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STRONG CONVERGENCE THEOREM OF COMMON ELEMENTS FOR EQUILIBRIUM PROBLEMS AND FIXED POINT PROBLEMS

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Abstract. In this paper, we introduce an iterative method for finding a common element of the set of solutions of an equilibrium problem, the set of common fixed points of an asymptotically strictly pseudocontractive mapping in a Hilbert space. We show that the iterative sequence converges strongly to a common element of the two sets.

1. Introduction

Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let f be a bifunction of $C \times C$ into R, where R is the set of real numbers. The equilibrium problem for $f : C \times C \to R$ is to find $x \in C$ such that

$$f(x, y) \geq 0, \forall y \in C. \tag{1.1}$$

The set of solutions of (1.1) is denoted by $EP(f)$. Given a mapping $T : C \to H$, let $f(x, y) = \langle Tx, y - x\rangle$ for all $x, y \in C$. Then, $\hat{x} \in EP(f)$ if and only if $\langle T\hat{x}, y - \hat{x}\rangle \geq 0$ for all $y \in C$, i.e., $\hat{x}$ is a solution of the variational inequality. Numerous problems in physics, optimization, and economics reduce to find a solution of (1.1). Some methods have been proposed to solve the equilibrium problem; see [1].

A mapping $T : C \to C$ is said to be asymptotically λ-strictly pseudocontractive if there exist $\lambda \in [0, 1)$ and a sequence $\{k_n\}$ with $k_n \geq 1$ for all n and $\lim_{n\to\infty} k_n = 1$ and such that $\|T^n x-T^n y\|^2 \leq k_n\|x-y\|^2+\lambda\|(I-T^n)x-(I-T^n)y\|^2$ for all $n \geq 1$ and $x, y \in C$. This class of mappings has been studied by several authors, and it includes the important class of asymptotically nonexpansive maps ($\lambda = 0$). It is well known that if T is asymptotically strictly pseudocontractive, then T is uniformly L-Lipschitzian, i.e., $\|T^n x - T^n y\| \leq L\|x - y\|$,

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see [2]. A point $x \in C$ is a fixed point of T provided $Tx = x$. Denoted by $F(T)$ the set of fixed points of T.

Recently, many authors studied the problem of finding a common element of the set of fixed points of a nonexpansive mapping and the set of solutions of an equilibrium problem; for instance [3]. Inspired and motivated by these facts, we prove strong convergence theorems for finding a common element of the set of solutions of an equilibrium problem and the set of fixed points of an asymptotically strictly pseudocontractive mapping.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\|\cdot\|$. $\{x_n\}$ is a sequence in H, $x_n \rightharpoonup x$ implies that $\{x_n\}$ converges weakly to x and $x_n \to x$ means the strong convergence. In a real Hilbert space H, we have

$$\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2, \tag{1.2}$$

for all $x, y \in H$ and $\lambda \in [0,1]$. Let C be a nonempty closed convex subset of H. Then, for any $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \forall y \in C.$$

Such a P_C is called the metric projection of H onto C. We know that P_C is nonexpansive. Further, for $x \in H$ and $z \in C$, $z = P_C x \Leftrightarrow \langle x - z, z - y \rangle \geq 0$ for all $y \in C$. We also know that for any sequence $\{x_n\} \subset H$ with $x_n \rightharpoonup x$, the inequality

$$\liminf_{n\to\infty} \|x_n - x\| < \liminf_{n\to\infty} \|x_n - y\|$$

holds for every $y \in H$ with $x \neq y$;

For solving the equilibrium problem for a bifunction $f : C \times C \to R$, we assume that f satisfies the following conditions:

(A1) $f(x,x) = 0$ for all $x \in C$;

(A2) f is monotone, i.e., $f(x,y) + f(y,x) \leq 0$ for all $x, y \in C$;

(A3) for each $x, y, z \in C$,

$$\lim_{t\downarrow 0} f(tz + (1-t)x, y) \leq f(x,y);$$

(A4) for each $x \in C$, $y \mapsto f(x,y)$ is convex and lower semicontinuous.

Lemma 2.1. ([4]) *Let C be a nonempty closed subset of H and f be a bifunction of $C \times C$ into R satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$f(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall y \in C.$$

Lemma 2.2. ([5]) *Assume that $f : C \times C \to R$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $\Phi_r : H \to C$ as follows:*

$$\Phi_r(x) = \{z \in C : f(z,y) + \frac{1}{r}\langle y - z, z - x \rangle \geq 0, \forall\ y \in C\}$$

for all $x \in H$. Then the following hold:

(1) Φ_r is single-valued;

(2) Φ_r is firmly nonexpansive, i.e., for any $x, y \in H$,

$$\|\Phi_r(x) - \Phi_r(y)\|^2 \le \langle \Phi_r(x) - \Phi_r(y), x - y\rangle;$$

(3) $F(\Phi_r) = EP(f)$;

(4) $EP(f)$ is closed and convex.

Lemma 2.3. ([6]) *Let E be a real q-uniformly smooth Banach space which is also uniformly convex. Let C be a nonempty closed convex subset of E and $T : C \to C$ an asymptotically k-strictly pseudocontractive mapping with a nonempty fixed point set. Then $(I - T)$ is demiclosed at zero.*

Lemma 2.4. ([7]) *Let H be a real Hilbert space. Given a closed convex subset $C \subset H$ and points $x, y, z \in H$. Given also a real number $a \in R$. The set $D := \{v \in C : \|y - v\|^2 \le \|x - v\|^2 + \langle z, v\rangle + a\}$ is convex and closed.*

3. Main result

Theorem 3.1. *Let C be a bounded closed convex subset of a real Hilbert space H. Let $T : C \to C$ is an asymptotically λ-strict pseudocontraction mapping. Let f be a bifunction from $C \times C$ into R satisfying (A1)-(A4). Assume that $\{\alpha_n\}$ is a sequence in $(0, 1)$ satisfying the condition: $0 < a + \lambda \le \alpha_n \le 1 - b, \forall n \ge 0$ and for some $a, b \in (0, 1)$, $\{r_n\} \subset [m, \infty)$ for some $m > 0$. If $F := F(T) \cap EP(f) \neq \emptyset$, then the sequence $\{x_n\}$ generated by*

$$\begin{cases} x_0 \in C, \\ y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ u_n \in C \text{ such that } f(u_n, y) + \dfrac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \forall y \in C, \\ C_{n+1} = \{z \in C_n : \|u_n - z\|^2 \le \|x_n - z\|^2 - (1 - \alpha_n)(\alpha_n - \lambda)\|x_n - T^n x_n\|^2 \\ \qquad\qquad + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}} x_0, \end{cases} \tag{3.1}$$

where $\theta_n = (1 - \alpha_n)(k_n - 1)(diamC)^2 \to 0$ as $n \to \infty$, converges in norm to $P_F x_0$.

Proof. Firstly, We observe that C_n is convex by Lemma 2.4.
Next observe that $F \subset C_n$ for all n. Indeed, for all $p \in F$, we have

$$\begin{aligned} \|u_n - p\|^2 &= \|p - \Phi_{r_n} y_n\|^2 \\ &\le \|y_n - p\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)\|T^n x_n - p\|^2 - \alpha_n(1 - \alpha_n)\|x_n - T^n x_n\|^2 \\ &\le \alpha_n\|x_n - p\|^2 + (1 - \alpha_n)k_n\|x_n - p\|^2 + (1 - \alpha_n)\lambda\|x_n - T^n x_n\|^2 \end{aligned}$$

$$
\begin{aligned}
&- \alpha_n(1-\alpha_n)\|x_n - T^n x_n\|^2 \\
&\le [1+(1-\alpha_n)(k_n-1)]\|x_n - p\|^2 - (1-\alpha_n)(\alpha_n - \lambda)\|x_n - T^n x_n\|^2 \\
&\le \|x_n - p\|^2 - (1-\alpha_n)(\alpha_n-\lambda)\|x_n - T^n x_n\|^2 + \theta_n.
\end{aligned}
$$

So $p \in C_{k+1}$. This implies that $F \subset C_n$ for all n.

From $x_n = P_{C_n}x_0$, we have $\langle x_0 - x_n, x_n - y\rangle \ge 0$ for all $y \in C_n$. Using $F \subset C_n$, we also have $\langle x_0 - x_n, x_n - p\rangle \ge 0$ for all $p \in F$.
So, for $p \in F$ we have

$$
\begin{aligned}
0 &\le \langle x_0 - x_n, x_n - p\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - p\rangle \\
&\le -\|x_0 - x_n\|^2 + \|x_0 - p\|\|x_0 - x_n\|.
\end{aligned}
$$

This implies that

$$\|x_0 - x_n\| \le \|x_0 - p\|.$$

From $x_n = P_{C_n}x_0$ and $x_{n+1} = P_{C_{n+1}}x_0 \in C_{n+1} \subset C_n$, we also have $\langle x_{n+1} - x_n, x_n - x_0\rangle \ge 0$, from the above inequality, we have for all n,

$$
\begin{aligned}
0 &\le \langle x_0 - x_n, x_n - x_{n+1}\rangle \\
&= \langle x_0 - x_n, x_n - x_0 + x_0 - x_{n+1}\rangle \\
&\le -\|x_0 - x_n\|^2 + \|x_0 - x_{n+1}\|\|x_0 - x_n\|
\end{aligned}
$$

and hence

$$\|x_0 - x_n\| \le \|x_0 - x_{n+1}\|.$$

Since $\{\|x_0 - x_n\|\}$ is bounded, $\lim_{n\to\infty}\|x_0 - x_n\|$ exists. Next, we show that $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$. In fact that $x_n = P_{C_n}x_0$ and $x_{n+1} \in C_n$ which imply that

$$
\begin{aligned}
\|x_{n+1} - x_n\|^2 &= \|(x_{n+1} - x_0) - (x_n - x_0)\|^2 \\
&= |x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2 - 2\langle x_{n+1} - x_n, x_n - x_0\rangle \\
&\le \|x_{n+1} - x_0\|^2 - \|x_n - x_0\|^2.
\end{aligned}
$$

So we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.
Since $x_{n+1} \in C_{n+1} \subset C_n$, we have

$$
\begin{aligned}
\|u_n - x_{n+1}\|^2 &\le \|x_n - x_{n+1}\|^2 - (1-\alpha_n)(\alpha_n - \lambda)\|x_n - T^n x_n\|^2 + \theta_n \\
&\le \|x_n - x_{n+1}\|^2 + \theta_n.
\end{aligned}
$$

So we have $\lim_{n\to\infty}\|x_{n+1} - u_n\| = 0$ and $\lim_{n\to\infty}\|x_n - u_n\| = 0$. Observe that

$$
\begin{aligned}
\|y_n - x_n\|^2 &= (1-\alpha_n)^2\|x_n - T^n x_n\|^2 \le [\|y_n - u_n\| + \|u_n - x_n\|]^2 \\
&= \|y_n - u_n\|^2 + 2\|y_n - u_n\|\|u_n - x_n\| + \|u_n - x_n\|^2.
\end{aligned}
$$

Since Φ_r is firmly nonexpansive, for all $p \in F$, we have

$$\begin{aligned}\|u_n - p\|^2 &= \|\Phi_{r_n} y_n - \Phi_{r_n} p\|^2 \le \langle \Phi_{r_n} y_n - \Phi_{r_n} p, y_n - p\rangle \\ &= \langle u_n - p, y_n - p\rangle = \frac{1}{2}(\|u_n - p\|^2 + \|y_n - p\|^2 - \|u_n - y_n\|^2),\end{aligned}$$

and hence

$$\begin{aligned}\|u_n - y_n\|^2 &\le \|y_n - p\|^2 - \|u_n - p\|^2 \\ &\le \|x_n - p\|^2 - \|u_n - p\|^2 + \theta_n - (1-\alpha_n)(\alpha_n - \lambda)\|x_n - T^n x_n\|^2.\end{aligned}$$

So we have

$$\begin{aligned}&(1-\alpha_n)^2 \|x_n - T^n x_n\|^2 \\ &\le \|x_n - p\|^2 - \|u_n - p\|^2 + \theta_n - (1-\alpha_n)(\alpha_n - \lambda)\|x_n - T^n x_n\|^2 \\ &+ 2\|y_n - u_n\|\|u_n - x_n\| + \|u_n - x_n\|^2.\end{aligned}$$

Thus

$$\begin{aligned}&b(1-\lambda)\|x_n - T^n x_n\|^2 \\ &\le \|u_n - x_n\|(\|u_n - p\| + \|x_n - p\|) + \theta_n + 2\|y_n - u_n\|\|u_n - x_n\| + \|u_n - x_n\|^2.\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - T^n x_n\| = 0$, $\lim_{n\to\infty} \|x_n - y_n\| = 0$, $\lim_{n\to\infty} \|u_n - y_n\| = 0$. Observing that

$$\begin{aligned}&\|x_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + \|T^{n+1}x_{n+1} - T^{n+1}x_n\| \\ &\quad + \|T^{n+1}x_n - Tx_n\| \\ &\le \|x_n - x_{n+1}\| + \|x_{n+1} - T^{n+1}x_{n+1}\| + L\|x_{n+1} - x_n\| + L\|T^n x_n - x_n\|.\end{aligned}$$

Hence $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Since $\{x_n\}$ is bounded, there exists a subsequence $\{x_{n_k}\}$ of $\{x_n\}$ such that $x_{n_k} \rightharpoonup \hat{x}$. By Lemma 2.3, we have that $\hat{x} \in F(T)$. From $x_{n_k} \rightharpoonup \hat{x}$, $\|u_n - y_n\| \to 0$ and $\|u_n - x_n\| \to 0$, we have $y_{n_k} \rightharpoonup \hat{x}$ and $u_{n_k} \rightharpoonup \hat{x}$. From $r_n \ge m$, we have $\lim_{n\to\infty} \frac{\|u_n - y_n\|}{r_n} = 0$. By $u_n = \Phi_{r_n} y_n$, we have

$$f(u_n, y) + \frac{1}{r_n}\langle y - u_n, u_n - y_n\rangle \ge 0, \forall y \in C.$$

Replacing n by n_k, we have from (A2) that

$$\frac{1}{r_{n_k}}\langle y - u_{n_k}, u_{n_k} - y_{n_k}\rangle \ge -f(u_{n_k}, y) \ge f(y, u_{n_k}), \forall y \in C.$$

Letting $k \to \infty$, we have from (A4) that $f(y, \hat{x}) \le 0, \forall y \in C$. For $0 < t < 1$ and $y \in C$, define $y_t = ty + (1-t)\hat{x}$. Since $y \in C$ and $\hat{x} \in C$, we have $y_t \in C$ and hence $f(y_t, \hat{x}) \le 0$. So, from (A1) we have

$$\begin{aligned}0 = f(y_t, y_t) &\le tf(y_t, y) + (1-t)f(y_t, \hat{x}) \\ &\le tf(y_t, y).\end{aligned}$$

Dividing by t, we have

$$f(y_t, y) \geq 0, \forall y \in C.$$

Letting $t \downarrow 0$, from (A3) we have

$$f(\hat{x}, y) \geq 0, \forall y \in C.$$

Therefore, $\hat{x} \in EP(f)$.

Let $w = P_F x_0$. From $x_n = P_{C_n} x_0$ and $w \in F \subset C_n$, we have

$$\|x_0 - x_n\| \leq \|x_0 - w\|.$$

Since the norm is weakly lower semicontinuous, we have

$$\begin{aligned} \|x_0 - w\| &\leq \|x_0 - \hat{x}\| \\ &\leq \liminf_{k\to\infty} \|x_0 - x_{n_k}\| \\ &\leq \limsup_{k\to\infty} \|x_0 - x_{n_k}\| \\ &\leq \|x_0 - w\|. \end{aligned}$$

This implies that $\|x_0 - w\| = \|x_0 - \hat{x}\|$ and $\|x_0 - x_{n_k}\| \to \|x_0 - w\|$. It follows that $w = \hat{x}$ and $x_{n_k} \to w$. Therefore $\{x_n\}$ converges strongly to w. □

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