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APPROXIMATING COMMON FIXED POINTS OF ONE-STEP ITERATIVE SCHEME WITH ERROR FOR NON-SELF ASYMPTOTICALLY NONEXPANSIVE IN THE INTERMEDIATE SENSE MAPPINGS

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Abstract. In this paper, we study a new one-step iterative scheme with error for approximating common fixed points of non-self asymptotically nonexpansive in the intermediate sense mappings in uniformly convex Banach spaces. Also we have proved weak and strong convergence theorems for above said scheme. The results obtained in this paper extend and improve the recent ones, announced by Zhou et al. [27] and many others.

1. Introduction and Preliminaries

Let E be a real Banach space and let K be a nonempty subset of E. Let $T\colon K \to K$ be a mapping and $F(T) = \{x : Tx = x\}$ be the set of fixed points of T.

A mapping $T\colon K \to K$ is called asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$; $k_n \to 1$ as $n \to \infty$ such that for all $x, y \in K$, the following inequality holds:

$$\|T^n x - T^n y\| \quad \le \quad k_n \|x - y\|, \quad \forall n \ge 1 \tag{1}$$

T is called uniformly L-Lipschitzian if there exists a constant $L > 0$ such that for all $x, y \in K$,

$$\|T^n x - T^n y\| \quad \le \quad L\|x - y\|, \quad \forall n \ge 1 \tag{2}$$

T is called asymptotically nonexpansive type [11] if the following inequality holds:

$$\limsup_{n\to\infty} \sup_{y\in K} \Big\{ \|T^n x - T^n y\| - \|x - y\| \Big\} \quad \le \quad 0, \tag{3}$$

for every $x \in K$, and that T^N be continuous for some $N \ge 1$.

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T is called asymptotically nonexpansive in the intermediate sense [1] if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big\{\|T^n x - T^n y\| - \|x-y\|\Big\} \quad \leq \quad 0. \tag{4}$$

$T\colon K \to E$ is called completely continuous [20] if for all bounded sequence $\{x_n\} \subset K$ there exists a convergent subsequence of $\{Tx_n\}$.

Recall that a Banach space E is called uniformly convex [13] if for every $0 < \varepsilon \leq 2$, there exists $\delta = \delta(\varepsilon) > 0$ such that $\|\frac{x+y}{2}\| \leq 1-\delta$ for every $x, y \in S_E$ and $\|x-y\| \geq \varepsilon$, $S_E = \{x \in E : \|x\| = 1\}$.

A Banach space E is said to be smooth [27] if

$$\lim_{t\to 0} \frac{\|x+ty\| - \|x\|}{t}$$

exists for all $x, y \in S_E$, $S_E = \{x \in E : \|x\| = 1\}$.

The class of asymptotically nonexpansive maps was introduced by Goebel and Kirk [8] as an important generalization of the class of nonexpansive maps (i.e., mappings $T\colon K \to K$ such that $\|Tx - Ty\| \leq \|x-y\|$, $\forall x, y \in K$) who proved that if K is a nonempty closed convex subset of a real uniformly convex Banach space and T is an asymptotically nonexpansive self-mapping of K, then T has a fixed point.

Iterative techniques for approximating fixed points of nonexpansive mappings and asymptotically nonexpansive mappings have been studied by various authers (see e.g., [2], [3, 4], [6], [10], [16], [17], [18], [19], [20, 21], [24], [25, 22, 23, 26]) using the Mann iteration method (see e.g., [12]) or the Ishikawa iteration method (see e.g., [9]).

In [20, 21], Schu introduced a modified Mann process to approximate fixed points of asymptotically nonexpansive self-maps defined on nonempty closed convex and bounded subsets of a Hilbert space H.

In 1994, Rhoades [19] extended the Schu's result to uniformly convex Banach space using a modified Ishikawa iteration method.

In all the above results, the operator T remains a self-mapping of a nonempty closed convex subset K of a uniformly convex Banach space. If, however, the domain of T, $D(T)$ is a proper subset of E (and this is the case in several applications), and T maps $D(T)$ into E, then the iteration processes of Mann and Ishikawa studied by these authors; and their modifications introduced by Schu may fail to be well defined.

In 2003, Chidume et al [5] studied the iterative scheme defined by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geq 1, \end{aligned} \tag{5}$$

in the framework of uniformly convex Banach space, where K is a closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retract. $T\colon K \to E$ is an non-self asymptotically nonexpansive with sequence $\{k_n\} \subset [1, \infty)$, $k_n \to 1$ as $n \to \infty$. $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in

$[0,1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1-\epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. They proved strong and weak convergence theorems for non-self asymptotically nonexpansive maps.

In 2004, Chidume et al [7] studied the iterative scheme defined by

$$\begin{aligned} x_1 &\in K, \\ x_{n+1} &= P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \quad n \geq 1, \end{aligned} \tag{6}$$

in the framework of uniformly convex Banach space, where K is a closed convex nonexpansive retract of a real uniformly convex Banach space E with P as a nonexpansive retract. $T\colon K \to E$ be non-self asymptotically nonexpansive in the intermediate sense mapping and $\{\alpha_n\}_{n=1}^{\infty}$ is a real sequence in $[0,1]$ satisfying the condition $\epsilon \leq \alpha_n \leq 1-\epsilon$ for all $n \geq 1$ and for some $\epsilon > 0$. They proved strong and weak convergence theorems for said mapping.

Recently, Zhou et al. [27] gave a new iterative scheme for approximating common fixed point of two non-self asymptotically nonexpansive mappings with respect to P and proving some strong and weak convergence theorems for such mappings in uniformly convex Banach spaces.

The aim of this paper is to study the approximating common fixed point of two non-self asymptotically nonexpansive in the intermediate sense mappings with respect to P and to prove weak and strong convergence theorems, and to define one-step iterative scheme with error which modified Zhou et al. [27] iteration scheme as follows:

Let E be a real normed linear space and let K be a nonempty closed convex subset of E. Let $P\colon E \to K$ be the nonexpansive retraction of E onto K and let $T_1, T_2\colon K \to E$ be two non-self asymptotically nonexpansive in the intermediate sense mappings.

Algorithm 1.1. *For a given $x_1 \in K$, compute the sequence $\{x_n\}$ by the iterative scheme*

$$x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n + \lambda_n u_n, \quad \forall n \geq 1, \tag{7}$$

where $\{\alpha_n\}$, $\{\beta_n\}$, $\{\gamma_n\}$ and $\{\lambda_n\}$ are real sequences in $[0,1]$ and satisfying $\alpha_n + \beta_n + \gamma_n + \lambda_n = 1$ and $\{u_n\}$ is a bounded sequence in K. The iterative scheme (7) is called the one-step iterative scheme with errors.

If $\lambda_n = 0$, then (7) reduces to the iterative scheme, defined by Zhou et al. [27], as follows:

Algorithm 1.2. *For a given $x_1 \in K$, compute the sequence $\{x_n\}$ by the iterative scheme*

$$x_{n+1} = \alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n, \quad \forall n \geq 1, \tag{8}$$

where $\{\alpha_n\}$, $\{\beta_n\}$ and $\{\gamma_n\}$ are real sequences in $[0,1]$ and satisfying $\alpha_n + \beta_n + \gamma_n = 1$.

A subset K of E is called retract of E if there exists a continuous mapping $P\colon E \to K$ such that $Px = x$ for all $x \in K$. Every closed convex subset of a uniformly convex Banach space is a retract. A mapping $P\colon E \to K$ is called retraction if $P^2 = P$. Note that if a mapping P is a retraction, then $Pz = z$ for all z in the range of P.

Let C, D be subsets of a Banach space E. Then a mapping $P\colon C \to D$ is said to be sunny if $P(Px + t(x - Px)) = Px$, whenever $Px + t(x - Px) \in C$ for all $x \in C$ and $t \geq 0$.

Let K be a subset of a Banach space E. For all $x \in K$ is defined a set $I_K(x)$ by $I_K(x) = \{x + \lambda(y - x) : \lambda > 0, y \in K\}$.

A non-self mapping $T\colon K \to E$ is said be inward if $Tx \in I_K(x)$ for all $x \in K$, and T is said to be weakly inward if $Tx \in \overline{I_K(x)}$ for all $x \in K$.

A Banach space E is said to satisfy Opial's condition [15] if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$ it follows that $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ for all $y \in E$ with $y \neq x$.

A mapping $T\colon K \to E$ is said to be demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in K and each $x \in E$, x_n converging weakly to x and Tx_n converging strongly to y, it follows that $x \in K$ and $Tx = y$.

Now, we recall the well known concept and the following useful lemmas to prove our main results:

Definition 1. ([27, Definition 1.5]) Let K be a nonempty subset of real normed linear space E. Let $P\colon E \to K$ be nonexpansive retraction of E onto K.

(1) A non-self mapping $T\colon K \to E$ is called asymptotically nonexpansive with respect to P if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $k_n \to 1$ as $n \to \infty$ such that

$$\|(PT)^n x - (PT)^n y\| \quad \leq \quad k_n \|x - y\|, \ \forall x, y \in K, \ n \geq 1. \tag{9}$$

(2) T is said to be uniformly L-Lipschitzian with respect to P if there exists a constant $L > 0$ such that

$$\|(PT)^n x - (PT)^n y\| \quad \leq \quad L\|x - y\|, \ \forall x, y \in K, \ n \geq 1. \tag{10}$$

Lemma 1.3. ([23, Lemma 1]) *Let $\{\alpha_n\}_{n=1}^{\infty}$, $\{\beta_n\}_{n=1}^{\infty}$ be two sequences of non-negative real numbers satisfying the inequality*

$$\alpha_{n+1} \leq \alpha_n + \beta_n, \quad \forall n \geq 1.$$

If $\sum_{n=1}^{\infty} \beta_n < \infty$, then $\lim_{n\to\infty} \alpha_n$ exists.

Lemma 1.4. ([14, Lemma 3]) *Let E be a uniformly convex Banach space and $B_r = \{x \in E : \|x\| < r, r > 0\}$. Then there exists a continuous strictly increasing convex function $g\colon [0, \infty) \to [0, \infty)$ with $g(0) = 0$ such that*

$$\|\lambda x + \mu y + \xi z + \nu w\|^2 \leq \lambda\|x\|^2 + \mu\|y\|^2 + \xi\|z\|^2 + \nu\|w\|^2 - \lambda\mu g(\|x - y\|)$$

for all $x, y, z, w \in B_r$ and $\lambda, \mu, \xi, \nu \in [0, 1]$ with $\lambda + \mu + \xi + \nu = 1$.

Lemma 1.5. ([27, Lemma 2.2]) *Let E be a real smooth Banach space, let K be a nonempty closed subset of E with P as a sunny nonexpansive retraction, and let $T\colon K \to E$ be a mapping satisfying weakly inward condition. Then $F(PT) = F(T)$.*

The following demiclosedness principle for non-self mapping follows from ([7], Theorem 3.5).

Lemma 1.6. *Let E be a real smooth and uniformly convex Banach space and K a nonempty closed convex subset of E with P as a sunny nonexpansive retraction. Let $T\colon K \to E$ be a weakly inward and asymptotically nonexpansive in the intermediate sense mapping with respect to P. Then $I-T$ is demiclosed at zero, that is, sequence $\{x_n\}$ converging weakly to x^* and $x_n - Tx_n$ converging strongly to 0 imply that $Tx^* = x^*$.*

Proof. Suppose that $\{x_n\} \subset K$ converges weakly to $x^* \in K$ and $x_n - Tx_n \to 0$ as $n \to \infty$. We will prove that $Tx^* = x^*$. Indeed, since $\{x_n\} \subset K$, by the property of P, we have $Px_n = x_n$ for all $n \geq 1$ and so $x_n - PTx_n \to 0$ as $n \to \infty$. By Chidume et al. [7, Theorem 3.5], we conclude that $x^* = PTx^*$. Since $F(PT) = F(T)$ by Lemma 1.3, we have $Tx^* = x^*$. This completes the proof. $\square$

2. Main results

In this section, we prove, weak and strong convergence theorems of one-step iterative scheme with errors for two non-self asymptotically nonexpansive in the intermediate sense mappings. In order to prove our main results, we need the following definition and lemmas.

Definition 2. Let K be a nonempty subset of real normed linear space E. Let $P\colon E \to K$ be nonexpansive retraction of E onto K.

(1) A mapping $T\colon K \to E$ is called non-self asymptotically nonexpansive type with respect to P if the following inequality holds:

$$\limsup_{n\to\infty} \sup_{y\in K} \Big\{ \|(PT)^n x - (PT)^n y\| - \|x-y\| \Big\} \quad \leq \quad 0, \tag{11}$$

for every $x \in K$, and that T^N be continuous for some $N \geq 1$.

(2) T is called non-self asymptotically nonexpansive in the intermediate sense with respect to P if T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in K} \Big\{ \|(PT)^n x - (PT)^n y\| - \|x-y\| \Big\} \quad \leq \quad 0. \tag{12}$$

Remark 1. If T is self mapping, then P becomes the identity mapping, so that (11) and (12) are reduced to (3) and (4), respectively.

Lemma 2.1. *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $T_1, T_2\colon K \to E$ be two non-self asymptotically* *nonexpansive in the intermediate sense mappings with respect to a nonexpansive retraction P of E onto K. Put*

$$A_n = \max\Big\{\sup_{x,y\in K}(\|(PT_1)^n x - (PT_1)^n y\| - \|x-y\|),\ 0\Big\}, \qquad \forall n \geq 1,$$

and

$$B_n = \max\Big\{\sup_{x,y\in K}(\|(PT_2)^n x - (PT_2)^n y\| - \|x-y\|),\ 0\Big\}, \qquad \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$, respectively. Suppose that $\{x_n\}$ is the sequence defined by (7) with $\sum_{n=1}^{\infty} \lambda_n < \infty$. If $F(T_1)\cap F(T_2) \neq \emptyset$, then $\lim_{n\to\infty}\|x_n - q\|$ *exists for any $q \in F(T_1)\cap F(T_2)$.*

Proof. Let $q \in F(T_1)\cap F(T_2)$, using the fact that P is a nonexpansive retraction and (7), then we have

$$\begin{aligned}
\|x_{n+1}-q\| &= \|\alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n + \lambda_n u_n - q\| \\
&\leq \alpha_n\|x_n - q\| + \beta_n\|(PT_1)^n x_n - q\| + \gamma_n\|(PT_2)^n x_n - q\| \\
&\quad + \lambda_n\|u_n - q\| \\
&\leq \alpha_n\|x_n - q\| + \beta_n[\|x_n - q\| + A_n] + \gamma_n[\|x_n - q\| + B_n] \\
&\quad + \lambda_n\|u_n - q\| \\
&\leq (\alpha_n + \beta_n + \gamma_n)\|x_n - q\| + \beta_n A_n + \gamma_n B_n + \lambda_n\|u_n - q\| \\
&= (1-\lambda_n)\|x_n - q\| + \beta_n A_n + \gamma_n B_n + \lambda_n\|u_n - q\| \\
&\leq \|x_n - q\| + \beta_n A_n + \gamma_n B_n + \lambda_n\|u_n - q\| \\
&= \|x_n - q\| + t_n
\end{aligned} \tag{13}$$

where $t_n = \beta_n A_n + \gamma_n B_n + \lambda_n\|u_n - q\|$. Since $\sum_{n=1}^{\infty} A_n < \infty$, $\sum_{n=1}^{\infty} B_n < \infty$ and $\sum_{n=1}^{\infty} \lambda_n < \infty$, we see that $\sum_{n=1}^{\infty} t_n < \infty$. It follows from Lemma 1.1 that $\lim_{n\to\infty}\|x_n - q\|$ exists. This completes the proof. □

Lemma 2.2. *Let E be a uniformly convex Banach space, K be a nonempty closed convex subset of E and $T_1, T_2\colon K \to E$ be two non-self uniformly L-Lipschitzian and non-self asymptotically nonexpansive in the intermediate sense mappings with respect to a nonexpansive retraction P of E onto K. Put*

$$A_n = \max\Big\{\sup_{x,y\in K}(\|(PT_1)^n x - (PT_1)^n y\| - \|x-y\|),\ 0\Big\}, \qquad \forall n \geq 1,$$

and

$$B_n = \max\Big\{\sup_{x,y\in K}(\|(PT_2)^n x - (PT_2)^n y\| - \|x-y\|),\ 0\Big\}, \qquad \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$, respectively. Suppose that $\{x_n\}$ is the sequence defined by (7) with the following restrictions:

(i) $\sum_{n=1}^{\infty} \lambda_n < \infty$,

(ii) $0 < \liminf_n \alpha_n$, $0 < \liminf_n \beta_n$ *and* $0 < \liminf_n \gamma_n$.

If $F(T_1) \cap F(T_2) \neq \emptyset$, *then*

$$\lim_{n\to\infty} \|x_n - (PT_1)x_n\| = \lim_{n\to\infty} \|x_n - (PT_2)x_n\| = 0.$$

Proof. By Lemma 2.1 $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F(T_1) \cap F(T_2)$. It follows that $\{x_n - q\}$, $\{(PT_1)^n x_n - q\}$ and $\{(PT_2)^n x_n - q\}$ are bounded. Also $\{u_n - q\}$ is bounded by the assumption. We may assume that such sequences belong to B_r, where $B_r = \{x \in E : \|x\| < r,\ r > 0\}$.

From (7), by the property of P and Lemma 1.2, we have

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
= \ & \|[\alpha_n x_n + \beta_n (PT_1)^n x_n + \gamma_n (PT_2)^n x_n + \lambda_n u_n] - q\|^2 \\
= \ & \|\alpha_n (x_n - q) + \beta_n ((PT_1)^n x_n - q) + \gamma_n ((PT_2)^n x_n - q) + \lambda_n (u_n - q)\|^2 \\
\leq \ & \alpha_n \|x_n - q\|^2 + \beta_n \|(PT_1)^n x_n - q\|^2 + \gamma_n \|(PT_2)^n x_n - q\|^2 \\
& + \lambda_n \|u_n - q\|^2 - \alpha_n \beta_n g(\|x_n - (PT_1)^n\|) \\
= \ & \alpha_n \|x_n - q\|^2 + \beta_n [\|(PT_1)^n x_n - q\|^2 - \|x_n - q\|^2] + \beta_n \|x_n - q\|^2 \\
& + \gamma_n [\|(PT_2)^n x_n - q\|^2 - \|x_n - q\|^2] + \gamma_n \|x_n - q\|^2 + \lambda_n \|u_n - q\|^2 \\
& - \alpha_n \beta_n g(\|x_n - (PT_1)^n\|) \qquad (14)
\end{aligned}$$

and

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
\leq \ & \alpha_n \|x_n - q\|^2 + \beta_n \|(PT_1)^n x_n - q\|^2 + \gamma_n \|(PT_2)^n x_n - q\|^2 \\
& + \lambda_n \|u_n - q\|^2 - \alpha_n \gamma_n g(\|x_n - (PT_2)^n\|) \\
= \ & \alpha_n \|x_n - q\|^2 + \beta_n [\|(PT_1)^n x_n - q\|^2 - \|x_n - q\|^2] + \beta_n \|x_n - q\|^2 \\
& + \gamma_n [\|(PT_2)^n x_n - q\|^2 - \|x_n - q\|^2] + \gamma_n \|x_n - q\|^2 + \lambda_n \|u_n - q\|^2 \\
& - \alpha_n \gamma_n g(\|x_n - (PT_2)^n\|). \qquad (15)
\end{aligned}$$

From (14);

$$\begin{aligned}
& \|x_{n+1} - q\|^2 \\
\leq \ & \alpha_n \|x_n - q\|^2 + \beta_n (\|(PT_1)^n x_n - q\| + \|x_n - q\|)(\|(PT_1)^n x_n - q\| \\
& - \|x_n - q\|) + \beta_n \|x_n - q\|^2 + \gamma_n (\|(PT_2)^n x_n - q\| + \|x_n - q\|) \\
& (\|(PT_2)^n x_n - q\| - \|x_n - q\|) + \gamma_n \|x_n - q\|^2 + \lambda_n \|u_n - q\|^2 \\
& - \alpha_n \beta_n g(\|x_n - (PT_1)^n\|) \\
\leq \ & \alpha_n \|x_n - q\|^2 + \beta_n (\|(PT_1)^n x_n - q\| + \|x_n - q\|) \\
& \sup_{x\in K} (\|(PT_1)^n x - q\| - \|x - q\|) + \beta_n \|x_n - q\|^2 \\
& + \gamma_n (\|(PT_2)^n x_n - q\| + \|x_n - q\|) \sup_{x\in K} (\|(PT_2)^n x - q\| - \|x - q\|) \\
& + \gamma_n \|x_n - q\|^2 + \lambda_n \|u_n - q\|^2 - \alpha_n \beta_n g(\|x_n - (PT_1)^n\|)
\end{aligned}$$

$$\begin{aligned}
&\leq (\alpha_n+\beta_n+\gamma_n)\|x_n-q\|^2+\beta_n(\|(PT_1)^nx_n-q\|+\|x_n-q\|)A_n \\
&\quad +\gamma_n(\|(PT_2)^nx_n-q\|+\|x_n-q\|)B_n+\lambda_n\|u_n-q\|^2 \\
&\quad -\alpha_n\beta_n g(\|x_n-(PT_1)^n\|) \\
&= (1-\lambda_n)\|x_n-q\|^2+\beta_n(\|(PT_1)^nx_n-q\|+\|x_n-q\|)A_n \\
&\quad +\gamma_n(\|(PT_2)^nx_n-q\|+\|x_n-q\|)B_n+\lambda_n\|u_n-q\|^2 \\
&\quad -\alpha_n\beta_n g(\|x_n-(PT_1)^n\|) \\
&\leq \|x_n-q\|^2+\beta_n(\|(PT_1)^nx_n-q\|+\|x_n-q\|)A_n \\
&\quad +\gamma_n(\|(PT_2)^nx_n-q\|+\|x_n-q\|)B_n+\lambda_n\|u_n-q\|^2 \\
&\quad -\alpha_n\beta_n g(\|x_n-(PT_1)^n\|) \\
&\leq \|x_n-q\|^2+\eta_nA_n+\theta_nB_n+\lambda_n\|u_n-q\|^2 \\
&\quad -\alpha_n\beta_n g(\|x_n-(PT_1)^n\|) \qquad (16)
\end{aligned}$$

where $\eta_n=\beta_n(\|(PT_1)^nx_n-q\|+\|x_n-q\|)$ and $\theta_n=\gamma_n(\|(PT_2)^nx_n-q\|+\|x_n-q\|)$.

Then

$$\begin{aligned}
\alpha_n\beta_n g(\|x_n-(PT_1)^n\|) \leq\ & \|x_n-q\|^2-\|x_{n+1}-q\|^2+\eta_nA_n \\
&+\theta_nB_n+\lambda_n\|u_n-q\|^2. \qquad (17)
\end{aligned}$$

Since $\sum_{n=1}^{\infty}A_n<\infty$, $\sum_{n=1}^{\infty}B_n<\infty$, $\sum_{n=1}^{\infty}\lambda_n<\infty$ and $\lim_{n\to\infty}\|x_n-q\|$ exists, it follows (17) that

$$\limsup_{n\to\infty}\alpha_n\beta_n g(\|x_n-(PT_1)^nx_n\|)=0.$$

Since g is continuous strictly increasing function with $g(0)=0$ and $0<\liminf_n\alpha_n$, $0<\liminf_n\beta_n$, we conclude that

$$\lim_{n\to\infty}\|x_n-(PT_1)^nx_n\|=0. \qquad (18)$$

By using a similar method, together with inequality (15), it can be shown that

$$\lim_{n\to\infty}\|x_n-(PT_2)^nx_n\|=0. \qquad (19)$$

Next, we show that $\lim_{n\to\infty}\|x_n-(PT_1)x_n\|=0$ and $\lim_{n\to\infty}\|x_n-(PT_2)x_n\|=0$. Using (7), we have

$$\|x_{n+1}-x_n\| \quad\leq\quad \beta_n\|(PT_1)^nx_n-x_n\|+\gamma_n\|(PT_2)^nx_n-x_n\|+\lambda_n\|u_n-x_n\|.$$

Using (18), (19) and $\lim_{n\to\infty}\lambda_n=0$, we have

$$\lim_{n\to\infty}\|x_{n+1}-x_n\|=0. \qquad (20)$$

Now, consider

$$\begin{aligned}
&\|x_n - (PT_1)x_n\| \\
\leq\ &\|x_n - x_{n+1}\| + \|x_{n+1} - (PT_1)^{n+1}x_{n+1}\| \\
&+\|(PT_1)^{n+1}x_{n+1} - (PT_1)^{n+1}x_n\| + \|(PT_1)^{n+1}x_n - (PT_1)x_n\| \\
\leq\ &\|x_n - x_{n+1}\| + \|x_{n+1} - (PT_1)^{n+1}x_{n+1}\| \\
&+L\|x_{n+1} - x_n\| + L\|(PT_1)^n x_n - x_n\|, \qquad (21)
\end{aligned}$$

using (18) and (20) in (21), we have

$$\lim_{n\to\infty} \|x_n - (PT_1)x_n\| = 0. \qquad (22)$$

Similarly, we have

$$\begin{aligned}
&\|x_n - (PT_2)x_n\| \\
\leq\ &\|x_n - x_{n+1}\| + \|x_{n+1} - (PT_2)^{n+1}x_{n+1}\| \\
&+\|(PT_2)^{n+1}x_{n+1} - (PT_2)^{n+1}x_n\| + \|(PT_2)^{n+1}x_n - (PT_2)x_n\| \\
\leq\ &\|x_n - x_{n+1}\| + \|x_{n+1} - (PT_2)^{n+1}x_{n+1}\| \\
&+L\|x_{n+1} - x_n\| + L\|(PT_2)^n x_n - x_n\|, \qquad (23)
\end{aligned}$$

using (19) and (20) in (23), we have

$$\lim_{n\to\infty} \|x_n - (PT_2)x_n\| = 0. \qquad (24)$$

This completes the proof. □

Theorem 2.3. *Let K be a nonempty closed convex subset of a real smooth and uniformly convex Banach space E. Let $T_1, T_2 \colon K \to E$ be two weakly inward and non-self asymptotically nonexpansive in the intermediate sense mappings with respect to P as a sunny nonexpansive retraction of E onto K and T_1, T_2 be non-self uniformly L-Lipschitzian mappings with respect to P. Put*

$$A_n = \max\Big\{ \sup_{x,y\in K} (\|(PT_1)^n x - (PT_1)^n y\| - \|x - y\|),\ 0\Big\}, \qquad \forall n \geq 1,$$

and

$$B_n = \max\Big\{ \sup_{x,y\in K} (\|(PT_2)^n x - (PT_2)^n y\| - \|x - y\|),\ 0\Big\}, \qquad \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$, respectively. Suppose that $\{x_n\}$ is the sequence defined by (7) with the following restrictions:

(i) $\sum_{n=1}^{\infty} \lambda_n < \infty$,

(ii) $0 < \liminf_n \alpha_n$, $0 < \liminf_n \beta_n$ and $0 < \liminf_n \gamma_n$.

If one of T_1 and T_2 is completely continuous and $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges strongly to a common fixed point of the mappings T_1 and T_2.

Proof. From Lemma 2.1, we know that $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F(T_1) \cap F(T_2)$, then $\{x_n\}$ is bounded. By Lemma 2.2, we have

$$\lim_{n\to\infty} \|x_n - (PT_1)x_n\| = 0 \quad \text{and} \quad \lim_{n\to\infty} \|x_n - (PT_2)x_n\| = 0. \qquad (25)$$

Suppose that T_1 is completely continuous and note that $\{x_n\}$ is bounded. We conclude that there exists subsequence $\{PT_1x_{n_j}\}$ of $\{PT_1x_n\}$ such that $\{PT_1x_{n_j}\}$ converges. Therefore, from (25), $\{x_{n_j}\}$ converges. Let $x_{n_j} \to q^*$ as $j \to \infty$. By the continuity of P, T_1, T_2 and (25), we have $q^* = PT_1q^* = PT_2q^*$. Since $F(PT_1) = F(T_1)$ and $F(PT_2) = F(T_2)$ by Lemma 1.3, we have $q^* = T_1q^* = T_2q^*$. Thus $\{x_n\}$ converges strongly to a common fixed point q^* of T_1 and T_2. This completes the proof. □

Theorem 2.4. *Let K be a nonempty closed convex subset of a real smooth and uniformly convex Banach space E satisfying Opial's condition. Let $T_1, T_2 \colon K \to E$ be two weakly inward and non-self asymptotically nonexpansive in the intermediate sense mappings with respect to P as a sunny nonexpansive retraction of E onto K and T_1, T_2 be non-self uniformly L-Lipschitzian mappings with respect to P. Put*

$$A_n = \max\Big\{\sup_{x,y\in K}(\|(PT_1)^n x - (PT_1)^n y\| - \|x-y\|),\ 0\Big\}, \qquad \forall n \geq 1,$$

and

$$B_n = \max\Big\{\sup_{x,y\in K}(\|(PT_2)^n x - (PT_2)^n y\| - \|x-y\|),\ 0\Big\}, \qquad \forall n \geq 1,$$

such that $\sum_{n=1}^{\infty} A_n < \infty$ and $\sum_{n=1}^{\infty} B_n < \infty$, respectively. Suppose that $\{x_n\}$ is the sequence defined by (7) with the following restrictions:

(i) $\sum_{n=1}^{\infty} \lambda_n < \infty$,

(ii) $0 < \liminf_n \alpha_n$, $0 < \liminf_n \beta_n$ and $0 < \liminf_n \gamma_n$.

If $F(T_1) \cap F(T_2) \neq \emptyset$, then $\{x_n\}$ converges weakly to a common fixed point of the mappings T_1 and T_2.

Proof. For any $q \in F(T_1) \cap F(T_2)$, by Lemma 2.1, we know $\lim_{n\to\infty} \|x_n - q\|$ exists. We now prove that $\{x_n\}$ has a unique weakly subsequential limit in $F(T_1) \cap F(T_2)$. To prove this, let q_1 and q_2 be weak limits of the subsequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$ of $\{x_n\}$ respectively. By Lemma 2.2 and $I - T_1$ is demiclosed with respect to zero by Lemma 1.4, we obtain $T_1q_1 = q_1$. Similarly, $T_2q_1 = q_1$. Again in the same way as above, we can prove that $q_2 \in F(T_1) \cap F(T_2)$. Next, we prove the uniqueness. For this we suppose that $q_1 \neq q_2$, then by the Opial's condition

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - q_1\| &= \lim_{i\to\infty} \|x_{n_i} - q_1\| \\
&< \lim_{i\to\infty} \|x_{n_i} - q_2\| \\
&= \lim_{n\to\infty} \|x_n - q_2\| \\
&= \lim_{j\to\infty} \|x_{n_j} - q_2\| \\
&< \lim_{j\to\infty} \|x_{n_j} - q_1\| \\
&= \lim_{n\to\infty} \|x_n - q_1\|.
\end{aligned}$$

This is a contradiction. Hence $\{x_n\}$ converges weakly to a common fixed point of T_1 and T_2. This completes the proof. □

Remark 2. Our results extend the corresponding results of Zhou et al. [27] to the case of more general class of non-self asymptotically nonexpansive mappings and one-step iteration scheme with error considered in this paper.

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