{\partial}{\partial p}a(x,p) \leq \gamma^* \quad \text{for any } (x,p)\in\Omega\times B.$$

2. $a$ and $f$ are uniformly Lipschitz continuous with respect to their second variable.

3. The model problem has a unique solution satisfying the following regularity conditions:

$$u \in L^2([0,T], H^s(\Omega)), \quad u_t \in L^2([0,T], H^s(\Omega)), \quad \text{for} \quad s \geq 2;$$
$$u_t \in L^\infty([0,T], L^\infty(\Omega)), \quad \nabla u \in L^\infty(\Omega \times [0,T]).$$

Let $\mathcal{E}_h = \{E_1, E_2, \cdots, E_{N_h}\}$ be a subdivision of $\Omega$ where $E_i$ is a triangle or a quadrilateral if $d = 2$ and $E_i$ is a 3-simplex or 3-rectangle if $d = 3$. Let $h_j = \text{diam}(E_j)$ and $h = \max_{1 \leq j \leq N_h} h_j$. We assume $\mathcal{E}_h$ satisfies the following regularity requirement : there exists a constant $\rho > 0$ such that each $E_j$ contains a ball of radius $\rho h_j$. And also we assume that $\mathcal{E}_h$ satisfies the quasi-uniformity requirement: there is a constant $\gamma > 0$ such that

$$\frac{h}{h_j} \leq \gamma \quad \text{for} \quad j = 1, 2, \cdots, N_h.$$

We denote the edges (resp., faces for $d = 3$) of the elements by $\{e_1, e_2, \cdots, e_{P_h}, e_{P_h+1}, \cdots, e_{N_h}\}$ where $e_l$ has positive $d-1$ dimensional Lebesque measure, $e_l \subset \Omega$, $1 \leq l \leq P_h$, and $e_l \subset \partial\Omega$, $P_h + 1 \leq l \leq N_h$. With each edge (or face) $e_l$, we take $n_l$ a unit normal vector to $E_i$ if $e_l = \partial E_i \cap \partial E_j$ and $i < j$. For $l \geq P_h + 1$, $n_l$ is taken to be the unit outward vector normal to $\partial\Omega$.

For an $s \geq 0$ and a domain $E \subset \mathbb{R}^d$, we denote by $H^s(E)$ the Sobolev space of order $s$ equipped with the usual Sobolev norm $\|\cdot\|_{s,E}$. We simply write $\|\cdot\|_s$ instead of $\|\cdot\|_{s,\Omega}$ if $E = \Omega$ and $\|\cdot\|_E$ instead of $\|\cdot\|_{s,E}$ if $s = 0$. And also the usual seminorm defined on $H^s(E)$ is denoted by $|\cdot|_{s,E}$.

Now for an $s \geq 0$, we let

$$H^s(\mathcal{E}_h) = \{v \in L^2(\Omega) \mid v|_{E_i} \in H^s(E_i), \quad i = 1, 2, \cdots, N_h\}.$$

For $\phi \in H^s(\mathcal{E}_h)$ with $s > \frac{1}{2}$, we define the average function $\{\phi\}$ and the jump function $[\phi]$ such that

$$\{\phi\} = \frac{1}{2}(\phi|_{E_i})|_{e_l} + \frac{1}{2}(\phi|_{E_j})|_{e_l}, \quad \forall x \in e_l, \ 1 \leq l \leq P_h$$
$$[\phi] = (\phi|_{E_i})|_{e_l} - (\phi|_{E_j})|_{e_l}, \quad \forall x \in e_l, \ 1 \leq l \leq P_h$$

where $e_l = \partial E_i \cap \partial E_j$ with $i < j$.

We define the following broken norms on the space $H^s(\mathcal{E}_h)$

$$|||\phi|||^2 = \sum_{i=1}^{N_h} \|\phi\|_{E_i}^2$$
$$|||\phi|||_1^2 = \sum_{i=1}^{N_h} (\|\phi\|_{1,E_i}^2 + h_i^2 \|\nabla^2 \phi\|_{E_i}^2) + J_\beta^\sigma(\phi, \phi)$$

where

$$J^{\sigma}_{\beta}(\phi,\psi) = \sum_{l=1}^{P_h} \frac{\sigma_l}{|e_l|^{\beta}} \int_{e_l} [\phi][\psi] ds, \quad \beta > 0$$

is an interior penalty term and $\sigma$ is a discrete positive function that takes the constant value $\sigma_l$ on the edge $e_l$ and is bounded below by $\sigma_0 > 0$ and above by $\sigma^* > 0$.

## 3. Finite element spaces and auxiliary projection

For a positive integer $r$, we define the following finite element spaces

$$D_r(\mathcal{E}_h) = \{v \in L^2(\Omega) \mid v|_{E_i} \in P_r(E_i), \quad i = 1, 2, \cdots, N_h\}$$

where $P_r(E_i)$ denotes the set of polynomials of degree less than or equal to $r$ on $E_i$.

We use the following $hp$-approximation results and trace inequality results whose proofs can be found in [2, 3].

**Theorem 3.1.** *If $E_j \in \mathcal{E}_h$ and $\phi \in H^s(E_j)$ then there exist a positive constant $C$ depending on $s$, $\gamma$, and $\rho$ but independent of $\phi$, $r$ and $h$ and a sequence $z^h_r \in P_r(E_j)$, $r = 1,\ 2, \cdots$ such that for any $0 \le q \le s$,*

$$\|\phi - z^h_r\|_{q,E_j} \le C \frac{h_j^{\mu-q}}{r^{s-q}} \|\phi\|_{s,E_j} \quad s \ge 0,$$

$$\|\phi - z^h_r\|_{0,e_j} \le C \frac{h_j^{\mu-\frac{1}{2}}}{r^{s-\frac{1}{2}}} \|\phi\|_{s,E_j} \quad s > \frac{1}{2},$$

$$\|\phi - z^h_r\|_{1,e_j} \le C \frac{h_j^{\mu-\frac{3}{2}}}{r^{s-\frac{3}{2}}} \|\phi\|_{s,E_j} \quad s > \frac{3}{2}$$

*where $\mu = \min(r+1, s)$ and $e_j$ denotes the diamater of $E_j$.*

**Theorem 3.2.** *For each $E_j \in \mathcal{E}_h$, there exists a positive constant $C$ depending only on $\gamma$ and $\rho$ such that*

$$\|\phi\|^2_{0,e_j} \le C\left(\frac{1}{h_j}|\phi|^2_{0,E_j} + h_j|\phi|^2_{1,E_j}\right), \quad \forall \phi \in H^1(E_j)$$

$$\left\|\frac{\partial \phi}{\partial n_j}\right\|^2_{0,e_j} \le C\left(\frac{1}{h_j}|\phi|^2_{1,E_j} + h_j|\phi|^2_{2,E_j}\right), \quad \forall \phi \in H^2(E_j)$$

*where $e_j$ is an edge or a face of $E_j$ and $n_j$ is the unit outward normal vector to $E_j$.*

Now we introduce the following bilinear mapping $A(\rho;\cdot,\cdot)$ defined on $H^s(\mathcal{E}_h)\times H^s(\mathcal{E}_h)$

$$A(\rho;\phi,\psi)=(a(\rho)\nabla\phi,\nabla\psi)-\sum_{l=1}^{P_h}\int_{e_l}\{a(\rho)\nabla\phi\cdot n_l\}[\psi]-\sum_{l=1}^{P_h}\int_{e_k l}\{a(\rho)\nabla\psi\cdot n_l\}[\phi]$$
$$+J_\beta^\sigma(\phi,\psi).$$

Using the bilinear mapping $A$ and (2.1), we construct the weak formulation as follows:

Find $u(\cdot,t)\in H^s(\mathcal{E}_h)$ such that

$$(u_t(t),v)+A(u(t);u(t),v)=(f(u(t)),v),\quad \forall\, v\in H^s(\mathcal{E}_h). \tag{3.1}$$

Now for a $\lambda>0$ we define the following bilinear form $A_\lambda(\rho;\cdot,\cdot)$ on $H^s(\mathcal{E}_h)\times H^s(\mathcal{E}_h)$ such that

$$A_\lambda(\rho;\phi,\psi)=A(\rho;\phi,\psi)+\lambda(\phi,\psi)$$

$A_\lambda$ satisfies the following boundedness and coercivity properties. The proofs can be found in [7, 8].

**Lemma 3.1.** *For a $\lambda>0$, there exists a constant $C>0$ satisfying*

$$|A_\lambda(\rho;\phi,\psi)|\le C|\!|\!|\phi|\!|\!|_1|\!|\!|\psi|\!|\!|_1\qquad\forall\phi,\psi\in H^s(\mathcal{E}_h).$$

**Lemma 3.2.** *For a $\lambda>0$, there exists a constant $\underset{\sim}{c}>0$ satisfying*

$$A_\lambda(\rho;\phi,\phi)\ge\underset{\sim}{c}|\!|\!|\phi|\!|\!|_1^2\qquad\forall\phi\in D_r(\mathcal{E}_h).$$

Now we define an elliptic projection initiated by Wheeler [13] to prove the optimal $L^2$ error estimates for Galerkin approximation to parabolic differential equations. We construct a projection $\widetilde{u}(t):[0,T]\to D_r(\mathcal{E}_h)$ such that

$$\begin{aligned}&A_\lambda(u;u-\widetilde{u},v)=0\qquad\forall v\in D_r(\mathcal{E}_h)\\&(\widetilde{u}(0),v)=(u(0),v).\end{aligned}\tag{3.2}$$

By Lemma 3.1 and Lemma 3.2, $\widetilde{u}(t)$ is well-defined.

## 4. The optimal $\ell^\infty(L^2)$ error estimates of fully discrete approximations

In this section by adopting the extrapolated Crank-Nicolson method we construct fully discrete discontinuous Galerkin approximations and prove the optimal convergence in $L^2$ normed space.

For a positive integer $N>0$ we let $k=\dfrac{T}{N}$ and for $0\le j\le N$ and we define $t_j=jk$ and $g_j=g(x,t_j)$. For $0\le j\le N-1$, we define $\Delta_t g_j=\dfrac{g_{j+1}-g_j}{k}$, $t_{j+\frac12}=\dfrac12(t_j+t_{j+1})$ and $g_{j+\frac12}=\dfrac12(g(t_j)+g(t_{j+1}))$.

Now we define fully discrete discontinuous Galerkin approximation $\{U_j\}_{j=0}^N \subset D_r(\mathcal{E}_h)$ as follows,

$$(\Delta_t U_j, v) + A(EU_j : U_{j+\frac{1}{2}}, v) = (f(EU_j), v), \quad \forall v \in D_r(\mathcal{E}_h) \tag{4.1}$$

where $EU_j = \dfrac{3}{2}U_j - \dfrac{1}{2}U_{j-1}$, $U_{j+\frac{1}{2}} = \dfrac{1}{2}(U_j + U_{j+1})$.

To apply (4.1), we need two intial stages $U_0$ and $U_1$ to be defined in the following

$$\begin{cases} (\Delta_t U_0, v) + A(U_{\frac{1}{2}}; U_{\frac{1}{2}}, v) = (f(U_{\frac{1}{2}}), v) \\ U_0 = \widetilde{u}(0) \end{cases}$$

where $U_{\frac{1}{2}} = \dfrac{1}{2}(U_0 + U_1)$.

To prove the optimal convergence of $u(t_j) - U_j$ in $L^2$ normed space we denote $\eta(x,t) = u(x,t) - \widetilde{u}(x,t)$ and $\xi(x,t_j) = \widetilde{u}(x,t_j) - U_j(x)$, $j = 0, 1, \cdots, N$.

Now we state the following approximations for $\eta$ whose proofs can be found in [7, 8].

**Theorem 4.1.** *If $u_t \in L^2(H^s)$ and $u_0 \in H^s$ then there exists a constant $C$ independent of $h$ and $k$ satisfying*

(i) $\|\eta_t\| + h\|\eta_t\|_1 \le Ch^s(\|u_t\|_{H^s} + \|u_0\|_s)$

(ii) $\|\eta\| + h\|\eta\|_1 \le Ch^s(\|u_t\|_{L^2(H^s)} + \|u_0\|_s)$.

**Theorem 4.2.** *If $u_t \in L^2(H^s)$, $u_{tt} \in L^\infty(H^s)$, $u_{ttt} \in L^\infty(H^s)$ and $u_0 \in H^s$ then there exists a constant $C$ independent of $h$ and $k$ satisfying*

(i) $\|\eta_{tt}\|_1 \le Ch^{s-1}\{\|u_t\|_{L^2(H^s)} + \|u_{tt}\|_s + \|u_0\|_s\}$

(ii) $\|\eta_{ttt}\|_1 \le Ch^{s-1}\{\|u_t\|_{L^2(H^s)} + \|u_{tt}\|_s + \|u_{ttt}\|_s + \|u_0\|_s\}$

*provided that* $\beta \ge \dfrac{1}{d-1}$.

By simple computations and the applications of Theorem 4.2 we obtain the following lemmas.

**Lemma 4.1.** *If $u_{tt} \in L^\infty(H^s)$, $u_{ttt} \in L^\infty(H^s)$ and $\rho$ satisfies*

$$\Delta_t \widetilde{u}_j - \widetilde{u}_t(t_{j+\frac{1}{2}}) = k\rho_{j+\frac{1}{2}}$$

*then there exists a constant $C$ independent of $h$ and $k$ such that*

$$\|\rho_{j+\frac{1}{2}}\| \le Ck(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{ttt}\|_{L^\infty(H^s)})$$
$$\|\rho_{j+\frac{1}{2}}\|_1 \le Ck(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{ttt}\|_{L^\infty(H^s)}).$$

*Consequently from Lemma 4.1 we conclude that there exists a constant $C$ independent of $k$ and $h$ such that*

$$\|\rho_{j+\frac{1}{2}}\| \le Ck \quad \text{and} \quad \|\rho_{j+\frac{1}{2}}\|_1 \le Ck$$

*if $u$ is sufficiently smooth.*

**Lemma 4.2.** *If we let* $u_{tt} \in L^\infty(H^s)$ *and* $r_{j+\frac{1}{2}} = \widetilde{u}(t_{j+\frac{1}{2}}) - \widetilde{u}_{j+\frac{1}{2}}$ *then there exists a constant* $C$ *independent of* $h$ *and* $k$ *such that*

$$\|r_{j+\frac{1}{2}}\| \le Ck^2(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)})$$
$$\|r_{j+\frac{1}{2}}\|_1 \le Ck^2(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)}).$$

*Consequently from Lemma* 4.2 *we conclude that there exists a constant* $C$ *independent of* $k$ *and* $h$ *such that*

$$\|r_{j+\frac{1}{2}}\| \le Ck^2, \quad \|r_{j+\frac{1}{2}}\|_1 \le Ck^2,$$

*if* $u$ *is sufficiently smooth.*

**Lemma 4.3.** *If we let* $u_{tt} \in L^\infty(H^s)$ *and* $\varphi_{j+\frac{1}{2}} = \widetilde{u}(t_{j+\frac{1}{2}}) - (\frac{3}{2}\widetilde{u}(t_j) - \frac{1}{2}\widetilde{u}(t_{j-1}))$ *then there exists a constant* $C$ *independent of* $h$ *and* $k$ *such that*

$$\|\varphi_{j+\frac{1}{2}}\| \le Ck^2(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)})$$
$$\|\varphi_{j+\frac{1}{2}}\|_1 \le Ck^2(\|u_0\|_s + \|u_t\|_{L^2(H^s)} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{tt}\|_{L^\infty(H^2)}).$$

*Consequently from Lemma* 4.3*, we induce that there exists a constant* $C$ *independent of* $k$ *and* $h$ *such that*

$$\|\varphi_{j+\frac{1}{2}}\| \le Ck^2, \quad \|\varphi_{j+\frac{1}{2}}\|_1 \le Ck^2$$

*if* $u$ *is sufficiently smooth.*

**Theorem 4.3.** *For* $0 < \lambda < 1$ *and* $\delta > 0$*, if* $u_t \in L^\infty(H^s)$*,* $u_{tt} \in L^\infty(H^s)$ *and* $u_{ttt} \in L^\infty(H^s)$ *then there exists a constant* $C > 0$ *independent of* $h$ *and* $k$ *such that for* $j = 1, 2, \cdots, N$

$$\|u(t_j) - U_j\| \le C(h^\mu + k^2)(\|u_0\|_s + \|u_t\|_{L^\infty(H^s)} + \|\nabla u_t\|_{L^\infty} + \|u_{tt}\|_{L^\infty(H^s)} + \|u_{ttt}\|_{L^\infty(H^s)})$$

*hold where* $s = \dfrac{d}{2} + 1 + \delta$ *and* $\mu = \min(r+1, s)$*.*

*Proof.* Applying (4.1) and (2.1), we get

$$\begin{aligned}&(u_t(t_{j+\frac{1}{2}}) - \Delta_t U_j, v) + A_\lambda(u(t_{j+\frac{1}{2}}); u(t_{j+\frac{1}{2}}), v) - A_\lambda(EU_j; U_{j+\frac{1}{2}}, v)\\ &= (f(u(t_{j+\frac{1}{2}})) - f(EU_j), v) + \lambda(u(t_{j+\frac{1}{2}}) - U_{j+\frac{1}{2}}, v).\end{aligned} \tag{4.2}$$

By the notations of $\eta$ and $\xi$, we obtain

$$\begin{aligned}u_t(t_{j+\frac{1}{2}}) - \Delta_t U_j &= u_t(t_{j+\frac{1}{2}}) - \Delta_t\widetilde{u}_j + \Delta_t\widetilde{u}_j - \Delta_t U_j\\ &= \eta_t(t_{j+\frac{1}{2}}) + k\rho_{j+\frac{1}{2}} + \Delta_t\xi_j.\end{aligned} \tag{4.3}$$

By applying the definition of $\eta$, we have

$$\begin{aligned}
&A_\lambda(u(t_{j+\frac{1}{2}});u(t_{j+\frac{1}{2}}),v) - A_\lambda(EU_j;U_{j+\frac{1}{2}},v)\\
&= A_\lambda(EU_j;\xi_{j+\frac{1}{2}},v) - A_\lambda(EU_j;\widetilde{u}_{j+\frac{1}{2}},v) + A_\lambda(u(t_{j+\frac{1}{2}});u(t_{j+\frac{1}{2}}),v)\\
&= A_\lambda(EU_j;\xi_{j+\frac{1}{2}},v) + A_\lambda(u(t_{j+\frac{1}{2}});\eta(t_{j+\frac{1}{2}}),v)\\
&\quad + A_\lambda(u(t_{j+\frac{1}{2}});\widetilde{u}(t_{j+\frac{1}{2}}) - \widetilde{u}_{j+\frac{1}{2}},v) + A_\lambda(u(t_{j+\frac{1}{2}});\widetilde{u}_{j+\frac{1}{2}},v)\\
&\quad - A_\lambda(EU_j;\widetilde{u}_{j+\frac{1}{2}},v).
\end{aligned} \tag{4.4}$$

Substituting (4.3) and (4.4) in (4.2) and choosing $v = \xi_{j+\frac{1}{2}}$ implies

$$\begin{aligned}
&(\Delta_t\xi_j,\xi_{j+\frac{1}{2}}) + A_\lambda(EU_j;\xi_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}})\\
&= -(\eta_t(t_{j+\frac{1}{2}}) + k\rho_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}}) - A_\lambda(u(t_{j+\frac{1}{2}});\eta(t_{j+\frac{1}{2}}),\xi_{j+\frac{1}{2}})\\
&\quad - A_\lambda(u(t_{j+\frac{1}{2}});r_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}}) - A_\lambda(u(t_{j+\frac{1}{2}});\widetilde{u}_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}})\\
&\quad + A_\lambda(EU_j;\widetilde{u}_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}}) + (f(u(t_{j+\frac{1}{2}})) - f(EU_j),\xi_{j+\frac{1}{2}})\\
&\quad + \lambda(u(t_{j+\frac{1}{2}}) - U_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}}).
\end{aligned} \tag{4.5}$$

Applying the Caucly-Schwarz's inequality, we have

$$(\Delta_t\xi_j,\xi_{j+\frac{1}{2}}) \geq \frac{1}{2k}(\|\xi_{j+1}\|^2 - \|\xi_j\|^2).$$

From (4.5) we obtain,

$$\begin{aligned}
&\frac{1}{2k}\Big[\|\xi_{j+1}\|^2 - \|\xi_j\|^2 + \underset{\sim}{c}\|\xi_{j+\frac{1}{2}}\|_1^2\\
&\leq -(\eta_t(t_{j+\frac{1}{2}}) + k\rho_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}}) + (f(u(t_{j+\frac{1}{2}})) - f(EU_j),\xi_{j+\frac{1}{2}})\\
&\quad + \lambda(u(t_{j+\frac{1}{2}}) - U_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}}) - A_\lambda(u(t_{j+\frac{1}{2}});r_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}})\\
&\quad - (A_\lambda(u(t_{j+\frac{1}{2}});\widetilde{u}_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}}) - A_\lambda(EU_j;\widetilde{u}_{j+\frac{1}{2}},\xi_{j+\frac{1}{2}}))\\
&= \sum_{i=1}^{5} L_i.
\end{aligned} \tag{4.6}$$

For a sufficiently small $\varepsilon > 0$ by applying Lemma 4.1 there exists a constant $C > 0$ such that

$$\begin{aligned}
|L_1| &\leq (\|\eta_t(t_{j+\frac{1}{2}})\| + \|k\rho_{j+\frac{1}{2}}\|)\|\xi_{j+\frac{1}{2}}\|\\
&\leq C(\|\eta_t(t_{j+\frac{1}{2}})\|^2 + k^2\|\rho_{j+\frac{1}{2}}\|^2 + \|\xi_{j+1}\|^2 + \|\xi_j\|^2)\\
&\leq C(\|\eta_t(t_{j+\frac{1}{2}})\|^2 + k^4 + \|\xi_{j+1}\|^2 + \|\xi_j\|^2).
\end{aligned}$$

Applying Lemmas 4.1 and 4.2, $L_2$ can be estimated as follows;

$$\begin{aligned}
|L_2| &\leq C\|u(t_{j+\frac{1}{2}}) - EU_j\|\|\xi_{j+\frac{1}{2}}\|\\
&\leq C(\|\eta(t_{j+\frac{1}{2}})\|^2 + k^4 + \|\xi_j\|^2 + \|\xi_{j+1}\|^2 + \|\xi_{j-1}\|^2).
\end{aligned}$$

We obtain the following estimates of $L_3$ and $L_4$.

$$\begin{aligned}|L_3| &\le \lambda(\|\eta(t_{j+\frac{1}{2}})\| + \|r_{j+\frac{1}{2}}\| + \|\xi_{j+\frac{1}{2}}\|)\|\xi_{j+\frac{1}{2}}\| \\ &\le C(\|\eta(t_{j+\frac{1}{2}})\|^2 + k^4 + \|\xi_j\|^2 + \|\xi_{j+1}\|^2)\end{aligned}$$

and

$$|L_4| \le C\|r_{j+\frac{1}{2}}\|_1\|\xi_{j+\frac{1}{2}}\|_1 \le Ck^4 + \varepsilon\|\xi_{j+\frac{1}{2}}\|_1^2.$$

From the definition of $L_5$, we can separate $L_5$ as follows

$$\begin{aligned}L_5 &= ((a(EU_j) - a(u(t_{j+\frac{1}{2}})))\nabla\widetilde{u}_{j+\frac{1}{2}}, \nabla\xi_{j+\frac{1}{2}}) \\ &\quad - \sum_{l=1}^{P_h}\int_{e_l}\{a(EU_j) - a(u(t_{j+\frac{1}{2}}))\nabla\widetilde{u}_{j+\frac{1}{2}}\cdot n_l\}[\xi_{j+\frac{1}{2}}] \\ &\quad - \sum_{l=1}^{P_h}\int_{e_l}\{a(EU_j) - a(u(t_{j+\frac{1}{2}}))\nabla\xi_{j+\frac{1}{2}}\cdot n_l\}[\widetilde{u}_{j+\frac{1}{2}}] \\ &= \sum_{i=1}^{3} L_{5i}.\end{aligned}$$

By applying Lemma 4.3, $L_{51}$ can be estimated in the following way

$$\begin{aligned}L_{51} &\le C\|\nabla\widetilde{u}_{j+\frac{1}{2}}\|_\infty(\|\eta(t_{j+\frac{1}{2}})\| + k^2 + \|\xi_{j-1}\| + \|\xi_j\|)\|\nabla\xi_{j+\frac{1}{2}}\| \\ &\le C(\|\eta(t_{j+\frac{1}{2}})\|^2 + k^4 + \|\xi_{j-1}\|^2 + \|\xi_j\|^2) + \varepsilon\|\xi_{j+\frac{1}{2}}\|_1^2.\end{aligned}$$

Similarly there exists a constant $C > 0$ such that

$$\begin{aligned}L_{52} &\le C\sum_{l=1}^{P_h}\|\nabla\widetilde{u}_{j+\frac{1}{2}}\|_{\infty,e_l}(\|\eta(t_{j+\frac{1}{2}})\|_{0,e_l} + \|\varphi_{j+\frac{1}{2}}\|_{0,e_l} + \|\xi_j\|_{0,e_l} + \|\xi_{j-1}\|_{0,e_l}) \\ &\qquad \|[\xi_{j+\frac{1}{2}}]\|_{0,e_l} \\ &\le C\sum_{i=1}^{N_h}\|\nabla\widetilde{u}_{j+\frac{1}{2}}\|_{\infty,E_i}(h^{-1/2}\|\eta(t_{j+\frac{1}{2}})\|_{0,E_i} + h^{1/2}\|\nabla\eta(t_{j+\frac{1}{2}})\|_{0,E_i} \\ &\qquad + h^{-1/2}\|\varphi_{j+\frac{1}{2}}\|_{0,E_i} + h^{-1/2}\|\xi_j\|_{0,E_i} + h^{-1/2}\|\xi_{j-1}\|_{0,E_i})\|\xi_{j+\frac{1}{2}}\|_1 h^{(d-1)/2} \\ &\le C(\|\eta(t_{j+\frac{1}{2}})\|^2 + h^2\|\nabla\eta(t_{j+\frac{1}{2}})\|^2 + k^4 + \|\xi_j\|^2 + \|\xi_{j-1}\|^2) + \varepsilon\|\xi_{j+\frac{1}{2}}\|_1^2.\end{aligned}$$

By applying the trace inequality we have

$$\begin{aligned}L_{53} &\le C\sum_{l=1}^{P_h}\|\nabla(\xi_{j+\frac{1}{2}})\|_{\infty,e_l}\big(\|\eta(t_{j+\frac{1}{2}})\|_{0,e_l} + \|\varphi_{j+\frac{1}{2}}\|_{0,e_l} + \|\xi_j\|_{0,e_l} + \|\xi_{j-1}\|_{0,e_l}\big) \\ &\qquad \|[\eta_{j+\frac{1}{2}}]\|_{0,e_l}\end{aligned}$$

$$\leq C\sum_{i=1}^{N_h} h^{-1/2}\|\nabla(\xi_{j+\frac{1}{2}})\|_{\infty,E_i}\big[\|\eta(t_{j+\frac{1}{2}})\|_{0,E_i}+h\|\nabla\eta(t_{j+\frac{1}{2}})\|_{0,E_i}$$

$$+\|\varphi_{j+\frac{1}{2}}\|_{0,E_i}+\|\xi_j\|_{0,E_i}+\|\xi_{j-1}\|_{0,E_i}\big]h^{-1/2}\big(\|\eta_{j+\frac{1}{2}}\|_{0,E_i}+h\|\nabla\eta_{j+\frac{1}{2}}\|_{0,E_i}\big)$$

$$\leq C\sum_{i=1}^{N_h}\|\nabla(\xi_{j+\frac{1}{2}})\|_{0,E_i}\big(\|\eta(t_{j+\frac{1}{2}})\|_{0,E_i}+h\|\nabla\eta(t_{j+\frac{1}{2}})\|_{0,E_i}$$

$$+\|\varphi_{j+\frac{1}{2}}\|_{0,E_i}+\|\xi_j\|_{0,E_i}+\|\xi_{j-1}\|_{0,E_i}\big)h^{-1-\frac{d}{2}+s}$$

$$\leq C\big(\|\eta(t_{j+\frac{1}{2}})\|^2+h^2\|\nabla\eta(t_{j+\frac{1}{2}})\|^2+k^4+\|\xi_j\|^2+\|\xi_{j-1}\|^2\big)+\varepsilon\|\xi_{j+\frac{1}{2}}\|_1^2.$$

By combining $L_{5i}$, $1\leq i\leq 3$, we have

$$|L_5|\leq C\big(\|\eta(t_{j+\frac{1}{2}})\|^2+h^2\|\nabla\eta(t_{j+\frac{1}{2}})\|^2+\|\xi_j\|^2+\|\xi_{j-1}\|^2+k^4)$$
$$+3\varepsilon\|\xi_{j+\frac{1}{2}}\|_1^2.$$

Substituting the estimations of $L_i$, $1\leq i\leq 5$ into (4.6), we get

$$\frac{1}{2k}\Big((\|\xi_{j+1}\|^2-\|\xi_j\|^2)\Big)+\|\xi_{j+\frac{1}{2}}\|_1^2$$
$$\leq C\Big(\|\eta_t(t_{j+\frac{1}{2}})\|^2+k^4+\|\xi_{j+1}\|^2+\|\xi_j\|^2+\|\eta(t_{j+\frac{1}{2}})\|^2+\|\xi_{j-1}\|^2 \tag{4.7}$$
$$+h^2\|\nabla\eta(t_{j+\frac{1}{2}})\|^2\Big).$$

If we sum both sides of (4.7) from $j=0$ to $N-1$, then we obtain

$$\|\xi_N\|^2-\|\xi_0\|^2+2k\sum_{j=0}^{N-1}\|\xi_{j+\frac{1}{2}}\|_1^2$$
$$\leq C\Bigg(k\sum_{j=0}^{N-1}\big(\|\eta(t_{j+\frac{1}{2}})\|^2+\|\eta_t(t_{j+\frac{1}{2}})\|^2+h^2\|\nabla\eta(t_{j+\frac{1}{2}})\|^2+k^4\big)$$
$$+k\sum_{j=0}^{N}\|\xi_j\|^2\Bigg)$$

which implies

$$\|\xi_N\|^2+2k\sum_{j=0}^{N-1}\|\xi_{j+\frac{1}{2}}\|_1^2$$
$$\leq\|\xi_0\|^2+Ck\sum_{j=0}^{N-1}\big[\|\eta(t_{j+\frac{1}{2}})\|^2+\|\eta_t(t_{j+\frac{1}{2}})\|^2+h^2\|\nabla\eta(t_{j+\frac{1}{2}})\|^2+k^4\big]$$
$$+Ck\sum_{j=0}^{N}\|\xi_j\|^2$$

where $k$ is sufficiently small. By applying the discrete version of Gronwall's inequality, we have

$$
\begin{aligned}
&\|\!|\xi_N\|\!|^2 + k\sum_{j=0}^{N-1}\|\!|\xi_{j+\frac{1}{2}}\|\!|_1^2 \\
&\leq C\Big\{\|\!|\xi_0\|\!|^2 + k\sum_{j=0}^{N-1}\big(\|\!|\eta(t_{j+\frac{1}{2}})\|\!|^2 + \|\!|\eta_t(t_{j+\frac{1}{2}})\|\!|^2 + h^2\|\!|\nabla\eta(t_{j+\frac{1}{2}})\|\!|^2 + k^4\big)\Big\}.
\end{aligned}
$$

Therefore by applying the result of the following Lemma 4.1 we have

$$
\begin{aligned}
\|\!|\xi\|\!|_{\ell^\infty(L^2)} &\leq C(h^s + k^2),\\
\|\!|e\|\!|_{\ell^\infty(L^2)} &\leq C(h^s + k^2),
\end{aligned}
$$

which proves the optimal $\ell^\infty(L^2)$ error estimation of the fully discrete solutions. □

The following Lemma 4.1 can be proved by the similar process of Theorem 4.3.

**Lemma 4.4.** *For* $0 < \lambda < 1$ *and* $\delta > 0$*, if* $u_t \in L^\infty(H^{\frac{d}{2}+1+\delta})$*,* $u_{tt} \in L^\infty(H^{\frac{d}{2}+1})$ *and* $h^{-\frac{d}{2}}k \leq C_0$ *for some constant* $C_0$ *then there exists a constant* $C > 0$ *independent of* $h$ *and* $k$

$$
\begin{aligned}
\|\!|\xi_1\|\!|_{L^2} &\leq C(h^s + k^2),\\
\|\!|e_1\|\!|_{L^2} &\leq C(h^s + k^2).
\end{aligned}
$$

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