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ANTI FUZZY IDEALS IN WEAK BCC-ALGEBRAS

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Abstract. We characterize the main types of anti fuzzy ideals in weak BCC-algebras.

1. Introduction

BCC-algebras (called also BIK$^+$-algebras) are an algebraic model of BIK$^+$-logic, i.e., implicational logic based on modus ponens and some axioms scheme containing the combinators B, I, and K. Weak BCC-algebras (called also BZ-algebras) have the same partial order as BCC-algebras and BCK-algebras but do not have a minimal element. Many mathematicians studied various types of algebras such as BCI-algebras, B-algebras, implication algebras, G-algebras, Hilbert algebras and do on. All these algebras have one distinguished element, satisfy some common identities and have a similar partial order. In fact, all these algebras are a generalization or a special case of weak BCC-algebras. So, results obtained for weak BCC-algebras are, in some sense, fundamental for these algebras, especially for BCC/BCH/BCI/BCK-algebras.

A very important role in the theory of such algebras plays ideals. In BCK-algebras ideals are induced by partial order or by homomorphisms. All ideals determine congruences. In BCC-algebras there are congruences which are not determined by ideals [3]. Moreover, in BCC-algebras relations determined by ideals (in the same way as in BCK-algebras) are not congruences, in general. So, in BCC-algebras the new concept of ideals should br introduced. Similarly in weak BCC-algebras.

2. Preliminaries

In this section, we give basic definitions and facts on weak BCC-algebras.

Definition 1. A *weak BCC-algebra* X is an abstract algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the following axioms

(i) $((x * y) * (z * y)) * (x * z) = 0$,
(ii) $x * x = 0$,

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(iii) $x * 0 = x$,
(iv) $x * y = y * x = 0 \longrightarrow x = y$.

Weak BCC-algebras also are called *BZ-algebras* (see [4] or [8]).

A weak BCC-algebra satisfying the identity

(v) $0 * x = 0$,

is called a *BCC-algebra*. A BCC-algebra with the condition

(vi) $(x * (x * y)) * y = 0$

is called a *BCK-algebra*.

An algebra $(X, *, 0)$ of type $(2, 0)$ satisfying the axioms (i), (ii), (iii), (iv) and (vi) is called a *BCI-algebra*.

In all these algebras one can define a natural partial order $\leqslant$ putting

$$x \leqslant y \longleftrightarrow x * y = 0.$$

In all BCC/BCK-algebras we have $0 \leqslant x$ for every $x \in X$. Moreover, from (i) it follows that in any (weak) BCC-algebra

$$x \leqslant y \longrightarrow z * y \leqslant z * x \text{ and } x * z \leqslant y * z, \tag{1}$$

for all $x, y, z \in X$.

A non-empty subset A of a weak BCC-algebra is an *ideal* if (I_1) $0 \in A$, (I_2) $y, x * y \in A$ imply $x \in A$. An ideal A such that $y \in A$ and $(x * y) * z \in A$ imply $x * z \in A$ is called a *BCC-ideal*. If $y \in A$ and $x \leqslant y$ then also $x \in A$. In BCK-algebras any ideal is a BCC-ideal, but in (weak) BCC-algebras there are ideals which are not BCC-ideals (see [3]). By a *p-ideal* is mean an ideal A in which $y \in A$ and $(x * z) * (y * z) \in A$ imply $x \in A$. More informations on various types of ideals and BCC-ideals of weak BCC-algebras one can find in [2], [4] and [8].

We use the following abbreviated notation: the expression $(...((x * y) * y) * ...) * y$, where y occurs n times is written as $x * y^n$. Similarly, $x^n * y$ denotes the expression $(x * (... * (x * (x * y))...)$, where x occurs n times. Also, $a \vee b = \max\{a, b\}$.

By a *fuzzy set* on X we mean a function $\mu : X \to [0, 1]$. For any fuzzy set μ defined on X and any $t \in [0, 1]$ we consider two subsets:

$$\mu_t = \{x \in X : \mu(x) \leqslant t\} \text{ and } \mu^t = \{x \in X : \mu(x) \geqslant t\}.$$

The first is called *lower*, the second *upper level set*.

Definition 2. A fuzzy set μ defined on a weak BCC-algebra X is called an *anti fuzzy subalgebra* of X if $\mu(x * y) \leqslant \mu(x) \vee \mu(y)$ for all $x, y \in X$, or an *anti fuzzy ideal* of X, if

(1) $\mu(0) \leqslant \mu(x)$,
(2) $\mu(x) \leqslant \mu(x * y) \vee \mu(y)$ $\forall x, y \in X$.

Note that the condition $\mu(0) \leqslant \mu(x)$ is satisfied by any anti fuzzy subalgebra. Indeed, $\mu(0) = \mu(x * x) \leqslant \mu(x) \vee \mu(x) = \mu(x)$ for every $x \in X$.

The study of anti fuzzy subalgebras and ideals (in BCK-algebras) was initiated by Hong and Jun [5]. Note that in the litarature, anti fuzzy subalgebras (ideals) are also called *doubt fuzzy subalgebras* (*ideals*) (cf. for example [6] or [7]).

Proposition 2.1. *Let μ be an anti fuzzy ideal of a weak BCC-algebra X. Then*

(1) $x \leqslant y \longrightarrow \mu(x) \leqslant \mu(y)$,
(2) $\mu(x * y) \leqslant \mu(x * z) \vee \mu(z * y)$,
(3) $\mu(x * y) = \mu(0) \longrightarrow \mu(x) \leqslant \mu(y)$,
(4) $\mu(x * x^n) \leqslant \mu(x)$,
(5) $\mu(x^n * x) = \mu(x)$ *if n is even,*
(6) $\mu(x^n * x) \leqslant \mu(x)$ *if n is odd,*
(7) $\mu(0 * (0 * x)) \leqslant \mu(x)$

for every $x, y, z \in X$ and all natural n.

Proof. (1) If $x \leqslant y$ then $x * y = 0$, hence $\mu(x) \leqslant \mu(x * y) \vee \mu(y) = \mu(0) \vee \mu(y) = \mu(y)$.

(2) From the definition of a weak BCC-algebra we obtain $(x*y)*(z*y) \leqslant x*z$, which, by (1), implies $\mu((x * y) * (z * y)) \leqslant \mu(x * z)$. Since μ is a an anti fuzzy ideal, the last gives $\mu(x * y) \leqslant \mu((x * y) * (z * y)) \vee \mu(z * y) \leqslant \mu(x * z) \vee \mu(z * y)$.

(3) Let $\mu(x * y) = \mu(0)$. Then $\mu(x) \leqslant \mu(x * y) \vee \mu(y) = \mu(0) \vee \mu(y) = \mu(y)$.

(4), (5), (6) By induction. (8) is obvious. □

3. Anti fuzzy p-ideals

Definition 3. A fuzzy subset μ of a weak BCC-algebra X is called an *anti fuzzy p-ideal* of X if

(1) $\mu(0) \leqslant \mu(x)$,
(2) $\mu(x) \leqslant \mu((x * z) * (y * z)) \vee \mu(y)$

for all $x, y, z \in X$.

Example 1. Consider on the set $X = \{0, a, b, c\}$ the binary operation defined by the following table:

$*$	0	a	b	c
0	0	0	b	b
a	a	0	c	c
b	b	b	0	0
c	c	c	a	0

Then $(X, *, 0)$ is a weak BCC-algebras (cf. [1]). Putting $\mu(0) = t_0$, $\mu(a) = t_1$, $\mu(b) = \mu(c) = t_2$, where $0 \leqslant t_0 < t_1 < t_2 \leqslant 1$, we obrain a fuzzy set μ defined on X. It is not difficult to see that μ is an anti fuzzy p-ideal of X.

Proposition 3.1. *If μ is an anti fuzzy p-ideal of X then $\mu(x) \leqslant \mu(0 * (0 * x))$ for every $x \in X$.*

Proof. Indeed, $\mu(x) \leqslant \mu((x * x) * (0 * x)) \vee \mu(0) = \mu(0 * (0 * x)) \vee \mu(0) = \mu(0 * (0 * x))$. □

Proposition 3.2. *Every anti fuzzy p-ideal is an anti fuzzy ideal.*

Proof. If μ is an anti fuzzy p-ideal of X, then for any $x, y \in X$ we have $\mu(x) \leqslant \mu((x*0)*(y*0)) \vee \mu(y) = \mu((x*y) \vee \mu(y)$, which means that μ is an anti fuzzy ideal. □

The following example shows that the converse is not true.

Example 2. Consider on the set $X = \{0, a, b, c, d\}$ with the operation:

$*$	0	a	b	c	d
0	0	0	b	b	b
a	a	0	b	b	b
b	b	b	0	0	0
c	c	b	a	0	a
d	d	b	a	a	0

Then $(X, *, 0)$ is a weak BCC-algebra. Consider the fuzzy set μ such that $\mu(0) = t_0$, $\mu(a) = t_1$, $\mu(b) = \mu(c) = \mu(d) = t_2$, where $0 \leqslant t_0 < t_1 < t_2 \leqslant 1$. By routine calculation we can verify that μ is an anti fuzzy ideal. It is not an anti fuzzy p-ideal because the inequality $t_1 = \mu(a) \leqslant \mu((a*b)*(0*b)) \vee \mu(0) = \mu(b*b) \vee \mu(0) = \mu(0) \vee \mu(0) = \mu(0) = t_0$ is not true.

Proposition 3.3. *An anti fuzzy ideal μ of a weak BCC-algebra X is its anti fuzzy p-ideal if and only if it satisfies the inequality*

$$\mu(x*y) \leqslant \mu((x*z)*(y*z)).$$

Proof. If μ is an anti fuzzy p-ideal of X, then, according to the definition on an anti fuzzy p-ideal and the axiom (i), we obtain

$$\mu((x*z)*(y*z)) \geqslant \mu(((x*z)*(y*z))*(x*y)) \vee \mu(x*y) = \mu(0) \vee \mu(x*y) = \mu(x*y),$$

i.e., $\mu(x*y) \leqslant \mu((x*z)*(y*z))$.

Assume now that $\mu(x*y) \leqslant \mu((x*z)*(y*z))$ for some anti fuzzy ideal of X. Then also $\mu(x*y) \vee \mu(y) \leqslant \mu((x*z)*(y*z)) \vee \mu(y)$. Since $\mu(x) \leqslant \mu(x*y) \vee \mu(y)$, the last implies $\mu(x) \leqslant \mu((x*z)*(y*z)) \vee \mu(y)$. This means that μ is an anti fuzzy p-ideal of X. □

Proposition 3.4. *Let μ be an anti fuzzy p-ideal* (*ideal*) *of a weak BCC-algebra X. Then the set $A = \{x \in X : \mu(x) = \mu(0)\}$ is a p-ideal* (*ideal*) *of X.*

Proof. Suppose that μ is an anti fuzzy p-ideal of X. Obviously $0 \in A$. Let also $(x*z)*(y*z), y \in A$ for some $x, y, z \in X$. Then $\mu(x) \leqslant \mu((x*z)*(y*z)) \vee \mu(y) = \mu(0) \vee \mu(0) = \mu(0)$. Hence $\mu(x) = \mu(0)$. Thus $x \in A$, i.e., A is a p-ideal of X.

For ideals the proof is analogous. □

Theorem 3.5. *A fuzzy set μ of a weak BCC-algebra X is its anti fuzzy ideal* (*p-ideal*) *if and only if each non-empty lower level set μ_t is an ideal* (*p-ideal*) *of X.*

Proof. Let μ be an anti fuzzy ideal of X. Assume that some lower level set μ_t is non-empty. If $x \in \mu_t$, then also $0 \in \mu_t$ because, according to the definition of μ, $\mu(0) \leqslant \mu(x) \leqslant t$. For $x * y, y \in \mu_t$ we have $\mu(x * y) \leqslant t$ and $\mu(y) \leqslant t$. Since μ is an anti fuzzy ideal of X, $\mu(x) \leqslant \mu(x * y) \vee \mu(y) \leqslant t$. Hence $x \in \mu_t$. So, μ_t is an ideal of X.

To prove the converse suppose that $\mu(0) > \mu(x_0)$ for some $x_0 \in X$. Then for $t_0 = \frac{1}{2}(\mu(0) + \mu(x_0))$ we have $0 \leqslant \mu(x_0) < t_0 < \mu(0)$. So, $x_0 \in \mu_{t_0}$, i.e., μ_{t_0} is non-empty. Since it is an ideal, $0 \in \mu_{t_0}$ which implies $\mu(0) < t_0$. This is a contradiction. Therefore $\mu(0) \leqslant \mu(x)$ for all $x \in X$. Similarly, $\mu(x_0) > \mu(x_0 * y_0) \vee \mu(y_0)$ for some $x_0, y_0 \in X$ means that for $t_0 = \frac{1}{2}(\mu(x_0) + (\mu(x_0 * y_0) \vee \mu(y_0))$ we have $0 \leqslant \mu(x_0 * y_0) \vee \mu(y_0) < t_0 < \mu(x_0)$ which shows that $\mu(x_0 * y_0) < t_0$ and $\mu(y_0) < t_0$, that is $(x_0 * y_0), y_0 \in \mu_{t_0}$. This implies $x_0 \in \mu_{t_0}$, i.e., $\mu(x_0) \leqslant t_0$ which is a contradiction. So, $\mu(x) \leqslant \mu(x * y) \vee \mu(y)$ for all $x, y \in X$. Hence μ is an anti fuzzy ideal of X.

For p-ideals the proof is analogous. □

Definition 4. Let f be a mapping defined on a set X. If μ is a fuzzy subset on X, then the fuzzy subset ν on $f(X)$ defined by

$$\nu(y) = \inf_{x \in f^{-1}(y)} \mu(x) \qquad \forall\, y \in f(X)$$

is called the *image of* μ *under* f. The fuzzy subset $\mu = \nu \circ f$ is called the *preimage of* ν *under* f.

Theorem 3.6. *If* $f : X \to Y$ *is a homomorphism of a weak BCC-algebra* X *onto a weak BCC-algebra* Y*, then the preimage of an anti fuzzy* p*-ideal of* Y *is an anti fuzzy* p*-ideal of* X*.*

Proof. Let $f : X \to Y$ be a homomorphism of a weak BCC-algebra X onto Y and let μ be the preimage of an anti fuzzy p-ideal ν under f. Then $\mu(x) = \nu(f(x))$ for all $x \in X$ and $f(0) = 0'$ is a zero of Y. Since ν is an anti fuzzy p-ideal of Y we have $\mu(0) = \nu(f(0)) = \nu(0') \leqslant \nu(f(x)) = \mu(x)$ for every $x \in X$. By the assumption f is onto Y so for each $y', z' \in Y$ there exist $y, z \in X$ such that $y' = f(y)$, $z' = f(z)$. Thus $\mu(x) = \nu(f(x)) \leqslant \nu((f(x) * z') * (y' * z')) \vee \nu(y') = \nu((f(x) * f(z)) * (f(y) * f(z))) \vee \nu(f(y)) = \nu(f((x * z) * (y * z))) \vee \nu(f(y)) = \mu((x * z) * (y * z)) \vee \mu(y))$, which proves that μ is an anti fuzzy p-ideal of X. □

Remark 1. Results proved in this section are also valid for anti fuzzy BCC-ideals, i.e., fuzzy sets μ such that $\mu(0) \leqslant \mu(x)$ and $\mu(x * z) \leqslant \mu((x * y) * z) \vee \mu(y)$ for all $x, y, z \in X$. The proofs are very similar.

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